Deformation Quantization and Cohomologies of Poisson, Graded, and Homotopy algebras

THESIS

Presented on 12 November 2008 in Luxembourg

by

Mourad Ammar
Born on 19 January 1980 in Sfax (Tunisia)

In order to receive the academic degree of

DOCTOR IN MATHEMATICS

OF THE UNIVERSITY OF LUXEMBOURG

and

OF THE PAUL-VERLAINE UNIVERSITY IN METZ

Doctoral committee

Prof. Dr Didier ARNAL, Referee
University of Bourgogne, France

Prof. Dr Martin BORDEMANN, Referee
University of Haute-Alsace Mulhouse, France

Prof. Dr Simone GUTT, Advisor
The Paul-Verlaine University in Metz, France

Prof. Dr Yvette KOSMANN-SCHWARZBACH, President
École Polytechnique, Paris, France

Prof. Dr Pierre LECOMTE, Referee
University of Liège, Belgium

Prof. Dr Norbert PONCIN, Advisor
University of Luxembourg, Luxembourg

Acknowledgements

This work benefited from the encouragement of many. So let me acknowledge my gratitude to those who inspired and who supported me throughout these 4 years, to those who pushed me to think harder and go further, and to all those who just were there when I needed them to be there.

My first thoughts therefore go out to Professor Simone Gutt. Thanks to your great support and all your encouragement, this work has had all the success I could have hoped and wished for. Thank you, Simone, for having enthused me to take the road I travel by and for making this the difference.

My second special thank you is dedicated to Professor Norbert Poncin. Thank you, Norbert, for this good training, your advice, your stimulation and understanding. What can I say when words are not enough to express all the gratitude I owe you. We both know that this work is what it is thanks to you. And for this I owe you my deepest respect and humblest THANK YOU.

Thirdly, I wish to thank those who supported my work over the years and who, through their intellectual contribution, pushed me to think further: Professors Véronique Chloup and Mohsen Mahsmoudi, as well as the belated Professor Guy Kass whom we all miss dearly.

A warm and hearty thank you to the referees and the members of the jury: Professors Didier Arnal, Martin Bordemann, Pierre Lecomte and Martin Schlichenmaier, as well as to Professor Yvette Kosmann-Schwarzbach who accepted to chair the doctoral committee. Thank you for all the time you invested in me, for the faith you gave to my work and to my abilities.

Lastly, on a more personal note, to my parents and brothers, whom I love, thank you for your enduring support and unconditional love.

To all my friends and colleagues in Luxembourg and Metz, for your friendship, all the discussions and the assistance, I owe you my gratitude.

Thank you to the Universities of Luxembourg and Metz for welcoming me in your warmest way. A special thank you to the Luxembourg mathematics research unit for having supported me financially and, thus giving me the opportunity to work in a such wonderful team.

And, finally, for her love and support, for the inspiration and the improvements, thank you, Melanie, for being as wonderful as you are.

Contents

Chapter 1

Introduction

1.1 General framework

This thesis "Deformation Quantization and Cohomologies of Poisson, Graded, and Homotopy algebras" is located at the interface of Deformation Quantization and Poisson Geometry.

1.1.1 Deformation Quantization

When switching from the Hamiltonian model of Classical Mechanics to the usual model of Quantum Mechanics, we completely change the nature of the observables. From functions on the phase space, we pass to operators on some Hilbert space. The commutator bracket $[-,-]$ of these operators substitutes for the classical Poisson bracket $\{-,-\}$ of functions.

The transition from Classical Mechanics to Quantum Mechanics is provided by Heisenberg's rules. These rules entail the uncertainty principle and Dirac's equation $[\hat{f},\hat{g}] = ih\widehat{\{f,g\}}$, where f,g are functions of the phase space, $\hat{}$ denotes the quantization map, and h is Planck's constant. However, Van Hove's theorem (1952) states that this quantization cannot be extended to all phase space functions—and even not to all polynomials—, in such a way that Heisenberg's rules and Dirac's equation be still valid. The way out is to look for a quantization that verifies the weakened Dirac equation

$$[\hat{f},\hat{g}] = ih\widehat{\{f,g\}} + h\varepsilon(h), \tag{1.1}$$

where $\varepsilon(h)$ tends to 0 with h. Weyl's quantization W meets both requirements, Heisenberg's rules and Dirac's weak equation. Indeed, it quantizes any monomial in positions q_α and momenta p^α by the symmetrized product of the corresponding operators $\hat{q}_\alpha$ and $\hat{p}^\alpha$, which are of course given by Heisenberg's rules. The map W is not a homomorphism from classical to quantum observables, i.e. in general $W(f.g) \neq W(f) \circ W(g)$, where . is the pointwise product. One observes that $W(f) \circ W(g) = W(f \star g)$. Here $f \star g$ denotes the Moyal-Vey product

$$f \star g = f.g + \nu\{f,g\} + \sum_{k\geq 2} \nu^k c_k(f,g), \tag{1.2}$$

where $\nu = ih/2$ and where the c_k are bidifferential operators on the function space, say N, which vanish on constants and verify $c_k(f,g) = (-1)^k c_k(g,f)$. It is now easily seen that

$$[W(f),W(g)] = W(f \star g - g \star f) = ihW(\{f,g\}) - \frac{ih^3}{4} W(c_3(f,g)) + \ldots, \tag{1.3}$$

so that condition (1.1) is actually satisfied. Moreover, the Moyal $\star$ product is a formal deformation of the associative algebra $(N,.)$ and leads via antisymmetrization to a formal deformation of the Poisson algebra $(N,\{-,-\})$.

A seminal idea of Flato is that our description of Physics, should evolve, when facing a paradox, to a higher level by means of an appropriate deformation. In this perspective, Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer suggest around 1975, in a founding article that appeared in "Annals of Physics", to abandon the representation of classical observables by linear operators, and to construct a model of Quantum Mechanics via deformation of the algebraic structure of the observable space N. Roughly speaking, the task is the construction, on any symplectic or Poisson manifold, of a $\star$-product similar to Moyal's product. This $\star$-product then allows endowing the space $N[[\nu]]$ of all formal series in ν with coefficients in N, with an associative noncommutative algebra structure, as well as with a Lie algebra structure; these algebras are formal deformations of $(N,.)$ and $(N,\{-,-\})$ respectively. Hence, in Deformation Quantization, Quantum Mechanics appears as a deformation of Classical Mechanics, as a deformation from commutativity to noncommutativity, such that the trace of noncommutativity on the classical level is the Poisson bracket.

1.1.2 Poisson Geometry

Poisson Geometry is the geometry of Poisson manifolds, i.e. smooth manifolds endowed with a bivector field that squares to zero under the Schouten-Nijenhuis bracket. This field allows defining Hamiltonian vector fields and thus leads to a general distribution, which turns out to be completely integrable. The Poisson structure induces on each maximal integral submanifold a symplectic structure, so that a Poisson manifold can be viewed as a smooth concatenation of symplectic manifolds. Of course, a Poisson manifold can also be thought of as a smooth manifold whose function space carries a Poisson bracket. Poisson structures were studied by Poisson, Jacobi and Lie, then by Kirillov and Lichnerowicz.

As from the 1970's, Poisson Geometry developed rapidly with connections with many fields of Mathematics and Theoretical Physics, such as variational calculus, geometric mechanics, noncommutative algebra, representation theory.

1.1.3 Investigated topics

There exist tight connections between Poisson Geometry and Deformation Quantization. First, Poisson Geometry is inter alia the natural frame of Deformation Quantization. Moreover, investigations on existence and uniqueness of star products on symplectic or Poisson manifolds are related with associative, Lie, Poisson, and strongly homotopy algebras, as well as with the corresponding Hochschild, Chevalley-Eilenberg, and Poisson-Lichnerowicz cohomologies.

This thesis is motivated by those connections. We study:

-graded and strongly homotopy algebraic structures (defining a Tensor Coalgebra for Graded Loday and Lod_∞ Structures which gives a unification for Graded and Infinity Cohomologies),

-Poisson and Koszul cohomologies (Formal Poisson cohomology of twisted r-matrix induced structures; Strongly r-matrix induced tensors, Koszul cohomology, and arbitrary-dimensional quadratic Poisson cohomology),

- universal star products on Poisson manifolds.

1.2 Developments, questions, results

1.2.1 Graded and strongly homotopy algebraic structures

In 1993, Stasheff, [Sta93], identified the Hochschild (resp. Chevalley-Eilenberg) cochain space with coderivations of the tensor (resp. symmetric tensor) coalgebra of the shifted underlying vector space, and observed that the Gerstenhaber (resp. Nijenhuis-Richardson) graded Lie bracket can be built from the commutator bracket of coderivations. Further, associative (resp. Lie) structures correspond to odd quadratic codifferentials. When replacing quadratic codifferentials with arbitrary odd codifferentials, we recover the concept of strongly homotopy associative or A_∞ (resp. strongly homotopy Lie or L_∞) algebra, see [Sta63] (resp. [SS85]). These algebras had originally been defined noncoalgebraically by means of a sequence of multilinear (resp. skew-symmetric multilinear) maps, which satisfy a sequence of relations that encode the fact that A_∞ (resp. L_∞) algebras are associative (resp. Lie) algebras up to homotopy. Kontsevich, [Kon03], described L_∞ algebras in terms of formal Q-manifolds, a view that allows proving proprieties of L_∞ algebras via geometric arguments.

The operadic theory, as developed in [MSS02], shows that the Stasheff's approach works on any quadratic operad $\mathscr{P}$. One associates a cofree (nilpotent) coalgebra over the dual operad whose quadratic codifferentials correspond to the $\mathscr{P}-$algebra structures on a graded vector space V. This result produces the homology or cohomology theories of $\mathscr{P}-$algebras on V and allows to define strongly homotopy $\mathscr{P}-$algebra structures on V as arbitrary codifferentials.

In this thesis we give an explicit coalgebraic approach to graded Loday and Loday infinity algebras, as well as to the corresponding cohomology theories.

Chapter 2 is devoted to these and related questions. We define a graded non-coassociative coproduct on the tensor space TW of any $\mathbb{Z}^n$–graded vector space W. If W is the desuspension space $\downarrow V$ of a graded vector space V, the coderivations (resp. codifferentials, quadratic codifferentials) of this coalgebra are 1-to-1 with sequences π_s, $s \geq 1$, of s-linear maps (resp. Loday infinity structures, $\mathbb{Z}^n$–graded Loday structures) on V. We prove a minimal model theorem for Loday infinity algebras, investigate Loday infinity morphisms, and observe that the Lod_∞ category contains the L_∞ category as a subcategory. Moreover, the graded Lie bracket of coderivations gives rise to a graded Lie "stem" bracket on the cochain spaces of graded Loday and Loday infinity algebras. These algebraic structures have square zero with respect to the stem bracket, so that we obtain natural cohomological theories that have good properties with respect to deformations. The stem bracket restricts to the graded Nijenhuis-Richardson and Grabowski-Marmo brackets (the last bracket extends the Schouten-Nijenhuis bracket to the space of graded first order differential operators), and it encodes the cohomologies of graded Loday, graded Lie, graded Poisson, graded Jacobi, Loday infinity, Lie infinity, as well as that of p-ary graded Lie algebras in the sense of Michor and Vinogradov.

1.2.2 Universal star product

In [Kon03], Kontsevich proved his formality theorem on $\mathbb{R}^d$ by explicitly determining an L_∞ morphism from the differential graded Lie algebra (DGLA) $\mathrm{T}_{\mathrm{poly}}(\mathbb{R}^d)$

of polyvector fields on $\mathbb{R}^d$ to the DGLA $\mathrm{D}_{\mathrm{poly}}(\mathbb{R}^d)$ of polydifferential operators on $\mathbb{R}^d$. The corestriction maps of this morphism, which are multilinear graded skew-symmetric maps from $\mathrm{T}_{\mathrm{poly}}(\mathbb{R}^d)$ to $\mathrm{D}_{\mathrm{poly}}(\mathbb{R}^d)$, associate to each collection of polyvector (resp. bivector) fields, a sequence of polydifferential (resp. bidifferential) operators. If in this collection each bivector field coincides with a same Poisson bivector, the resulting bidifferential operators are the coefficients of a star product on $\mathbb{R}^d$. Kontsevich then proved the existence of star products on a general Poisson manifold M by abstract gluing arguments that originate from the Gelfand-Kazhdan [GK71] formal geometry.

In [CFT02], Cattaneo, Felder and Tomassini proposed another globalization procedure for M. They observed that the so-called Grothendieck connection D^G on the jet bundle $E \to M$ can be used to build, in a spirit similar to Fedosov's construction, a flat connection $\mathscr{D}$ on E that allows transferring Kontsevich's fiberwize quantization to the base M.

Later on, in [Dol05], Dolgushev globalizes Kontsevich's L_∞ morphism to an arbitrary smooth manifold M. He constructs “à la Fedosov" a flat connection D^F on the jet bundle $E \to M$, which gives rise to a resolution of the function algebra $C^\infty(M)$ by differential forms on M valued in the sections of E. This resolution induces a resolution of the space of polyvectors fields (resp. polydifferential operators) on M by differential forms on M with values in the bundle of formal fiberwize polyvector fields (resp. polydifferential operators). The fiberwize Kontsevich L_∞ morphism is then twisted and contracted to yield an L_∞ morphism from $T_{\mathrm{poly}}(M)$ to $D_{\mathrm{poly}}(M)$.

In this thesis, we took an interest in the comparison of the star products implemented by the globalization procedures of [CFT02] and [Dol05], motivated by the quest of an intrinsic way to characterize and parameterize (at least in lower orders) some star products on a Poisson manifold.

Chapter 3 contains the needed preliminaries. We analyze the role of L_∞ algebras in deformation theory and review Kontsevich's formality theorem together with his star product formula. Further, we detail some algebraic proofs concerning L_∞ algebras that cannot easily be found in the literature.

In Chapter 4, we define the concept of universal formality L_∞ morphism: For any manifold M and any torsion-free linear connection on M, a universal formality L_∞ morphism is an L_∞ morphism from the DGLA $T_{\mathrm{poly}}(M)$ to the DGLA $D_{\mathrm{poly}}(M)$, such that the corestriction maps associate to each collection of polyvector fields a collection of polydifferential operators, whose coefficients are tensors given by universal polynomial expressions in the considered fields, the curvature tensor R and the covariant iterated derivatives. Existence of such a morphism is deduced from Dolgushev's formality globalization. It implies in particular the existence of a universal deformation quantization. Similarly, we stress that the globalization procedure of [CFT02] also induces a universal deformation quantization. We compare these procedures and prove that the Grothendieck connection D^G and the Fedosov-Dolgushev connection D^F, coincide. We show that universal quantizations essentially are unique up to order 3 in the deformation parameter, by computing the appropriate universal Poisson cohomology.

The results are published in [ACG08].

1.2.3 Poisson and Koszul cohomologies

Since Poisson cohomology computations are known to be quite difficult, many papers study the Euclidean plane or specific cases. In [MP06], the authors provide a general approach to Poisson cohomology of a broad set of isomorphism classes of the Dufour-Haraki classification for quadratic Poisson tensors of Euclidean three-space, [DH91]. More precisely, this quite powerful cohomological technique applies to all r-matrix induced Poisson tensors and allowed discovering main aspects of the structure of Poisson cohomology.

Hence, the questions whether it might be possible to construct a cohomological modus operandi for the more demanding remaining isomorphism classes and to understand the impact of the deviation from r-matrix implementation on the cohomological structure.

The answers to these questions are detailed in Chapter 5. More precisely, quadratic Poisson tensors of the Dufour-Haraki classification read as a sum of an r-matrix induced structure twisted by a (small) compatible exact quadratic tensor. This splitting is a remote variant of Liu and Xu's decomposition theorem. An algebraic bidegree of the space of formal Poisson cochains, that differs with the geometric bigrading used by Vaisman in the regular case, then leads to a vertically positive double complex. The associated spectral sequence allows to compute the Poisson-Lichnerowicz cohomology of the considered tensors. We depict this modus operandi, apply our technique to concrete examples of twisted Poisson structures, and obtain a complete description of their cohomology. Since richness of Poisson cohomology entails computation through the whole spectral sequence, we detail an entire model of this sequence. Finally, the chapter corroborates that largeness of Poisson cohomology can be viewed as a measure for deficiency of the considered Poisson tensor to be Koszul-exact.

The results are published in [AP07].

The methods presented in [MP06] and [AP07] allow computing the cohomology of any three-dimensional quadratic Poisson tensor. Hence, it seems natural to examine to which extent these techniques may be generalized to higher dimensional spaces.

In Chapter 6, we introduce the concept of strongly r-matrix induced (SRMI) Poisson structure, report on the relation of this property with the stabilizer dimension of the considered quadratic Poisson tensor, and classify the Poisson structures of the Dufour-Haraki classification (DHC) according to their membership of the family of SRMI tensors. One of the main results of this work is a generic cohomological procedure for SRMI Poisson structures in arbitrary dimension. This approach allows decomposing Poisson cohomology into, basically, a Koszul cohomology and a relative cohomology. Moreover, we investigate this associated Koszul cohomology, highlight its tight connections with Spectral Theory, and reduce the computation of this main building block of Poisson cohomology to a problem of linear algebra. We apply this to two structures of the DHC and provide an exhaustive description of their cohomology. We thus complete the list of data obtained in previous works, see [MP06] and [AP07]. This deepens our insight into the structure of Poisson cohomology, in particular as concerns Casimirs and the cohomological impact of the singularities and the stabilizer of the considered Poisson tensor.

Chapter 2

Coalgebraic Approach to the Loday Infinity Category, Stem Differential for $2p$-ary Graded and Homotopy Algebras

2.1 Introduction

Our initial investigations on deformation quantization of Poisson manifolds inspired us to explore two notions: the concept of strongly homotopy Lie algebras or L_∞ algebras, which plays a crucial role in Kontsevich's work [Kon03] on the existence of star products on a Poisson manifold and on the classification of these products; the second notion is that of Poisson cohomology, which appears in the problem of uniqueness of star products.

When realizing that an L_∞ [SS85] structure is a codifferential of a certain coalgebra, and that Poisson cohomology can be derived as a particular case of more general cohomologies of algebras, we first concentrated on studying the concepts of coalgebras and of cohomologies.

As our comprehension progressed, we understood that coalgebras provide a framework that allows constructing the appropriate cohomology theory of a given algebraic structure. To sum it up briefly, the technique consists in the identification of the cochain space of the investigated algebra with certain coderivations, in such a way that the algebraic structure can be identified to a homogenous odd quadratic codifferential. These identifications enable constructing a graded Lie bracket on the space of cochains by transfer of the commutator bracket of coderivations. The considered algebraic structures are then canonical elements of the transferred bracket, i.e. they square to zero with respect to this bracket. This propriety allows defining a natural cohomology operator for the algebraic structure, namely the adjoint action of this structure with respect to the transferred graded Lie bracket. These canonical coboundary operators have excellent properties as far as deformation theory is concerned. The procedure was first applied by Stasheff [Sta93] in the case of associative and Lie algebras.

In [MSS02], the authors generalize the Stasheff approach to any arbitrary quadratic operad $\mathscr{P}$. The chain complex for the homology or cohomology of $\mathscr{P}-$algebras on a graded vector space V is defined by means of constructing of a cofree (nilpotent) coalgebra over the dual operad, whereby quadratic codifferentials correspond to the $\mathscr{P}-$algebra structures on V. This naturally leads to define strongly homotopy $\mathscr{P}-$algebra structures on V as arbitrary codifferentials.

In the present work we provide explicitly a tensor coalgebra that induces the proper concepts of Loday infinity algebras and morphisms, and use Stasheff's modus operandi to define the cohomologies of $\mathbb{Z}^n$-graded Loday, Loday infinity, and $2p$-ary graded Loday algebras. This leads to a graded Lie "stem" bracket, in which are encrypted, in addition to the preceding cohomologies, the coboundary operators of graded Lie, graded Poisson, graded Jacobi, Lie infinity, and $2p$-ary graded Lie algebras [MV97].

This chapter is organized as follows.

In Section 2, we study the link between the cohomology induced by a canonical element in a graded Lie algebra with formal deformations of this element. Our investigations extend similar properties for the adjoint Hochschild (resp. Chevalley-Eilenberg, Leibniz) cohomology and deformations of associative (resp. Lie, Loday) structures, which were proved in [Ger64] (resp. [NR67], [Bal96]), and recovered in [Bal97].

Section 3 contains the definition of a graded dual Leibniz coalgebra structure Δ on the tensor algebra $T(W)$ of a $\mathbb{Z}^n$-graded vector space W. We provide explicit formulæ for the reconstruction of coderivations and cohomomorphisms from their corestriction maps.

In Section 4, we transfer the $\mathbb{Z}^n$-graded Lie bracket of coderivations of the mentioned tensor coalgebra $T(W)$ of the desuspension space $W := \downarrow V$ of an underlying $\mathbb{Z}^n$-graded vector space V, and get a $\mathbb{Z}^{n+1}$-graded (resp. $\mathbb{Z}^n$-graded)

Lie bracket on the $\mathbb{Z}^{n+1}$-graded vector space of weighted multilinear maps on V (resp. on the $\mathbb{Z}^n$-graded vector space of sequences of shifted weighted multilinear maps on V). We determine the explicit form of this pullback "stem" bracket and show that its $\mathbb{Z}^{n+1}$-graded version coincides in the case of a nongraded underlying space V (resp. of graded skew-symmetric multilinear mappings on V) with Rotkiewicz's bracket [Rot05] pertaining to left Loday structures [and corresponds to Balavoine's bracket [Bal97] concerning right Loday structures] (resp. with the graded Nijenhuis-Richardson bracket [LMS91]).

Codifferentials of our dual Leibniz coalgebra $T(W)$ are characterized in Section 5. We prove that $\mathbb{Z}^n$-graded Loday structures on V can be viewed as (resp. we define strongly homotopy Loday structures on V as) degree $e_1 := (1,0,\ldots,0) \in \mathbb{Z}^n$ quadratic (resp. odd degree) codifferentials of $(T(\downarrow V),\Delta)$. Loday infinity structures and Loday infinity morphisms are described in terms of sequences of weighted multilinear maps that satisfy explicitly depicted sequences of constraints: our Lod_∞ algebras are really differential graded Loday algebras up to homotopy and the Lod_∞ category contains the L_∞ category as a subcategory.

Loday infinity (quasi)-isomorphisms are investigated. In Section 6, we prove a minimal model theorem for strongly homotopy Loday algebras, and deduce that any Loday infinity quasi-isomorphism has a quasi-inverse – a theorem whose Lie infinity counterpart plays a key-role in Deformation Quantization.

In Section 7 we deal with graded and strongly homotopy cohomologies. $\mathbb{Z}^n$-graded Loday [resp. strongly homotopy Loday] structures are canonical for the $\mathbb{Z}^{n+1}$-graded [resp. $\mathbb{Z}^n$-graded] stem bracket, so that we obtain a natural cohomology theory and an explicit coboundary operator. In the nongraded (resp. the antisymmetric) [resp. the Lie infinity] case, our $\mathbb{Z}^n$-graded Loday [resp. Loday infinity] cohomology operator coincides with the Loday (resp. graded Chevalley-Eilenberg) [resp. Lie infinity] differential given in [DT97] and [Bal97] (resp. in [LMS91]) [resp. in [Pen01] and [FP02]].

Further, graded Poisson and Jacobi cohomologies were defined purely algebraically by Grabowski and Marmo in [GM03]. The authors prove existence and uniqueness of a $\mathbb{Z}^{n+1}$-graded Jacobi (resp. Poisson) bracket on the algebra of antisymmetric graded first order polydifferential operators (resp. of graded polyderivations). We compute this "Grabowski-Marmo" bracket explicitly and explain how the corresponding cohomologies are induced by our stem bracket.

Finally, essentially two p-ary extensions of the Jacobi identity were investigated during the last decades. The first, see e.g. [Fil85], leads to the Nambu-Lie structure, see [Nam73], the second, see [MV97], [VV98], [VV01], will in this text be referred to as p-ary Lie structure. We define analogously p-ary (p even) $\mathbb{Z}^n$-graded Loday structures and their cohomology. These graded p-ary Loday algebras are special strongly homotopy Loday algebras, so that we have to prove that the two stem bracket induced cohomologies coincide.

2.2 Canonical elements of graded Lie algebras

2.2.1 Definitions, cohomology and formal deformations

At the beginning of this thesis, we briefly analyze the well-known fact that in a graded Lie algebra (GLA) $(\mathfrak{g}, \{-,-\})$, any element $\pi \in \mathfrak{g}^1$, such that $\{\pi, \pi\} = 0$, generates a differential graded Lie algebra (DGLA) $(\mathfrak{g}, \{-,-\}, \partial_\pi)$, $\partial_\pi = \{\pi, -\}$, and a GLA in cohomology that allows controlling the formal deformations of π.

Unless otherwise stated, all vector spaces that we consider in this text are spaces over a field $\mathbb{K}$ of characteristic 0, and all graded vector spaces are $\mathbb{Z}^n$-graded, $n \in \mathbb{N}^*$. The $\mathbb{Z}^n$–degree $\deg(v)$ of a vector v or the $\mathbb{Z}^n$–weight $\deg(f)$ of a graded linear map f are often denoted by the same symbol v or f. If $v, f \in \mathbb{Z}^n$ are two such degrees, we set $\langle v, f \rangle = \sum_i v_i f_i$. A homogeneous vector or graded linear map w is termed *odd*, if $\langle w, w \rangle \in \mathbb{Z}$ is an odd number.

Definition 1. *A* graded Lie algebra $(\mathfrak{g}, \{-,-\})$ *(GLA) is a $\mathbb{Z}^n$–graded vector space $\mathfrak{g} = \oplus_{\alpha \in \mathbb{Z}^n} \mathfrak{g}^\alpha$ together with a bilinear bracket $\{-,-\} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ that satisfies the following conditions:*

1. *$\{-,-\}$ is compatible with the grading of $\mathfrak{g}$, i.e.*
$$\{\mathfrak{g}^\alpha, \mathfrak{g}^\beta\} \subset \mathfrak{g}^{\alpha+\beta}, \quad \forall \alpha, \beta \in \mathbb{Z}^n \tag{2.1}$$

2. *$\{-,-\}$ is anticommutative, i.e.*
$$\{a, b\} = -(-1)^{\langle a,b \rangle} \{b, a\}, \tag{2.2}$$
for all homogeneous $a, b \in \mathfrak{g}$

3. *Any homogeneous $a, b, c \in \mathfrak{g}$ verify the Jacobi identity*
$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{\langle a,b \rangle} \{b, \{a, c\}\} \tag{2.3}$$

Definition 2. *We call* canonical element *of a graded Lie algebra (GLA) $(V, \{-,-\})$, any odd element $\pi \in V$ that verifies $\{\pi, \pi\} = 0$.*

Definition 3. *A* differential graded Lie algebra $(\mathfrak{g}, d, \{-,-\})$ *(DGLA) is a GLA together with a graded linear map $d : \mathfrak{g} \longrightarrow \mathfrak{g}$ that is a differential (i.e. $d^2 = 0$) and a graded derivation of the graded Lie bracket,*

$$d\{a,b\} = \{da,b\} + (-1)^{\langle \deg(d),a \rangle}\{a,db\}. \tag{2.4}$$

Proposition 1. *Every GLA $(\mathfrak{g}, \{-,-\})$ equipped with a canonical element π is a DGLA $(\mathfrak{g}, \partial_\pi, \{-,-\})$, where $\partial_\pi := \{\pi, -\}$.*

Proof. When applying the Jacobi identity (2.3) with $a = b = \pi$, we obtain $2\{\pi,\{\pi,c\}\} = 0$ and, as field $\mathbb{K}$ is of characteristic zero, $\{\pi,\{\pi,c\}\} = 0$, so that ∂_π is a differential. It is also a derivation of the Lie bracket:

$$\begin{aligned} \partial_\pi\{a,b\} &= \{\pi,\{a,b\}\} = \{\{\pi,a\},b\} + (-1)^{\langle \pi,a \rangle}\{a,\{\pi,b\}\} \\ &= \{\partial_\pi a,b\} + (-1)^{\langle \pi,a \rangle}\{a,\partial_\pi b\}. \end{aligned} \tag{2.5}$$

∎

Since coboundary operator ∂_π has the form of a Hamiltonian vector field, we sometimes refer to it as a *Hamiltonian differential.*

Proposition 2. *The cohomology space $H(\mathfrak{g}, d, \{-,-\})$ (or $H(\mathfrak{g})$ for short) of any DGLA $(\mathfrak{g}, d, \{-,-\})$ is a GLA for the bracket that is canonically induced by $\{-,-\}$.*

Proof. Obvious. ∎

If the considered DGLA is implemented by a GLA $(\mathfrak{g}, \{-,-\})$ endowed with a canonical element π, we denote the corresponding cohomology GLA by $H_\pi(\mathfrak{g})$.

Next, we investigate the links between deformations of a canonical element π of a GLA $(\mathfrak{g}, \{-,-\})$ and the cohomology algebra $H_\pi(\mathfrak{g})$.

Definition 4. *Let π be a canonical element of a GLA $(\mathfrak{g}, \{-,-\})$ and set*

$$\mathfrak{g}[[\nu]] = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathfrak{g}^\alpha[[\nu]],$$

where $\mathfrak{g}^\alpha[[\nu]]$ is the space of formal power series in the formal parameter ν with coefficients in $\mathfrak{g}^\alpha$. A formal power series

$$\pi_\nu := \sum_{i=0}^{\infty} \nu^i \pi_i \in \mathfrak{g}^{\deg(\pi)}[[\nu]] \tag{2.6}$$

with first term $\pi_0 = \pi$ is a formal deformation *of the canonical element π, if it squares to zero w.r.t. the natural extension of the bracket $\{-,-\}$ to a bilinear map of the space $\mathfrak{g}[[\nu]]$, i.e. if*

$$\{\pi_\nu, \pi_\nu\} = \sum_{p=0}^{\infty} \nu^p \sum_{i+j=p} \{\pi_i, \pi_j\} = 0. \tag{2.7}$$

A formal deformation of order *q of π is a formal series (2.6) that is truncated at order q in ν and satisfies the condition*

$$\sum_{i+j=p} \{\pi_i, \pi_j\} = 0, \tag{2.8}$$

for each $1 \leq p \leq q$. We refer to formal deformations of order 1 as infinitesimal deformations.

Let us first focus on existence and construction of formal deformations.

Proposition 3. *The degree $2\deg(\pi)$ cohomology space $H_\pi^{2\deg(\pi)}(\mathfrak{g})$ of the DGLA implemented by a canonical element π of a GLA $(\mathfrak{g}, \{-,-\})$, contains the obstructions to extension of formal deformations of order at least 1 to higher order deformations. In particular, if $H_\pi^{2\deg(\pi)}(\mathfrak{g}) = 0$, any formal deformation of order $q \geq 1$ can be extended to a formal deformation of order $q+1$.*

Proof. Assume that π admits a formal deformation π_ν of order q, $q \geq 1$, and set

$$E_p := \sum_{\substack{i+j=p \\ i,j\neq 0}} \{\pi_i, \pi_j\} \in \mathfrak{g}^{2\deg(\pi)}, \quad 1 \leq p \leq q+1. \tag{2.9}$$

Note first that Condition (2.8) is equivalent to

$$E_p = -2\partial_\pi(\pi_p), \quad 1 \leq p \leq q, \tag{2.10}$$

since π_p is of odd degree $\deg(\pi_p) = \deg(\pi)$.

As for E_{q+1}, it is quite easy to see that it is a cocycle for ∂_π. Indeed, we have

$$\begin{aligned}
\partial_\pi(E_{q+1}) &= \sum_{\substack{i+j=q+1\\ i,j\neq 0}} \partial_\pi(\{\pi_i,\pi_j\}) \\
&= \sum_{\substack{i+j=q+1\\ i,j\neq 0}} \{\partial_\pi\pi_i,\pi_j\}+(-1)^{\langle\pi_i,\pi\rangle}\{\pi_i,\partial_\pi\pi_j\} \\
&= 2\sum_{\substack{i+j=q+1\\ i,j\neq 0}} \{\partial_\pi\pi_i,\pi_j\} \stackrel{(*)}{=} -\sum_{\substack{k+l+j=q+1\\ k,l,j\neq 0}} \{\{\pi_k,\pi_l\},\pi_j\} \\
&= \frac{1}{3}\sum_{\substack{k+l+j=q+1\\ k,l,j\neq 0}} (-\{\{\pi_k,\pi_l\},\pi_j\}-\{\{\pi_l,\pi_j\},\pi_k\}-\{\{\pi_j,\pi_k\},\pi_l\}) \\
&= 0,
\end{aligned}$$

where, at $(*)$, we used Equations (2.9) and (2.10).

In order to extend deformation π_ν to order $q+1$, we must find an element $\pi_{q+1}\in\mathfrak{g}^{\deg(\pi)}$ that satisfies the condition $E_{q+1}=\partial_\pi(-2\pi_{q+1})$, see Equation (2.10). Hence, cocycle E_{q+1} has to be a coboundary. Consequently, the obstruction to the extension of the formal deformation π_ν of π to the order $q+1$ is a (nonvanishing) cohomology class E_{q+1} in $H_\pi^{2\deg(\pi)}(\mathfrak{g})$. ∎

In order to define the equivalence of two formal deformations of a same canonical element of a GLA, we need the following

Lemma 1. *Let π be a canonical element of a GLA $(\mathfrak{g},\{-,-\})$. Consider a formal series*

$$\chi_\nu=\sum_{i=1}^{\infty}\nu^i\chi_i \tag{2.11}$$

with coefficients in $\mathfrak{g}^0$. If π_ν is a formal deformation of π, then

$$\exp(\operatorname{ad}\chi_\nu)\,\pi_\nu$$

is a formal deformation of π as well.

Proof. Let us first remark that here exp denotes the exponential series and that

$$\begin{gathered}
(\operatorname{ad}\chi_\nu)^k\pi_\nu=\{\chi_\nu\{\chi_\nu\ldots\{\chi_\nu,\pi_\nu\underbrace{\}\ldots\}\}}_{k} \\
=\textstyle\sum_{p=0}^{\infty}\nu^p\sum_{i_1+\ldots+i_k+j=p}\{\chi_{i_1}\{\chi_{i_2}\ldots\{\chi_{i_k},\pi_j\}\ldots\}\}.
\end{gathered}\tag{2.12}$$

It follows that the coefficient of ν^p in the exponential series over k is made up by a finite number of terms in $\mathfrak{g}^{\deg(\pi)}$; indeed, if $k \geq p+1$, at least one of the χ_{i_ℓ} vanishes. Moreover, the coefficient of ν^0 contains only the term $k=0$, and thus $(\exp(\mathrm{ad}\,\chi_\nu)\ \pi_\nu)_0 = \pi$. Eventually, as $\mathrm{ad}\,\chi_\nu$ is a derivation of the bracket $\{-,-\}$, we have

$$(\mathrm{ad}\,\chi_\nu)^k\{\pi_\nu,\pi_\nu\} = \sum_{\substack{r+s=k\\ r,s\geq 0}} C_k^r\{(\mathrm{ad}\,\chi_\nu)^r\pi_\nu, (\mathrm{ad}\,\chi_\nu)^s\pi_\nu\},$$

where $C_k^r = \frac{k!}{(k-r)!r!}$. Hence,

$$\exp(\mathrm{ad}\,\chi_\nu)\{\pi_\nu,\pi_\nu\} = \{\exp(\mathrm{ad}\,\chi_\nu)\pi_\nu, \exp(\mathrm{ad}\,\chi_\nu)\pi_\nu\},$$

which completes the proof of the lemma. ∎

Definition 5. *Let π be a canonical element of a GLA $(\mathfrak{g},\{-,-\})$. Two formal deformations π_ν and π'_ν of π are said to be* equivalent *(resp.* equivalent up to order *q, $q\geq 1$), if there is a formal series χ_ν of type (2.11), such that*

$$\exp(\mathrm{ad}\,\chi_\nu)\ \pi_\nu = \pi'_\nu \quad (\textit{resp. } \exp(\mathrm{ad}\,\chi_\nu)\ \pi_\nu = \pi'_\nu + \mathscr{O}(\nu^{q+1})). \tag{2.13}$$

A deformation π_ν of π is called trivial *(resp.* trivial up to order *q, $q\geq 1$), if π_ν is equivalent to π (resp. equivalent to π up to order q).*

Proposition 4. *Let π be a canonical element of a GLA $(\mathfrak{g},\{-,-\})$. If the cohomology space $H_\pi^{\deg(\pi)}(\mathfrak{g})$ vanishes, any formal deformation of π is trivial.*

Proof. Let $\pi_\nu := \pi + \sum_{i=1}^\infty \nu^i\pi_i$ be a formal deformation of π. We first prove that π_ν is trivial up to order 1, then we proceed by induction. Condition (2.8) implies that $2\partial_\pi(\pi_1)=0$, i.e. that π_1 is a cocycle; as $H_\pi^{\deg(\pi)}(\mathfrak{g})=0$, there exists a $\chi_1\in\mathfrak{g}^0$, such that $\pi_1 = \partial_\pi(\chi_1)$. When setting $\chi_\nu^{(1)} = \nu\chi_1$, we get

$$\exp(\mathrm{ad}\,\chi_\nu^{(1)})\pi_\nu = \pi + \mathscr{O}(\nu^2).$$

Suppose now that π_ν is trivial up to order q ($q\geq 1$), or, equivalently, that there exists a formal series

$$\chi_\nu^{(q)} = \sum_{i=1}^\infty \nu^i\chi_i$$

with coefficients in $\mathfrak{g}^0$, such that

$$\pi'_\nu := \exp(\mathrm{ad}\,\chi_\nu^{(q)})\pi_\nu = \pi + \nu^{q+1}\pi'_{q+1} + \mathscr{O}(\nu^{q+2}). \tag{2.14}$$

As above, since π'_ν is a deformation of π, $2\partial_\pi(\pi'_{q+1}) = 0$, so that there is $\chi'_{q+1} \in \mathfrak{g}^0$ that verifies $\pi'_{q+1} = \partial_\pi(\chi'_{q+1})$. Set now

$$\chi_\nu^{(q+1)} := \chi_\nu^{(q)} + \nu^{q+1}\chi'_{q+1} \quad \text{and} \quad \pi''_\nu := \exp(\mathrm{ad}\,\chi_\nu^{(q+1)})\pi_\nu.$$

It follows from Equation (2.12) that

$$\pi''_\nu - \pi'_\nu = -\nu^{q+1}\partial_\pi(\chi'_{q+1}) + \mathscr{O}(\nu^{q+2}).$$

Hence, $\pi''_\nu = \pi + \mathscr{O}(\nu^{q+2})$, which completes the proof. ∎

Concerning *infinitesimal deformations*, i.e. first order formal deformations, it is easily seen from the above explanations that

Proposition 5. *Infinitesimal deformations of a canonical element π of a GLA $(\mathfrak{g}, \{-,-\})$ are classified up to first order equivalence by $H_\pi^{\deg(\pi)}(\mathfrak{g})$.*

2.2.2 Examples

Below, we give two basic examples of canonical elements of graded Lie algebras.

Associative graded algebras, graded Gerstenhaber algebra

Let us recall the

Definition 6. *Let $V = \oplus_{A\in\mathbb{Z}^n} V^A$ be a $\mathbb{Z}^n$-graded vector space. An* associative graded algebra *structure on V is an associative bilinear map $\pi : V \times V \to V$ that respects the gradation, i.e. that verifies $\pi(V^A, V^B) \subset V^{A+B}$.*

We now define a $\mathbb{Z}^{n+1}$-graded Lie algebra, for which any associative graded algebra structure on the $\mathbb{Z}^n$-graded vector space V is a canonical element.

Set

$$M(V) = \bigoplus_{(A,a)\in\mathbb{Z}^n\times\mathbb{Z}} M^{(A,a)}(V),$$

where $M^{(A,a)}(V) = 0$ for all $a \le -2$, $M^{(A,-1)}(V) = V^A$, and where for each $a \ge 0$, $M^{(A,a)}(V)$ is the space of all $(a+1)$-multilinear maps $A : V^{\times(a+1)} \to V$ that have weight A. For notational ease multilinear maps and their weights are denoted, here and below, by the same symbol.

It is known, see [LMS91], that the $\mathbb{Z}^{n+1}$-graded vector space $M(V)$ admits a $\mathbb{Z}^{n+1}$-graded Lie algebra structure.

Theorem 1. *For any $A \in M^{(A,a)}(V)$, $B \in M^{(B,b)}(V)$, and $v_i \in V^{v_i} = M^{(v_i,-1)}(V)$, $i \in \{1,\ldots,a+b+1\}$, we set*

$$j_B^G A(v_1,\ldots,v_{a+b+1}) = \sum_{i=1}^{a+1} (-1)^{\langle A,B\rangle + \langle B, v_1+\ldots+v_{i-1}\rangle + b(i-1)} \\ A(v_1,\ldots,v_{i-1},B(v_i,\ldots,v_{i+b}),v_{i+b+1},\ldots,v_{a+b+1}). \tag{2.15}$$

Then,

1. *The pair $(\mathrm{M}(V),[-,-]^G)$, where*

$$[A,B]^G = j_A^G B - (-1)^{\langle (A,a),(B,b)\rangle} j_B^G A,$$

   is a $\mathbb{Z}^{n+1}$-graded Lie algebra.

2. *A bilinear map $\pi : V \times V \to V$ of weight $0 \in \mathbb{Z}^n$, i.e. a map $\pi \in \mathrm{M}^{(0,1)}(V)$, satisfies the condition $[\pi,\pi]^G = 0$ if and only if π is an associative graded algebra structure on V.*

Observe that for a non-graded vector space V, i.e. a vector space endowed with the trivial gradation, the GLA $(M(V),[-,-]^G)$ coincides with the Gerstenhaber (graded Lie) algebra of V. In the graded case, we refer to $(M(V),[-,-]^G)$ as the *graded Gerstenhaber algebra* of V.

Remark also that *the associative graded algebra structures on V are exactly the canonical elements of degree $(0,1)$ of the graded Gerstenhaber algebra of V.* If π is an associative graded multiplication on V, the cohomology $H_\pi(M(V))$ of the DGLA induced by the canonical element π will be denoted in the sequel by $H_\pi^{\mathrm{Ass}}(V)$. In the non-graded case, it coincides with the adjoint *Hochschild cohomology* of the associative algebra (V,π).

Graded Lie algebras, graded Nijenhuis-Richardson algebra

For any integer $p \in \mathbb{N}^*$, let $N^{(p)}$ be the p–tuple $(1,\ldots,p)$. An *unshuffle* $I = (i_1,\ldots,i_k)$ $(1 \le k \le p)$ is a naturally ordered subset of $N^{(p)}$, i.e. a subset, such that $1 \le i_1 < \ldots < i_k \le p$. The length of I will be denoted by $|I|$. If I and J are two unshuffles, such that $I \cap J = \emptyset$, we set $(I;J) = (i_1,\ldots,i_{|I|};j_1,\ldots,j_{|J|})$ and we denote by $I \cup J$ the unique unshuffle that coincides with $(I;J)$ as a set. Let $(-1)^{(I;J)}$ be the signature of the permutation $(I;J) \to I \cup J$. If V is a $\mathbb{Z}^n$-graded vector space, we also set $V_{(I;J)} = (v_{i_1},\ldots,v_{i_{|I|}};v_{j_1},\ldots,v_{j_{|J|}})$, $v_\ell \in V$, and denote by $\varepsilon_V(I;J)$ the Koszul sign (which is for any transposition $V_{(\ell+1,\ell)} \to V_{(\ell,\ell+1)}$ given by

$(-1)^{\langle v_\ell, v_{\ell+1}\rangle}$) implemented by the permutation $V_{(I;J)} \to V_{I\cup J}$.

Consider now the $\mathbb{Z}^{n+1}$-graded vector subspace $A(V)$ of $M(V)$ made up by the $\mathbb{Z}^n$-graded skew-symmetric multilinear maps

$$A(\ldots, v_i, v_{i+1}, \ldots) = -(-1)^{\langle v_i, v_{i+1}\rangle} A(\ldots, v_{i+1}, v_i, \ldots). \tag{2.16}$$

As $M(V)$, the $\mathbb{Z}^{n+1}$-graded vector space $A(V)$ admits a $\mathbb{Z}^{n+1}$-graded Lie algebra structure, see [LMS91].

Theorem 2. *For any $A \in A^{(A,a)}(V)$ and $B \in A^{(B,b)}(V)$, we set*

$$(i_B A)(V_{N^{(a+b+1)}}) =$$

$$(-1)^{\langle A,B\rangle} \sum_{\substack{I\cup J = N^{(a+b+1)} \\ |I| = b+1, |J| = a}} (-1)^{(I;J)} \varepsilon_V(I;J) A(B(V_I), V_J),$$

where notations such as $V_{N^{(a+b+1)}}$ mean of course

$$V_{N^{(a+b+1)}} = (v_1, \ldots, v_{a+b+1}), \quad v_\ell \in V.$$

Then,

1. *The pair $(A(V), [-,-]^{\mathrm{NR}})$, where*

$$[A,B]^{\mathrm{NR}} = i_A B - (-1)^{\langle (A,a),(B,b)\rangle} i_B A, \tag{2.17}$$

   is a $\mathbb{Z}^{n+1}$-graded Lie algebra.

2. *A bilinear $\mathbb{Z}^n$-graded antisymmetric map $\pi : V \times V \to V$ of weight $0 \in \mathbb{Z}^n$, i.e. $\pi \in A^{(0,1)}(V)$, satisfies the condition $[\pi,\pi]^{\mathrm{NR}} = 0$ if and only if π defines a $\mathbb{Z}^n$-graded Lie bracket on V.*

We refer to the $\mathbb{Z}^{n+1}$-graded Lie algebra $(A(V), [-,-]^{\mathrm{NR}})$ as the *graded Nijenhuis-Richardson algebra* of V. The preceding theorem points out that *the $\mathbb{Z}^n$-graded Lie algebra structures on V are exactly the canonical elements of degree $(0,1)$ of this graded Nijenhuis-Richardson algebra of V.* If π is such a $\mathbb{Z}^n$-graded Lie structure on V, the cohomology Lie algebra $H_\pi(A(V))$ of the DGLA induced by the canonical element π will be denoted by $H_\pi^{\mathrm{Lie}}(V)$. In the non-graded case, it coincides with the adjoint *Chevalley-Eilenberg cohomology* of (V,π).

2.3 Noncoassociative Tensor coalgebra

Let us briefly recall some well-known facts. A *graded coalgebra* (C,Δ) is a graded vector space $C=\bigoplus_{\alpha\in\mathbb{Z}^n}C^\alpha$ together with a *coproduct* Δ, i.e. a linear map $\Delta: C\to C\otimes C$ that verifies $\Delta(C^\alpha)\subset\bigoplus_{\beta+\gamma=\alpha}C^\beta\otimes C^\gamma$. A *cohomomorphism* from (C,Δ) to a graded coalgebra (C',Δ') is a weight 0 linear map $\mathscr{F}: C\to C'$, such that $\Delta'\mathscr{F}=(\mathscr{F}\otimes\mathscr{F})\Delta$. In this text, the tensor product of linear maps is defined by $(f\otimes g)(v_1\otimes v_2)=(-1)^{\langle g,v_1\rangle}f(v_1)\otimes g(v_2)$, with self-explaining notations. Further, a homogeneous *coderivation* of (C,Δ) is a linear map $Q: C\to C$ of weight $\deg(Q)$ that satisfies the *co-Leibniz identity* $\Delta Q=(Q\otimes\mathrm{id}+\mathrm{id}\otimes Q)\Delta$, where id is the identity map of C. Weight α coderivations form a vector space $\mathrm{CoDer}^\alpha(C)$, and the space $\mathrm{CoDer}(C)=\bigoplus_{\alpha\in\mathbb{Z}^n}\mathrm{CoDer}^\alpha(C)$ of all coderivations carries a natural $\mathbb{Z}^n$-graded Lie algebra structure provided by the graded commutator bracket.

To any $\mathbb{Z}^n$-graded vector space V, we associate the (reduced) associative tensor algebra $T(V)=\bigoplus_{p=1}^\infty V^{\otimes p}$ (the full tensor algebra includes the term $V^{\otimes 0}=\mathbb{K}$ as well), which carries two natural gradings, the $\mathbb{Z}$-gradation $T(V)=\bigoplus_{p=1}^\infty T^pV$, $T^pV:=V^{\otimes p}$, and the $\mathbb{Z}^n$-gradation $T(V)=\bigoplus_{\alpha\in\mathbb{Z}^n}T(V)^\alpha$, $T(V)^\alpha:=\bigoplus_{p=1}^\infty(T^pV)^\alpha$, $(T^pV)^\alpha=\bigoplus_{\beta_1+\ldots+\beta_p=\alpha}V^{\beta_1}\otimes\ldots\otimes V^{\beta_p}$. In the following, unless differently stated, we view $T(V)$ as $\mathbb{Z}^n$-graded vector space.

Proposition 6. *Let V be a $\mathbb{Z}^n-$graded vector space. The coproduct*

$$\Delta: T(V)\longrightarrow T(V)\bigotimes T(V),$$

defined by

$$\Delta(v_1\otimes\ldots\otimes v_p)=\sum_{\substack{I\cup J=N^{(p-1)}\\ I\neq\emptyset}}\varepsilon_V(I;J)\,V_I\bigotimes V_J\otimes v_p\quad(v_\ell\in V, p\geq 1),\tag{2.18}$$

provides a graded (noncoassociative) coalgebra structure on $T(V)$ and verifies

$$(\mathrm{id}\bigotimes\Delta)\Delta=(\Delta\bigotimes\mathrm{id})\Delta+(T\bigotimes\mathrm{id})(\Delta\bigotimes\mathrm{id})\Delta,\tag{2.19}$$

where we used above-detailed notations and where $T: T(V)\bigotimes T(V)\to T(V)\bigotimes T(V)$ is the twisting map, which exchanges two elements of the $\mathbb{Z}^n$-graded space $T(V)$ modulo the corresponding Koszul sign.

Proof. It suffices to develop the three terms of (2.19). The first one reads

$$
\begin{aligned}
(\mathrm{id}\bigotimes\Delta)\Delta(v_1\otimes\ldots\otimes v_p) &= \sum_{\substack{I\cup J=N^{(p-1)}\\ I\neq\emptyset}} \varepsilon_V(I;J)V_I\bigotimes\Delta(V_J\otimes v_p)\\
&= \sum_{\substack{I\cup J=N^{(p-1)}\\ I\neq\emptyset}} \varepsilon_V(I;J)\sum_{\substack{K\cup L=J\\ K\neq\emptyset}} \varepsilon_V(K;L)V_I\bigotimes V_K\bigotimes V_L\otimes v_p\\
&= \sum_{\substack{I\cup K\cup L=N^{(p-1)}\\ I,K\neq\emptyset}} \varepsilon_V(I;K;L)V_I\bigotimes V_K\bigotimes V_L\otimes v_p\,,
\end{aligned}
$$

the second is equal to

$$
\begin{aligned}
(\Delta\bigotimes\mathrm{id})\Delta(v_1\otimes\ldots\otimes v_p) &= \sum_{\substack{J\cup L=N^{(p-1)}\\ J\neq\emptyset}} \varepsilon_V(J;L)\Delta(V_J)\bigotimes V_L\otimes v_p\\
&= \sum_{\substack{J\cup L=N^{(p-1)}\\ J\neq\emptyset}} \varepsilon_V(J;L)\sum_{\substack{I\cup\widetilde{K}=J\setminus j_{|J|}\\ I\neq\emptyset}} \varepsilon_V(I;\widetilde{K})V_I\bigotimes V_{\widetilde{K}}\otimes v_{j_{|J|}}\bigotimes V_L\otimes v_p\\
&= \sum_{\substack{I\cup K\cup L=N^{(p-1)}\\ i_{|I|}<k_{|K|},\, I,K\neq\emptyset}} \varepsilon_V(I;K;L)V_I\bigotimes V_K\bigotimes V_L\otimes v_p
\end{aligned}
$$

($J\setminus j_{|J|}$ denotes J with $j_{|J|}$ omitted and $K=(\widetilde{K},j_{|J|})$), and for the third we get

$$
\begin{aligned}
&(T\bigotimes\mathrm{id})(\Delta\bigotimes\mathrm{id})\Delta(v_1\otimes\ldots\otimes v_p)\\
&= \sum_{\substack{I\cup K\cup L=N^{(p-1)}\\ i_{|I|}<k_{|K|},\, I,K\neq\emptyset}} \varepsilon_V(I;K;L)(-1)^{\langle V_I,V_K\rangle}V_K\bigotimes V_I\bigotimes V_L\otimes v_p\\
&= \sum_{\substack{I\cup K\cup L=N^{(p-1)}\\ k_{|K|}<i_{|I|},\, I,K\neq\emptyset}} \varepsilon_V(K;I;L)(-1)^{\langle V_I,V_K\rangle}V_I\bigotimes V_K\bigotimes V_L\otimes v_p\\
&= \sum_{\substack{I\cup K\cup L=N^{(p-1)}\\ k_{|K|}<i_{|I|},\, I,K\neq\emptyset}} \varepsilon_V(I;K;L)V_I\bigotimes V_K\bigotimes V_L\otimes v_p\,.
\end{aligned}
$$

Hence, the result. ∎

The following theorem provides a characterization of the coderivations of the graded coalgebra $(T(V),\Delta)$.

Theorem 3. *The mapping*

$$\psi_Q^V : \mathrm{CoDer}^Q(T(V)) \ni Q \to (Q_1, Q_2, \ldots) =: \sum_p Q_p \in \prod_{p\geq 1} M^{(Q,p-1)}(V), \quad (2.20)$$

which assigns to any (weight Q) coderivation Q its (weight Q) corestriction maps $Q_p : T^pV \hookrightarrow T(V) \xrightarrow{Q} T(V) \xrightarrow{\mathrm{pr}} V$, *where* pr *denotes the canonical projection, is a vector space isomorphism, the inverse of which associates to any sequence* $(Q_1, Q_2, \ldots)$ *the coderivation Q that is defined by*

$$Q(v_1 \otimes \ldots \otimes v_p) = \sum_{\substack{I\cup J\cup K=N^{(p)}\\ I,J<K}} \varepsilon_V(I;J)(-1)^{\langle Q,V_I\rangle} V_I \otimes Q_{|J|+1}(V_J \otimes v_{k_1}) \otimes V_{K\setminus k_1}, \quad (2.21)$$

where $I < K$ *means that* $i_{|I|} < k_1$ *and where* $v_\ell \in V^{v_\ell}$.

Remark 1. *The isomorphisms* ψ_Q^V, $Q \in \mathbb{Z}^n$, *(for inverses, see Equation (2.21)), induce an isomorphism* ψ^V *between* $\mathrm{CoDer}(T(V))$ *and the corresponding direct sum of direct products. Further, if we denote by* $\mathrm{CoDer}_p^Q(T(V))$, $Q \in \mathbb{Z}^n, p \in \mathbb{N}^*$, *the image by* $(\psi_Q^V)^{-1}$ *of* $M^{(Q,p-1)}(V)$, *isomorphism* ψ_Q^V *restricts to an isomorphism* $\psi_{(Q,p)}^V$ *between these spaces. If no confusion arises, we write* ψ *instead of* ψ^V, ψ_Q^V, *or* $\psi_{(Q,p)}^V$.

Proof. It suffices to show that Equation (2.21) defines a coderivation of degree Q of the graded coalgebra $(T(V), \Delta)$, and that the thus defined map Ψ is the inverse of ψ, i.e. that its compositions with ψ coincide with the corresponding identity maps.

Let us momentarily assume that Q verifies coderivation condition (2.3), so that map Ψ is actually valued in $\mathrm{CoDer}^{\mathrm{Q}}(\mathrm{T(V)})$. It is then easily seen that $\psi \circ \Psi = \mathrm{id}$. The second condition $\Psi \circ \psi = \mathrm{id}$ then means that ψ is injective. So let us prove that if the corestriction maps of a coderivation Q vanish, then Q vanishes as well.

As Δ vanishes if and only if its argument belongs to V, and as for any $v \in V$, we have

$$\Delta Q(v) = (Q \bigotimes \mathrm{id})\Delta(v) + (\mathrm{id} \bigotimes Q)\Delta(v) = 0,$$

it follows that $Q(v) \in V$. However, as $Q_1 = 0$, we get $Q(v) = 0$, for any $v \in V$. Assume now that $Q(v_1 \otimes \ldots \otimes v_k) = 0$, $v_\ell \in V$, for any $1 \leq k \leq p-1$ and proceed by induction. Since

$$\Delta(v_1 \otimes \ldots \otimes v_p) = \sum_{\substack{I\cup J=N^{(p-1)}\\ I\neq\emptyset}} \varepsilon_V(I;J)\, V_I \bigotimes V_J \otimes v_p,$$

we have

$$\Delta Q(v_1 \otimes \ldots \otimes v_p) = (Q \bigotimes \mathrm{id})\Delta(v_1 \otimes \ldots \otimes v_p) + (\mathrm{id} \bigotimes Q)\Delta(v_1 \otimes \ldots \otimes v_p) = 0.$$

Consequently, $Q(v_1 \otimes \ldots \otimes v_p) \in V$ and $Q(v_1 \otimes \ldots \otimes v_p) = Q_p(v_1 \otimes \ldots \otimes v_p) = 0$. Eventually, Q actually vanishes, if its corestrictions vanish.

Let us now come to the core of the proof and show that the map Q defined by Equation (2.21) verifies the coderivation condition

$$\Delta Q(v_1 \otimes \ldots \otimes v_p) = (Q \bigotimes \mathrm{id} + \mathrm{id} \bigotimes Q)\Delta(v_1 \otimes \ldots \otimes v_p), \tag{2.22}$$

for any $p \geq 1$.

Although this result is quite easily checked for $p \leq 3$, the general proof is rather technical. It is better understood, if we are aware of the special role played by v_p in the definition of Δ. Thus, in the subsequent proof, we examine the terms of type

$$\ldots Q(\ldots \otimes v_p) \quad \text{and} \quad \ldots \bigotimes v_p \tag{2.23}$$

separately. Below, we refer to this idea as Remark ($\star$).

The next lemma allows placing in the definition of $\Delta(v_1 \otimes \ldots \otimes v_t \otimes \ldots \otimes v_p)$ any factor $v_t, t \in \{1, \ldots, p-1\}$, on the left of $\bigotimes$ and thus simplifies the comparison the LHS and RHS of Equation (2.22).

Lemma 2. *Set* $\overline{T} = \mathrm{id} + T \otimes \mathrm{id}$*, where T is the twisting map. For any fixed integer* $t \in \{1, \ldots, p-1\}$,

$$\begin{aligned}\Delta(v_1 \otimes \ldots \otimes v_t \otimes \ldots \otimes v_p) = {} & V_{N^{(p-1)}} \bigotimes v_p + \\ & \overline{T}\Big(\sum_{\substack{K \cup R \cup J = N^{(p-1)} \setminus t \\ K \subset \{1,\ldots,t-1\}, R \subset \{t+1,\ldots,p-1\}, J \neq \emptyset}} c \ V_K \otimes v_t \otimes V_R \bigotimes V_J \otimes v_p \Big),\end{aligned} \tag{2.24}$$

with $c = (-1)^{\langle v_t, V_{N^{(t-1)}} - V_K \rangle} \varepsilon_V(K;J) \varepsilon_V(R;J)$.

Proof. We have

$$\begin{aligned}\Delta(v_1 \otimes \ldots \otimes v_t \otimes \ldots \otimes v_p) &= V_{N^{(p-1)}} \bigotimes v_p + \\ & \sum_{\substack{I \cup J = N^{(p-1)} \\ t \in I,\, J \neq \emptyset}} \varepsilon_V(I;J) V_I \bigotimes V_J \otimes v_p + \sum_{\substack{I \cup J = N^{(p-1)} \\ t \in J,\, I \neq \emptyset}} \varepsilon_V(I;J) V_I \bigotimes V_J \otimes v_p \\ &= V_{N^{(p-1)}} \bigotimes v_p + \overline{T}\Big(\sum_{\substack{I \cup J = N^{(p-1)} \\ t \in I,\, J \neq \emptyset}} \varepsilon_V(I;J) V_I \bigotimes V_J \otimes v_p \Big).\end{aligned} \tag{2.25}$$

Since the unshuffles I can be written as the unshuffles (K,t,R), where $K \subset \{1,\dots,t-1\}$ and $R \subset \{t+1,\dots,p-1\}$, the sign $\varepsilon_V(I;J)$ reads

$$\varepsilon_V(I;J) = \varepsilon_V(t;J)\varepsilon_V(K;J)\varepsilon_V(R;J).$$

When observing that the elements of J that are implicated in $\varepsilon_V(t;J)$ are exactly those that belong to the unshuffle $N^{(t-1)} \setminus K$, we see that

$$\varepsilon_V(t;J) = (-1)^{\langle v_t, V_{N^{(t-1)}} - V_K\rangle},$$

which completes the proof of the lemma. ∎

We are now prepared to finish the proof of Theorem 3.

Set $V_{N^{(p)}} = v_1 \otimes \dots \otimes v_p$, $v_\ell \in V$, and develop the RHS of Equation (2.22). We have

$$\begin{aligned}
(Q \bigotimes \mathrm{id})\Delta(V_{N^{(p)}}) &= \sum_{\substack{I\cup J=N^{(p-1)}\\ I\neq\emptyset}} \varepsilon_V(I;J) Q(V_I) \bigotimes V_J \otimes v_p \\
&\overset{(\star)}{=} \sum_{\substack{I\cup J=N^{(p-1)}\\ I,J\neq\emptyset}} \varepsilon_V(I;J) Q(V_I) \bigotimes V_J \otimes v_p + Q(V_{N^{(p-1)}}) \bigotimes v_p \\
&=: A_1 + A_2 \qquad (2.26)
\end{aligned}$$

and

$$\begin{aligned}
(\mathrm{id} \bigotimes Q)\Delta(V_{N^{(p)}}) &= \sum_{\substack{I\cup J=N^{(p-1)}\\ I\neq\emptyset}} \varepsilon_V(I;J)(-1)^{\langle V_I, Q\rangle} V_I \bigotimes Q(V_J \otimes v_p) \\
&\overset{(\star)}{=} \sum_{\substack{I\cup J=N^{(p-1)}\\ I,J\neq\emptyset}} \varepsilon_V(I;J)(-1)^{\langle V_I, Q\rangle} V_I \bigotimes Q(V_J) \otimes v_p + \\
&\quad \sum_{\substack{I\cup J=N^{(p-1)}\\ I\neq\emptyset}} \sum_{K\cup L=J} \varepsilon_V(I;J)\varepsilon_V(K;L)(-1)^{\langle V_K+V_I, Q\rangle} \\
&\qquad\qquad V_I \bigotimes V_K \otimes Q_{|L|+1}(V_L \otimes v_p) \\
&=: B_1 + B_2, \qquad (2.27)
\end{aligned}$$

where the terms B_1 and B_2 correspond to $k_1 \neq p$ and $k_1 = p$ in the definition (2.21) of Q. As $(T \otimes \mathrm{id})A_1 = B_1$, we get

$$(Q \bigotimes \mathrm{id} + \mathrm{id} \bigotimes Q)\Delta(V_{N^{(p)}}) = \overline{T}A_1 + A_2 + B_2, \qquad (2.28)$$

where

$$B_2 = \sum_{\substack{I\cup K\cup L=N^{(p-1)}\\ I\neq\emptyset}} \varepsilon_V(I;K;L)(-1)^{\langle V_I+V_K,Q\rangle} V_I \bigotimes V_K \otimes Q_{|L|+1}(V_L\otimes v_p). \tag{2.29}$$

Let us now examine the LHS

$$\Delta Q(V_{N^{(p)}}) = \sum_{\substack{M\cup L\cup S=N^{(p)}\\ M,L<S}} \varepsilon_V(M;L)(-1)^{\langle V_M,Q\rangle}\Delta(V_M\otimes Q_{|L|+1}(V_L\otimes v_{s_1})\otimes V_{S\setminus s_1}) \tag{2.30}$$

of Equation (2.22).

In this sum, the term C_1 that corresponds to $s_1 = p$, see Remark $(\star)$, reads

$$C_1 = \sum_{\substack{M\cup L=N^{(p-1)}\\ M\neq\emptyset}} \varepsilon_V(M;L)(-1)^{\langle V_M,Q\rangle}\Delta(V_M\otimes Q_{|L|+1}(V_L\otimes v_p)),$$

where unshuffle M is not empty, since term $M=\emptyset$ vanishes, due to the fact that $\Delta(v)=0$, for any $v\in V$. The definition of Δ yields

$$C_1 = \sum_{\substack{M\cup L=N^{(p-1)}\\ M\neq\emptyset}} \varepsilon_V(M;L)(-1)^{\langle V_M,Q\rangle} \sum_{\substack{I\cup K=M\\ I\neq\emptyset}} \varepsilon_V(I;K) V_I \bigotimes V_K\otimes Q_{|L|+1}(V_L\otimes v_p) =$$

$$\sum_{\substack{I\cup K\cup L=N^{(p-1)}\\ I\neq\emptyset}} \varepsilon_V(I;K;L)(-1)^{\langle V_I+V_K,Q\rangle} V_I\bigotimes V_K\otimes Q_{|L|+1}(V_L\otimes v_p),$$

which is exactly B_2, see Equation (2.29).

In view of Equation (2.28), it thus suffices to show that the remaining term C_2 in Equation (2.30), which corresponds to $s_1\neq p$, i.e.

$$C_2 = \sum_{\substack{M\cup L\cup S=N^{(p-1)}\\ M,L<S}} \varepsilon_V(M;L)(-1)^{\langle V_M,Q\rangle}\Delta(V_M\otimes Q_{|L|+1}(V_L\otimes v_{s_1})\otimes V_{S\setminus s_1}\otimes v_p),$$

verifies

$$C_2 = \overline{T}A_1 + A_2. \tag{2.31}$$

When applying Lemma 2 with $v_t = Q_{|L|+1}(V_L \otimes v_{s_1})$, we obtain

$$
\begin{aligned}
&C_2 = \\
&\textstyle\sum_{\substack{M\cup L\cup S=N^{(p-1)}\\ M,L<S}} \varepsilon_V(M;L)(-1)^{\langle V_M,Q\rangle} \\
&\qquad\qquad V_M \otimes Q_{|L|+1}(V_L \otimes v_{s_1}) \otimes V_{S\setminus s_1} \bigotimes v_p \\
&+\overline{T}(\textstyle\sum_{\substack{M\cup L\cup S=N^{(p-1)}\\ M,L<S}} \sum_{\substack{K\cup R\cup J=M\cup S\setminus s_1\\ K\subset M, R\subset S\setminus s_1,\, J\neq\emptyset}} \\
&\qquad\qquad C\, V_K \otimes Q_{|L|+1}(V_L \otimes v_{s_1}) \otimes V_R \bigotimes V_J \otimes v_p),
\end{aligned}
$$

where

$$C = (-1)^{\langle V_M,Q\rangle}(-1)^{\langle V_L+v_{s_1}+Q, V_M-V_K\rangle}\varepsilon_V(M;L)\varepsilon_V(K;J)\varepsilon_V(R;J).$$

Equation (2.21) entails that the first term in C_2 is nothing but A_2. If D denotes the second term in C_2, it suffices to prove that $D = \overline{T}A_1$.

Let us simplify D. The double sum in D is computed over unshuffles M,L,S,K,R,J, such that

$$
\begin{aligned}
&(a) M\cup L\cup S = N^{(p-1)}, \quad (b) K\cup R\cup J = M\cup S\setminus s_1,\\
&(c) M<S, \quad (d) L<S, \quad (e) K\subset M, \quad (f) R\subset S\setminus s_1.
\end{aligned}
$$

We now examine the impact of these conditions on C.

- Condition (b) implies that

$$\varepsilon_V(M;L) = \varepsilon_V(K;L)\varepsilon_V(R;L)\varepsilon_V(J;L)\varepsilon_V(S\setminus s_1;L).$$

- Conditions (d) and (f) imply that

$$
\begin{aligned}
\varepsilon_V(R;L)\varepsilon_V(S\setminus s_1;L) &= (-1)^{\langle V_L, V_{S\setminus s_1}+V_R\rangle}\varepsilon_V(L;R)\varepsilon_V(L;S\setminus s_1)\\
&= (-1)^{\langle V_L, V_{S\setminus s_1}-V_R\rangle}, \qquad (2.32)
\end{aligned}
$$

  since for any unshuffles I,J, where $I<J$, we have $\varepsilon_V(I;J)=1$.

- Condition (f) implies that $\widetilde{R} = \{s_1, R\}$ is an unshuffle. Then, in view of (2.32),

$$\varepsilon_V(M;L) = (-1)^{\langle V_L, V_{S\setminus\widetilde{R}}\rangle}\varepsilon_V(K;L)\varepsilon_V(J;L).$$

- Conditions (b), (e) and (f) imply that

$$J = (M \backslash K) \cup (S \backslash \widetilde{R}) \tag{2.33}$$

and so that

$$V_J = V_{M \backslash K} + V_{S \backslash \widetilde{R}}.$$

Thus,

$$\varepsilon_V(M;L) = (-1)^{\langle V_L, V_J - V_{M \backslash K} \rangle} \varepsilon_V(K;L) \varepsilon_V(J;L).$$

When replacing in C, we get

$$\begin{aligned} C &= (-1)^{\langle V_K, Q \rangle}(-1)^{\langle v_{s_1}, V_{M \backslash K} \rangle}(-1)^{\langle V_L, V_J \rangle} \varepsilon_V(K;J;L) \varepsilon_V(R;J) \\ &= (-1)^{\langle V_K, Q \rangle}(-1)^{\langle v_{s_1}, V_{M \backslash K} \rangle} \varepsilon_V(K;L;J) \varepsilon_V(R;J). \end{aligned}$$

Finally, when using (2.33) and (c), we obtain

$$\varepsilon_V(s_1;J) = (-1)^{\langle v_{s_1}, V_{M \backslash K} \rangle},$$

$$\varepsilon_V(\widetilde{R};J) = \varepsilon_V(s_1;J) \varepsilon_V(R;J) = (-1)^{\langle v_{s_1}, V_{M \backslash K} \rangle} \varepsilon_V(R;J),$$

and

$$C = (-1)^{\langle V_K, Q \rangle} \varepsilon_V(K;L;J) \varepsilon_V(\widetilde{R};J).$$

Hence,

$$D = \overline{T}\Big(\sum_{\substack{K \cup L \cup \widetilde{R} \cup J = N^{(p-1)} \\ K, L < \widetilde{R},\ J \neq \emptyset}} C\, V_K \otimes Q_{|L|+1}(V_L \otimes v_{s_1}) \otimes V_{\widetilde{R} \backslash s_1} \bigotimes V_J \otimes v_p \Big).$$

If we set now $I = K \cup L \cup \widetilde{R}$, we deduce that

$$C = (-1)^{\langle V_K, Q \rangle} \varepsilon_V(K;L) \varepsilon_V(I;J)$$

and, if we use again (2.21), we see that

$$D = \overline{T}\Big(\sum_{\substack{I \cup J = N^{(p-1)} \\ I, J \neq \emptyset}} \varepsilon_V(I;J) Q(V_I) \bigotimes V_J \otimes v_p \Big) = \overline{T} A_1. \blacksquare$$

Like coderivations, cohomomorphisms from $(T(V), \Delta)$ to $(T(V'), \Delta)$ are characterized by their corestriction maps.

Theorem 4. *Let V and V' be two $\mathbb{Z}^n$-graded vector spaces. A coalgebra cohomomorphism $\mathscr{F} : (T(V),\Delta) \longrightarrow (T(V'),\Delta)$ is uniquely defined by its (weight 0) corestriction maps $\mathscr{F}_p : T^pV \to V'$, $p \geq 1$, via the equation*

$$\mathscr{F}(v_1 \otimes \ldots \otimes v_p) = \sum_{s=1}^{p} \sum_{\substack{I^1\cup\ldots\cup I^s=N^{(p)} \\ I^1,\ldots,I^s\neq\emptyset \\ i^1_{|I^1|}<\ldots<i^s_{|I^s|}}} \varepsilon_V(I^1;\ldots;I^s)\mathscr{F}_{|I^1|}(V_{I^1}) \otimes \ldots \otimes \mathscr{F}_{|I^s|}(V_{I^s}), \tag{2.34}$$

where $v_\ell \in V^{v_\ell}$.

Proof. The proof of this theorem is similar to that of Theorem 3 and will not be detailed here. ∎

Set now $e_1 = (1,0,\ldots,0) \in \mathbb{Z}^n$ and consider the desuspension operator $\downarrow : V \to \downarrow V$, where $\downarrow V$ is the same space as V up to the shift $(\downarrow V)^\alpha = V^{\alpha+e_1}$ of gradation. The inverse map of $\downarrow$ is denoted by $\uparrow$. The mapping $\downarrow^{\otimes p} := \downarrow \otimes \ldots \otimes \downarrow$, p factors, i.e. the mapping

$$\downarrow^{\otimes p} : V^{\otimes p} \ni v_1 \otimes \ldots \otimes v_p \to (-1)^{\sum_{s=1}^{p}\langle (p-s)e_1, v_s\rangle} \downarrow v_1 \otimes \ldots \otimes \downarrow v_p \in (\downarrow V)^{\otimes p}, \tag{2.35}$$

is an isomorphism of weight $-pe_1$, whose inverse is $(-1)^{\frac{p(p-1)}{2}} \uparrow^{\otimes p}$.

Remark 2. *The isomorphisms*

$$\sigma^{\downarrow V}_{(Q,p)} : M^{(Q,p-1)}(\downarrow V) \ni Q_p \to \pi_p := \uparrow \circ Q_p \circ \downarrow^{\otimes p} \in M^{(Q+(1-p)e_1,p-1)}(V), \tag{2.36}$$

$Q \in \mathbb{Z}^n, p \in \mathbb{N}^*$, (their inverses are defined by $(-1)^{\frac{p(p-1)}{2}} \downarrow \circ \pi_p \circ \uparrow^{\otimes p}$) generate isomorphisms $\sigma^{\downarrow V}_Q$ and $\sigma^{\downarrow V}$ between the corresponding direct products and direct sums of direct products. If no confusion is possible, we omit super- and subscripts and denote these isomorphisms simply by σ. Isomorphisms (2.36) extend of course to multilinear maps on $\downarrow V$ valued in $\downarrow V'$.

Remark 3. *Theorem 3-4 and Remarks1-2 show that weight Q coderivations $Q : (T(\downarrow V),\Delta) \to (T(\downarrow V),\Delta)$ can be viewed as sequences $\pi = (\pi_1,\pi_2,\ldots) = \sum_p \pi_p$ of weight $Q+(1-p)e_1$ multilinear maps $\pi_p : V^{\times p} \to V$, and that cohomomorphisms $\mathscr{F} : (T(\downarrow V),\Delta) \to (T(\downarrow V'),\Delta)$ (which by definition have weight 0) can be seen as sequences $f = (f_1,f_2,\ldots) = \sum_p f_p$ of weight $(1-p)e_1$ multilinear maps $f_p : V^{\times p} \to V'$.*

2.4 Stem bracket

When combining the isomorphisms σ^{-1} and ψ^{-1}, we get, for $A \in \mathbb{Z}^n, a \in \mathbb{N}$, a vector space isomorphism

$$\phi_{(A,a)} : M^{(A,a)}(V) \ni A \to Q^A = (0,\ldots,0,Q^A_{a+1},0,\ldots) \in \mathrm{CoDer}^{A+ae_1}_{a+1}(T(\downarrow V)), \tag{2.37}$$

where $Q^A_{a+1} = (-1)^{\frac{a(a+1)}{2}} \downarrow \circ A \circ \uparrow^{\otimes(a+1)}$ and where Q^A is the coderivation that is obtained from Q^A_{a+1} via extension equation (2.21).

Theorem 5. *The $\mathbb{Z}^{n+1}$-graded vector space of multilinear maps $M_r(V) = M(V) \ominus V$ is a $\mathbb{Z}^{n+1}$-graded Lie algebra, when endowed with the bracket*

$$[A,B]^{\otimes} := (-1)^{1+\langle ae_1, be_1+B\rangle} \phi^{-1}_{(A+B,a+b)}([\phi_{(A,a)}(A), \phi_{(B,b)}(B)]), \tag{2.38}$$

$A \in M^{(A,a)}(V)$, $B \in M^{(B,b)}(V)$, where $[-,-]$ denotes the $\mathbb{Z}^n$-graded Lie bracket of the space of coderivations of $(T(\downarrow V),\Delta)$.

Proof. It follows from Equation (2.21) that the p-th corestriction map $[\phi_{(A,a)}(A), \phi_{(B,b)}(B)]_p$ vanishes if $p \neq a+b+1$, so that the bracket $[\phi_{(A,a)}(A), \phi_{(B,b)}(B)]$ is really a coderivation in $\mathrm{CoDer}^{A+B+(a+b)e_1}_{a+b+1}(T(\downarrow V))$. The sign $(-1)^{1+\langle ae_1, be_1+B\rangle}$ ensures that the $\mathbb{Z}^n$-graded Lie bracket of coderivations induces a $\mathbb{Z}^{n+1}$-graded Lie bracket $[-,-]^{\otimes}$. ∎

As the map $\phi = \psi^{-1} \circ \sigma^{-1}$ is also a (weight 0) $\mathbb{Z}^n$-graded vector space isomorphism

$$\phi : C(V) := \bigoplus_{Q\in\mathbb{Z}^n} \prod_{p\geq 1} M^{(Q+(1-p)e_1, p-1)}(V) \to \mathrm{CoDer}(T(\downarrow V)) = \bigoplus_{Q\in\mathbb{Z}^n} \mathrm{CoDer}^Q(T(\downarrow V)), \tag{2.39}$$

the next proposition is obvious.

Proposition 7. *The $\mathbb{Z}^n$-graded vector space $C(V)$ of sequences of multilinear maps is a $\mathbb{Z}^n$-graded Lie algebra for the bracket*

$$[\pi,\rho]^{\bar{\otimes}} = \phi^{-1}[\phi\pi, \phi\rho] = \sum_{q\geq 1} \sum_{s+t=q+1} (-1)^{1+(s-1)\langle e_1,\rho\rangle} [\pi_s, \rho_t]^{\otimes}, \tag{2.40}$$

where $\pi = \sum_s \pi_s \in C^{\pi}(V)$ and $\rho = \sum_t \rho_t \in C^{\rho}(V)$ are two homogeneous $C(V)$-elements of $\mathbb{Z}^n$-degree π and ρ respectively.

Remark 4. *In the following, we refer to $[-,-]^{\bar{\otimes}}$ (resp. $[-,-]^{\otimes}$) as the $\mathbb{Z}^n$-graded (resp. $\mathbb{Z}^{n+1}$-graded) stem bracket.*

Theorem 6. *The $\mathbb{Z}^{n+1}$-graded stem bracket on $M_r(V)$ explicitly reads*

$$[A,B]^{\otimes} = j_A B - (-1)^{\langle (A,a),(B,b)\rangle} j_B A, \tag{2.41}$$

where

$$(j_B A)(v_1 \otimes \ldots \otimes v_{a+b+1}) = (-1)^{\langle A,B\rangle} \sum_{\substack{I\cup J\cup K = N^{(a+b+1)} \\ I,J<K,\,|J|=b}} (-1)^{\langle B,V_I\rangle + b|I|}$$
$$(-1)^{(I;J)} \varepsilon_V(I;J) A(V_I \otimes B(V_J \otimes v_{k_1}) \otimes V_{K\setminus k_1}), \tag{2.42}$$

for any $A \in M^{(A,a)}(V)$, $B \in M^{(B,b)}(V)$, $a,b \geq 0$, and $v_\ell \in V^{v_\ell}$.

Proof. It follows from Equation (2.21) that the $(a+b+1)$-th corestriction of $[\phi_{(A,a)}(A), \phi_{(B,b)}(B)] = [Q^A, Q^B]$ is obtained by just restricting the involved composite maps to $(\downarrow V)^{\otimes(a+b+1)}$. The description of the isomorphisms $\phi^{-1}_{(A,a)}$ then shows that

$$\begin{aligned} [A,B]^{\otimes} &= -(-1)^{\langle ae_1, be_1+B\rangle} \Big(\uparrow \circ Q^A \circ Q^B \circ \downarrow^{\otimes(a+b+1)} \\ &\quad -(-1)^{\langle A+ae_1, B+be_1\rangle} \uparrow \circ Q^B \circ Q^A \circ \downarrow^{\otimes(a+b+1)} \Big). \end{aligned} \tag{2.43}$$

If $V_{N^{(a+b+1)}} = v_1 \otimes \ldots \otimes v_{a+b+1}$, with $v_\ell \in V^{v_\ell}$, we get

$$\begin{aligned} \left(\uparrow \circ Q^A \circ Q^B \circ \downarrow^{\otimes(a+b+1)}\right)(V_{N^{(a+b+1)}}) &= (-1)^{\beta_1} \left(\uparrow \circ Q^A \circ Q^B\right)(\downarrow V_{N^{(a+b+1)}}), \\ \beta_1 &= \langle e_1, \sum_{s\geq 1} (a+b+1-s) v_s \rangle, \end{aligned}$$

where $\downarrow V_{N^{(a+b+1)}} = \downarrow v_1 \otimes \ldots \otimes \downarrow v_{a+b+1}$. Formula (2.21) yields

$$\begin{aligned} Q^B(\downarrow V_{N^{(a+b+1)}}) &= \sum_{\substack{I\cup J\cup K = N^{(a+b+1)} \\ I,J<K,|J|=b}} (-1)^{\beta_2} \varepsilon_{\downarrow V}(I;J) \downarrow V_I \otimes Q^B_{b+1}(\downarrow V_J \otimes \downarrow v_{k_1}) \otimes \downarrow V_{K\setminus k_1}, \\ \beta_2 &= \langle B + be_1, V_I + |I| e_1 \rangle. \end{aligned} \tag{2.44}$$

Moreover,

$$Q^B_{b+1}(\downarrow V_J \otimes \downarrow v_{k_1}) = (-1)^{\beta_3} \downarrow B(V_J \otimes v_{k_1}), \beta_3 = \langle e_1, \sum_{s\geq 1} (b+1-s) v_{j_s} \rangle,$$

as the sign $(-1)^{\frac{b(b+1)}{2}}$ inside Q^B_{b+1} and the sign due to the shift of the $\mathbb{Z}^n$-gradation cancel each other out. When noticing that Q^A evaluated on an element of $(\downarrow V)^{\otimes(a+1)}$ is nothing but Q^A_{a+1}, we obtain

$$\left(\uparrow \circ Q^A \circ Q^B \circ \downarrow^{\otimes(a+b+1)}\right)(V_{N^{(a+b+1)}}) = \sum_{\substack{I\cup J\cup K=N^{(a+b+1)}\\ I,J<K,\ |J|=b}} (-1)^{\ell}\varepsilon_{\downarrow V}(I;J)A(V_I\otimes B(V_J\otimes v_{k_1})\otimes V_{K\setminus k_1}), \tag{2.45}$$

with $\ell=\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6$, where

$\beta_4=\langle e_1,\sum_{s\geq 1}(a+1-s)v_{i_s}\rangle$, $\beta_5=(a-|I|)\langle e_1,B+V_J+v_{k_1}\rangle$, and $\beta_6=\langle e_1,\sum_{s\geq 2}(a-|I|-s+1)v_{k_s}\rangle$

are generated by $\uparrow^{\otimes(a+1)}$ and where, again, the sign inside Q^A_{a+1} and the sign due to the shift neutralize.

We will prove that

$$(-1)^{\ell}\varepsilon_{\downarrow V}(I;J)=(-1)^{\langle B,ae_1\rangle+\langle B,V_I\rangle+b|I|}(-1)^{(I;J)}\varepsilon_V(I;J). \tag{2.46}$$

Observe first that an appropriate regrouping of terms yields

$$\begin{aligned}\ell=\ell_1+\ell_2 &:= (\langle B,ae_1\rangle+\langle B,V_I\rangle+b|I|)+\Bigg(\langle e_1,\sum_{s=1}^{a+b+1}(a+b+1-s)v_s\rangle\\ &+\langle e_1,\sum_{s=1}^{|I|}(a+b+1-s)v_{i_s}\rangle+\langle e_1,\sum_{s=|I|+1}^{|I|+|J|}(a+b+1-s)v_{j_{s-|I|}}\rangle\\ &\quad+\langle e_1,\sum_{s=|I|+|J|+1}^{a+b+1}(a+b+1-s)v_{k_{s-|I|-|J|}}\rangle\Bigg),\end{aligned}$$

where, since in view of the conditions $I,J<K$ the concatenation $(I;J)$ is a permutation of $1,\ldots,|I|+|J|$, the term ℓ_2 reads (modulo even terms)

$$\ell_2=\langle e_1,\sum_{s=1}^{|I|}(s+i_s)v_{i_s}\rangle+\langle e_1,\sum_{s=|I|+1}^{|I|+|J|}(s+j_{s-|I|})v_{j_{s-|I|}}\rangle.$$

If permutation $(I;J)$ is a transposition $(I;J)=(1,\ldots,q-1,q+1,q,q+2,\ldots,|I|+|J|)$, then $\ell_2=\langle e_1,v_q+v_{q+1}\rangle$. It is now easily checked that for a transposition

$$(-1)^{\ell_2}\varepsilon_{\downarrow V}(I;J)=(-1)^{(I;J)}\varepsilon_V(I;J) \tag{2.47}$$

and that for a composition of transpositions all the factors of this last equation are the products of the corresponding factors induced by the involved transpositions. It follows that Equations (2.47) and (2.46) hold true for any permutation $(I;J)$.

Eventually Equation (2.45) may be written

$$\left(\uparrow \circ Q^A \circ Q^B \circ \downarrow^{\otimes(a+b+1)}\right)(V_{N^{(a+b+1)}}) =$$
$$\sum_{\substack{I\cup J\cup K=N^{(a+b+1)}\\ I,J<K,\ |J|=b}} (-1)^m (-1)^{(I;J)} \varepsilon_V(I;J) A(V_I \otimes B(V_J \otimes v_{k_1}) \otimes V_{K\setminus k_1}),$$

where $m = \langle B, ae_1\rangle + \langle B, V_I\rangle + b|I|$.

If we define the insertion operator

$$j_B A = (-1)^{\langle ae_1 + A, B\rangle} \uparrow \circ Q^A \circ Q^B \circ \downarrow^{\otimes(a+b+1)},$$

$A \in M^{(A,a)}(V)$, $B \in M^{(B,b)}(V)$, $a,b \geq 0$, we finally get the announced result. ∎

Remark 5. *It is easily checked that the restriction of the stem bracket* $[-,-]^{\otimes}$ *to the subspace* $\mathrm{A}_r(V)$ *of* $M_r(V)$*, made up by graded skew-symmetric multilinear maps on* V*, coincides with the graded Nijenhuis-Richardson bracket* $[-,-]^{\mathrm{NR}}$.

2.5 Graded and strongly homotopy Loday structures

Let us recall that a *codifferential* of a coalgebra is a coderivation that squares to 0.

Proposition 8. *A homogenous odd weight coderivation* Q *of the coalgebra* $(T(V),\Delta)$ *is a codifferential if and only if, for any* $p \geq 1$*, the following equation holds identically:*

$$\sum_{\substack{I\cup J\cup K=N^{(p)}\\ I,J<K}} \varepsilon_V(I;J)(-1)^{\langle Q,V_I\rangle} Q_{|I|+|K|}(V_I \otimes Q_{|J|+1}(V_J \otimes v_{k_1}) \otimes V_{K\setminus k_1}) = 0. \quad (2.48)$$

Proof. As Q is an odd weight coderivation, $2Q^2 = [Q,Q]$ and so Q^2 is also a coderivation. Thus, according to Theorem 3, the condition $Q^2 = 0$ is satisfied if and only if the corestriction maps Q^2_p, $p \geq 1$, of Q^2 vanish. It is easily seen that $Q^2_p(V_{N^{(p)}})$ is exactly the LHS of (2.48). ∎

Definition 7. *A* graded Loday algebra $(\mathfrak{g}, \{-,-\})$ *(GLodA for short) is made up by a* $\mathbb{Z}^n$*-graded vector space* $\mathfrak{g}$ *and a weight* 0 *bilinear bracket* $\{-,-\}$ *that satisfies the graded Jacobi identity (2.3).*

Non-graded Loday algebras were introduced by Loday in [Lod93] and are also known as *Leibniz algebras*.

Theorem 7. *Let* $\mathrm{Lod}(V)$ *be the set of* $\mathbb{Z}^n$*-graded Loday structures on a* $\mathbb{Z}^n$*-graded vector space* V*, and denote by* $\mathrm{CoDiff}_p^Q(T(\downarrow V))$*,* $Q \in \mathbb{Z}^n$*,* $p \in \mathbb{N}^*$*, the set of codifferentials* Q *of* $(T(\downarrow V), \Delta)$*, which have weight* Q *and whose corestriction maps all vanish except* Q_p*. Then,*

1. *The restriction of* $\phi_{(0,1)}$ *to* $\mathrm{Lod}(V)$ *is a bijection and*
$$\mathrm{Lod}(V) \simeq \mathrm{CoDiff}_2^{e_1}(T(\downarrow V)), \tag{2.49}$$

2. *The graded Loday structures on* V *are exactly the canonical elements of weight* $(0,1)$ *of the graded Lie algebra* $(M_r(V), [-,-]^{\otimes})$*.*

Proof. Since $\phi_{(0,1)}$ is a bijection between $M^{(0,1)}(V)$ and $\mathrm{CoDer}_2^{e_1}(T(\downarrow V))$, it suffices, in order to account for Point 1, to prove that $\phi_{(0,1)}(\mathrm{Lod}(V)) = \mathrm{CoDiff}_2^{e_1}(T(\downarrow V))$.

If $\pi \in \mathrm{Lod}(V)$, its image $\phi_{(0,1)}(\pi) = Q^\pi = (0, Q_2^\pi, 0, \ldots) \in \mathrm{CoDer}_2^{e_1}(T(\downarrow V))$ is a codifferential, if

$$\begin{aligned} Q_2^\pi(Q_2^\pi(\downarrow v_1 \otimes \downarrow v_2) \otimes \downarrow v_3) + (-1)^{\langle e_1, \downarrow v_1\rangle} Q_2^\pi(\downarrow v_1 \otimes Q_2^\pi(\downarrow v_2 \otimes \downarrow v_3)) \\ + (-1)^{\langle e_1 + \downarrow v_1, \downarrow v_2\rangle} Q_2^\pi(\downarrow v_2 \otimes Q_2^\pi(\downarrow v_1 \otimes \downarrow v_3)) = 0, \end{aligned} \tag{2.50}$$

see Proposition 8 and note that Condition (2.48) is trivial for $p \neq 3$. When remembering that $Q_2^\pi = -\downarrow \circ \pi \circ (\uparrow \otimes \uparrow)$, we easily check that Condition (2.50) reads

$$\begin{aligned} (-1)^{\langle v_2, e_1\rangle} \downarrow [\pi(\pi(v_1 \otimes v_2) \otimes v_3) \\ -\pi(v_1 \otimes \pi(v_2 \otimes v_3)) + (-1)^{\langle v_1, v_2\rangle} \pi(v_2 \otimes \pi(v_1 \otimes v_3))] = 0. \end{aligned}$$

As π verifies the graded Jacobi identity, the last requirement is fulfilled.

Conversely, if $Q = (0, Q_2, 0, \ldots) \in \mathrm{CoDiff}_2^{e_1}(T(\downarrow V))$, then $\pi := \phi_{(0,1)}^{-1}(Q)$ is a graded Loday structure.

As regards Point 2, note that any graded Loday structure π is odd. Furthermore,

$$[\pi, \pi]^{\otimes} = -(-1)^{\langle e_1, e_1\rangle} \phi_{(0,2)}^{-1}([\phi_{(0,1)}(\pi), \phi_{(0,1)}(\pi)]) = 2\phi_{(0,2)}^{-1}(\phi_{(0,1)}^2(\pi)),$$

so that the graded Loday structures are exactly the canonical elements of weight $(0,1)$. ∎

In the sequel, we extend graded Loday structures, see Equation (2.49), by introducing the homotopy version of Loday algebras.

Definition 8. *A* strongly homotopy Loday algebra *or a* Loday infinity algebra *is a $\mathbb{Z}^n$-graded vector space V endowed with a codifferential of odd weight of the tensor coalgebra $(T(\downarrow V), \Delta)$.*

We denote by $\mathrm{Lod}_\infty^Q(V)$, $Q \in \mathbb{Z}^n$, $\langle Q, Q\rangle$ odd, the set

$$\mathrm{Lod}_\infty^Q(V) \simeq \mathrm{CoDiff}^Q(T(\downarrow V))$$

of weight Q Loday infinity (Lod_∞^Q for short) structures on V, whereas $\mathrm{Lod}_\infty(V)$ denotes the set $\mathrm{Lod}_\infty^{e_1}(V)$ – as most infinity structures considered below have weight e_1. Since Q is odd, $Q \in \mathrm{Lod}_\infty^Q(V)$ if and only if $Q \in \mathrm{CoDer}^Q(T(\downarrow V))$ and $[Q, Q] = 2Q^2 = 0$. Hence, in view of Remark 3 and Proposition 7, the sequence maps "definition" of Loday infinity algebras:

Proposition 9. *A Loday infinity algebra is a $\mathbb{Z}^n$-graded vector space V together with a sequence of* structure maps

$$\pi = (\pi_1, \pi_2, \ldots) = \sum_p \pi_p \in C^Q(V) = \prod_{p\geq 1} M^{(Q+(1-p)e_1, p-1)}(V)$$

of odd degree Q, such that

$$\sum_{s+t=p} (-1)^{1+(s-1)\langle e_1, Q\rangle} [\pi_s, \pi_t]^{\otimes} = 0, \forall p \geq 2. \tag{2.51}$$

In the usual case of Lod_∞ structures π on V, the first three conditions (2.51) mean that (V, π_1) is a chain complex, that π_1 is a $\mathbb{Z}^n$-graded derivation of the bilinear map π_2, and that π_2 is a $\mathbb{Z}^n$-graded Loday structure modulo homotopy π_3.

Example 1. *If the structure maps of a* Lod_∞ *algebra (V, π) all vanish, except π_1 (resp. except π_2, except π_1 and π_2), (V, π) is a chain complex (resp. a $\mathbb{Z}^n$-graded Loday algebra (GLodA), a differential graded Loday algebra(DGLodA)).*

Example 2. *Let (V, π) and (V', π') be two* Lod_∞ *algebras. Their direct sum $(V \oplus V', \pi \oplus \pi')$ is a* Lod_∞ *algebra, where the structure maps are defined by*

$$(\pi \oplus \pi')_p(v_1 + v_1', \ldots, v_p + v_p') := \pi_p(v_1, \ldots, v_p) + \pi_p'(v_1', \ldots, v_p'),$$

for any $p \geq 1$.

It is obvious from the structure of the explicit form of the stem bracket $[-,-]^{\otimes}$, see Theorem 6, that if π and π' verify the Lod_∞ conditions (2.51) in V and V' respectively, then $\pi \oplus \pi'$ verifies the same conditions in $V \oplus V'$.

As a Lod_∞ structure on V is a weight e_1 codifferential Q of the coalgebra $(T(\downarrow V), \Delta)$, it is natural to give the following coalgebraic

Definition 9. *Let (V,Q) and (V',Q') be two* Lod$_\infty$ *algebras. A* Lod$_\infty$ morphism *from (V,Q) to (V',Q') is a coalgebra cohomomorphism $\mathscr{F} : (T(\downarrow V),\Delta) \longrightarrow (T(\downarrow V'),\Delta)$, such that $Q'\mathscr{F} = \mathscr{F}Q$.*

Since the composition of Lod$_\infty$ morphisms is again a Lod$_\infty$ morphism, and as, for any Lod$_\infty$ algebra, the identity map is a morphism (say $\mathscr{I}$, with corestriction maps $\mathscr{I}_1 = \mathrm{id}$ and $\mathscr{I}_p = 0$, for $p \geq 2$), Lod$_\infty$ algebras and their morphisms form a category, which we denote by **Lod**$_\infty$.

Next we give the sequence maps "definition" of Lod$_\infty$ morphisms, see Remark 3.

Proposition 10. *Let (V,π) and (V',π') be two* Lod$_\infty$ *algebras. A* Lod$_\infty$ *morphism $f : (V,\pi) \to (V',\pi')$ is a sequence of weight $(1-p)e_1$ multilinear maps $f_p : V^{\times p} \to V'$, $p \geq 1$, which satisfy the condition*

$$\sum_{s=1}^{p} \sum_{\substack{I^1\cup\ldots\cup I^s=N^{(p)}\\ I^1,\ldots,I^s\neq\emptyset\\ i^1_{|I^1|}<\ldots<i^s_{|I^s|}}} (-1)^{\omega}(-1)^{(I^1;\ldots;I^s)}\varepsilon_V(I^1;\ldots;I^s)\pi'_s(f_{|I^1|}(V_{I^1}),\ldots,f_{|I^s|}(V_{I^s}))$$
$$= \sum_{\substack{I\cup J\cup K=N^{(p)}\\ I,J<K}} (-1)^{\lambda}(-1)^{(J;I)}\varepsilon_V(I;J)f_{|I|+|K|}(V_I,\pi_{|J|+1}(V_J,v_{k_1}),V_{K\setminus k_1}), \quad (2.52)$$

where

$$\omega = \tfrac{s(s-1)}{2} + \textstyle\sum_{1\leq r\leq s}(s-r)|I^r| + \sum_{2\leq r\leq s}\langle(|I^r|+1)e_1, V_{|I^1|}+\ldots+V_{|I^{r-1}|}\rangle \quad (2.53)$$

and

$$\lambda = \langle(1+|J|)e_1, V_I + (p+1)e_1\rangle, \quad (2.54)$$

for all $p \geq 1$.

Proof. When using (2.21) and (2.34), we see that $Q'\mathscr{F}$ and $\mathscr{F}Q$ are similarly composed of the respective corestriction maps, so that they coincide, if their corestrictions do. This leads to a reformulation of the intertwining condition $Q'\mathscr{F} = \mathscr{F}Q$ in terms of $Q'_p,\mathscr{F}_p,Q_p$, $p \geq 1$. The translation of this reformulation by means of the sequences π', f, π, and in particular the above signs, are obtained by a direct computation that is based upon similar arguments than those used in the proof of Theorem 6. ∎

Remark 6. *The first constraint* (2.52) *states that f_1 is a chain map between (V,π_1) and (V',π'_1), whereas the second means that f_2 measures the deviation from f_1*

being a $(V,\pi_2)-(V',\pi_2')$ homomorphism. If (V,π) and (V',π') are DGLodAs, map f_1 is a DGLodA morphism. In Chapter 3, we shall recall the category $\mathbf{L}_\infty$ of L_∞ *algebras and morphisms and show that the category $\mathbf{L}_\infty$ is a subcategory of* $\mathbf{Lod}_\infty$.

Definition 10. *A* Lod_∞ quasi-isomorphism *from a* Lod_∞ *algebra (V,π) to a* Lod_∞ *algebra (V',π') is a* Lod_∞ *morphism $f:(V,\pi)\to(V',\pi')$, such that the chain map $f_1:(V,\pi_1)\to(V',\pi_1')$ induces an isomorphism $f_{1\,\sharp}:H(V,\pi_1)\to H(V',\pi_1')$ in cohomology. In particular, f is called a* Lod_∞ isomorphism*, if $f_1:V\to V'$ is an isomorphism.*

If $\mathscr{F}\simeq f$ and $\mathscr{G}\simeq g$ are two composable Lod_∞ morphisms, we denote by $g\circ f$ the sequence of multilinear maps that corresponds to the Lod_∞ morphism $\mathscr{G}\mathscr{F}$. Similarly, $\pi'\circ f$ and $f\circ\pi$ are the sequences that represent $Q'\mathscr{F}$ and $\mathscr{F}Q$. The Lod_∞ morphism condition (2.52) then reads $\pi'\circ f=f\circ\pi$. We use these and analogous notations below.

The next proposition will be needed in the following.

Proposition 11.

1. *Any coalgebra cohomomorphism $f:(T(\downarrow V),\Delta)\to(T(\downarrow V'),\Delta)$, which corresponds to a sequence $f=(f_1,f_2,\ldots)$, whose first element f_1 is bijective, is invertible, i.e. there is a coalgebra cohomomorphism $f^{-1}:(T(\downarrow V'),\Delta)\to(T(\downarrow V),\Delta)$, such that $f\circ f^{-1}=\mathrm{Id}$ and $f^{-1}\circ f=\mathrm{Id}$, where* Id *is the unit cohomomorphism* $\mathrm{Id}=(\mathrm{id},0,0,\ldots)$.

2. *If (V,π) denotes a* Lod_∞ *algebra, any sequence $f=(f_1,f_2,\ldots)$ of weight $(1-p)e_1$ multilinear maps $f_p:V^{\times p}\to V$, whose first element f_1 is the identity map of V, induces a new* Lod_∞ *structure $f\circ\pi\circ f^{-1}$ on V and f is a* Lod_∞ *isomorphism between (V,π) and $(V,f\circ\pi\circ f^{-1})$.*

3. *Any* Lod_∞ *isomorphism $f:(V,\pi)\to(V',\pi')$ admits an inverse f^{-1} that is a* Lod_∞ *isomorphism as well.*

Proof. 1. Let $\mathscr{F}:(T(\downarrow V),\Delta)\to(T(\downarrow V'),\Delta)$ be a coalgebra cohomomorphism, whose first corestriction $\mathscr{F}_1:\downarrow V\to\downarrow V'$ is bijective. If there is an inverse cohomomorphism $\mathscr{G}:(T(\downarrow V'),\Delta)\to(T(\downarrow V),\Delta)$, it follows from the condition $\mathscr{F}\mathscr{G}=\mathscr{I}$ and from Equation (2.34) that $\mathscr{G}_1=\mathscr{F}_1^{-1}$ and that, for any $p\geq 2$,

$$\begin{aligned}&\mathscr{G}_p(\downarrow v_1',\ldots,\downarrow v_p')\\&=-\sum_{s=2}^{p}\sum_{\substack{I^1\cup\ldots\cup I^s=N^{(p)}\\ I^1,\ldots,I^s\neq\emptyset\\ i^1_{|I^1|}<\ldots<i^s_{|I^s|}}}\varepsilon_{\downarrow V'}(I^1;\ldots;I^s)\mathscr{F}_1^{-1}\mathscr{F}_s\left(\mathscr{G}_{|I^1|}(\downarrow V'_{I^1})\otimes\ldots\otimes\mathscr{G}_{|I^s|}(\downarrow V'_{I^s})\right).\end{aligned}$$

The last equation provides inductively the corestriction maps of a cohomomorphism $\mathscr{G}$. One can check that $\mathscr{G}$ not only verifies $\mathscr{F}\mathscr{G} = \mathscr{I}$, but also $\mathscr{G}\mathscr{F} = \mathscr{I}$.

2. Take a Lod_∞ structure Q on V and a cohomomorphism $\mathscr{F} : (T(\downarrow V), \Delta) \to (T(\downarrow V), \Delta)$, such that $\mathscr{F}_1 = \mathrm{id}$. Since $(f \otimes g) \circ (h \otimes k) = (-1)^{\langle g,h \rangle}(f \circ h) \otimes (g \circ k)$, with self-explaining notations, it is easily seen that $\mathscr{F} Q \mathscr{F}^{-1}$, where $\mathscr{F}^{-1}$ is the inverse cohomomorphism $\mathscr{G}$ given by Item 1, is a weight e_1 codifferential of $T(\downarrow V)$, i.e. a Lod_∞ structure on V. Eventually, $\mathscr{F}$ is obviously a Lod_∞ morphism, and, in view of the assumption $\mathscr{F}_1 = \mathrm{id}$, even a Lod_∞ isomorphism.

3. Consider two Lod_∞ algebras (V, Q), (V', Q') and a Lod_∞ isomorphism $\mathscr{F}$, i.e. a cohomomorphism, such that $\mathscr{F}_1$ is bijective and $Q'\mathscr{F} = \mathscr{F} Q$ $(\star)$. It then follows from Item 1 that there is an inverse cohomomorphism $\mathscr{F}^{-1}$, such that $(\mathscr{F}^{-1})_1 = (\mathscr{F}_1)^{-1}$, and from Equation $(\star)$ that $Q\mathscr{F}^{-1} = \mathscr{F}^{-1} Q'$. ∎

The following key-theorem generalizes the last item of Proposition 11.

Theorem 8. *If $f : (V, \pi) \to (V', \pi')$ is a* Lod_∞ *quasi-isomorphism, it admits a* quasi-inverse, *i.e. there exists a* Lod_∞ *quasi-isomorphism $g : (V', \pi') \to (V, \pi)$, which induces the inverse isomorphism in cohomology, i.e. $g_{1\sharp} = (f_{1\sharp})^{-1}$.*

We prove this theorem, which does not hold true in the category of DGLodAs, in the next section.

2.6 Minimal model theorem for Loday infinity algebras

Definition 11. *A* Lod_∞ *algebra (V, π) is* minimal*, if $\pi_1 = 0$. It is* contractible*, if $\pi_p = 0$, for $p \geq 2$, and if in addition $H(V, \pi_1) = 0$.*

Theorem 9. *Each* Lod_∞ *algebra is* Lod_∞ *isomorphic to the direct sum of a minimal* Lod_∞ *algebra and a contractible* Lod_∞ *algebra.*

Theorem 9 was proved for L_∞ algebras in [Kon03] and e.g. [AMM02]. In the sequel, we provide a proof in the Lod_∞ case.

Proof. Let (V, π) be a Lod_∞ algebra. For any $\alpha \in \mathbb{Z}^n$, denote by Z^α and B^α the trace on V^α of the kernel and the image of π_1. Consider a supplementary vector subspace V_m^α of B^α in Z^α and a supplementary subspace W^α of Z^α in V^α. Let Z, B, V_m, and W be the corresponding graded spaces. Then, the complex (V, π_1) decomposes into the direct sum of the complex $(V_m, 0)$, with vanishing differential, and the complex $(V_c := B \oplus W, \pi_1)$, with trivial cohomology. It follows that the sequence $f^{(1)} := (\mathrm{id}, 0, \ldots)$ is a Lod_∞ isomorphism from (V, π) to the Lod_∞ algebra $L_1 := (V_m \oplus V_c, 0 \oplus \pi_1, \pi_2, \pi_3, \ldots)$. We will transform inductively the maps π_p, $p \geq 2$, via Lod_∞ isomorphisms, into mappings of the form $\pi_p^m \oplus 0$, such that $\pi^m := (0, \pi_2^m, \pi_3^m, \ldots)$ be a minimal Lod_∞ structure on V_m.

Lemma 3. *Consider the operator $\delta : V \to V$ that is defined, for any $v \in V_m \oplus W$, by $\delta(v) = 0$, and, for any $v \in B$, by $\delta(v) = w$, where w is the unique element $w \in W$, such that $\pi_1(w) = v$. Let P be the projection onto V_m with respect to the decomposition $V = V_m \oplus V_c$. Then, δ is a homotopy operator between the complex endomorphisms P and* id *of (V, π_1), i.e. $\pi_1 \delta + \delta \pi_1 = \mathrm{id} - P$.*

Proof. Obvious. ∎

Let us construct π_2^m. Consider a sequence $f^{(2)} := (\mathrm{id}, f_2, 0, 0, \ldots)$, where f_2 is a weight $-e_1$ bilinear map on V. According to Item 2 of Proposition 11, $f^{(2)}$ defines a Lod_∞ isomorphism

$$L_1 \to (V_m \oplus V_c, \pi_1^{(2)}, \pi_2^{(2)}, \pi_3^{(2)}, \ldots) =: (V_m \oplus V_c, \pi^{(2)}),$$

and $\pi^{(2)}$ is a Lod_∞ structure on $V_m \oplus V_c$, if and only if $\pi^{(2)} \circ f^{(2)} = f^{(2)} \circ \pi$. In view of Equation (2.52), this condition implies that (take $p = 1$) $\pi_1^{(2)} = \pi_1 = 0 \oplus \pi_1$, that (take $p = 2$), for $v_1 \in V^{v_1}, v_2 \in V$,

$$\pi_2^{(2)}(v_1, v_2) = -\pi_1 f_2(v_1, v_2) + \pi_2(v_1, v_2) - f_2(\pi_1 v_1, v_2) - (-1)^{\langle e_1, v_1 \rangle} f_2(v_1, \pi_1 v_2), \tag{2.55}$$

and (take $p \geq 3$) it provides the $\pi_p^{(2)}$, $p \geq 3$, in terms of f_2.

It suffices to find a weight $-e_1$ bilinear map f_2, such that the resulting $\pi_2^{(2)}$ maps $V_m \times V_m$ to V_m and vanishes elsewhere. Indeed, the restriction π_2^m of $\pi_2^{(2)}$ to $V_m \times V_m$ is then a weight 0 bilinear map on V_m and $\pi_2^{(2)} = \pi_2^m \oplus 0$. If we choose the $\pi_p^{(2)}$, $p \geq 3$, given by f_2, intertwining condition $\pi^{(2)} \circ f^{(2)} = f^{(2)} \circ \pi$ is satisfied and $f^{(2)}$ is a Lod_∞ isomorphism between L_1 and $(V_m \oplus V_c, 0 \oplus \pi_1, \pi_2^m \oplus 0, \pi_3^{(2)}, \pi_4^{(2)}, \ldots)$. When continuing step by step, we finally get a Lod_∞ isomorphism $\ldots f^{(3)} \circ f^{(2)} \circ f^{(1)}$ between (V, π) and $(V_m \oplus V_c, 0 \oplus \pi_1, \pi_2^m \oplus 0, \pi_3^m \oplus 0, \ldots) =: L_2$. It eventually follows from the explicit form of the stem bracket, see Example 2 and subsequent explanation, that, since $(0 \oplus \pi_1, \pi_2^m \oplus 0, \pi_3^m \oplus 0, \ldots)$ verifies the Lod_∞ structure condition on $V_m \oplus V_c$, the two terms of this direct sum verify the same condition on V_m and V_c respectively.

Let us define f_2 as follows:

$$f_2(v_1,v_2) = \begin{cases} \delta\pi_2(v_1,v_2) + P\pi_2(w,v_2), \text{ if } (v_1,v_2) \in B^\alpha \times Z^\beta,\ v_1 = \pi_1(w), \\ \delta\pi_2(v_1,v_2) + \frac{1}{2}P\pi_2(w,v_2), \text{ if } (v_1,v_2) \in B^\alpha \times W^\beta,\ v_1 = \pi_1(w), \\ \delta\pi_2(v_1,v_2) + (-1)^{\langle e_1,\alpha\rangle}P\pi_2(v_1,w'), \text{ if } (v_1,v_2) \in Z^\alpha \times B^\beta, \\ \hfill v_2 = \pi_1(w'), \\ \delta\pi_2(v_1,v_2) + (-1)^{\langle e_1,\alpha\rangle}\frac{1}{2}P\pi_2(v_1,w'), \text{ if } (v_1,v_2) \in W^\alpha \times B^\beta, \\ \hfill v_2 = \pi_1(w'), \\ \delta\pi_2(v_1,v_2), \text{ otherwise.} \end{cases} \tag{2.56}$$

Map f_2 is well-defined, i.e. the two definitions on $B^\alpha \times B^\beta \subset (B^\alpha \times Z^\beta) \cap (Z^\alpha \times B^\beta)$ coincide, as the Lod_∞ structure condition (2.51) implies

$$0 = P\pi_1\pi_2(v_1,v_2) = P\pi_2(\pi_1 v_1, v_2) + (-1)^{\langle e_1,\alpha\rangle}P\pi_2(v_1,\pi_1 v_2),$$

for any $v_1 \in V^\alpha, v_2 \in V$. Indeed, when writing this upshot for $w \in W^{\alpha-e_1}$ and w', see Equation (2.56), we get the announced result.

It remains to show that $\pi_2^{(2)}$ sends $V_m \times V_m$ to V_m and vanishes elsewhere.

If $(v_1,v_2) \in Z \times Z$, Equation (2.55) and Lemma 3 yield $\pi_2^{(2)}(v_1,v_2) = \delta\pi_1\pi_2(v_1,v_2) + P\pi_2(v_1,v_2)$. But, since the Lod_∞ condition entails that $\pi_1\pi_2(v_1,v_2) = 0$, we get $\pi_2^{(2)}(v_1,v_2) = P\pi_2(v_1,v_2)$. Furthermore, for any $(w,v_2) \in W \times Z$, Condition (2.51) implies that $P\pi_2(\pi_1 w, v_2) = 0$, whereas for any $(v_1,w') \in Z \times W$, we obtain $P\pi_2(v_1,\pi_1 w') = 0$. Therefore,

$$\pi_2^{(2)}(v_1,v_2) = \begin{cases} P\pi_2(v_1,v_2) =: \pi_2^m(v_1,v_2) \in V_m, \text{ if}(v_1,v_2) \in V_m \times V_m, \\ 0, \text{ if } (v_1,v_2) \in B \times V_m \text{ or } (v_1,v_2) \in V_m \times B \text{ or } (v_1,v_2) \in B \times B. \end{cases}$$

If $(v_1,v_2) \in (W \times Z) \cup (Z \times W) \cup (W \times W)$, Equation (2.55), Lemma 3, and Condition (2.51) allow checking that $\pi_2^{(2)}(v_1,v_2) = 0$.

Hence,

$$\pi_2^{(2)}(v_1,v_2) = \begin{cases} \pi_2^m(v_1,v_2) = P\pi_2(v_1,v_2) \in V_m, \text{ if}(v_1,v_2) \in V_m \times V_m, \\ 0, \text{ otherwise.} \end{cases}$$

More generally, it can be shown that any Lod_∞ algebra L_{k-1}, $k \geq 2$, of the form

$$(V_m \oplus V_c, 0 \oplus \pi_1, \pi_2^m \oplus 0, \ldots, \pi_{k-1}^m \oplus 0, \pi_k^{(k-1)}, \pi_{k+1}^{(k-1)}, \ldots)$$

is Lod_∞ isomorphic to a Lod_∞ algebra

$$L_k := (V_m \oplus V_c, 0 \oplus \pi_1, \pi_2^m \oplus 0, \ldots, \pi_k^m \oplus 0, \pi_{k+1}^{(k)}, \ldots),$$

where $\pi_k^m = P\pi_k^{(k-1)}$ (of course $\pi_2^{(1)} := \pi_2$). Indeed, let f_k be the weight $(1-k)e_1$ k-linear map on V defined by:

1. For any $(k-1)$-tuple $(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_k)$ of $V^{\times(k-1)}$, $1\le i\le k$, which has j elements v_ℓ in W, $0\le j\le k-1$, and all other elements in Z, and for any $w\in W$, set

$$f_k(v_1,\ldots,v_{i-1},\pi_1 w,v_{i+1},\ldots,v_k)=\delta\pi_k^{(k-1)}(v_1,\ldots,v_{i-1},\pi_1 w,v_{i+1},\ldots,v_k)$$
$$+\frac{1}{j+1}(-1)^{\langle e_1,v_1+\ldots+v_{i-1}\rangle}P\pi_k^{(k-1)}(v_1,\ldots,v_{i-1},w,v_{i+1},\ldots,v_k).$$

2. Otherwise, set $f_k=\delta\pi_k^{(k-1)}$.

When combining Lemma 3 and the Lod_∞ condition as previously, we can check that the sequence $f^{(k)} := (\mathrm{id},0,\ldots,0,f_k,0,\ldots)$ is a Lod_∞ isomorphism from L_{k-1} to L_k.

Finally, we get, as announced, a Lod_∞ isomorphism from the Lod_∞ algebra (V,π) to the Lod_∞ algebra

$$(V_m\oplus V_c, 0\oplus\pi_1,\pi_2^m\oplus 0,\pi_3^m\oplus 0,\ldots).\ \blacksquare$$

We are now prepared to prove Theorem 8.

Proof. Let $f:(V,\pi)\to(V',\pi')$ be a Lod_∞ quasi-isomorphism. According to the minimal model theorem, there is a Lod_∞ isomorphism h (resp. h') that identifies the Lod_∞ algebra (V,π) (resp. (V',π')) to a direct sum $(V_m\oplus V_c,\pi^m\oplus\pi^c)$ (resp. $(V'_m\oplus V'_c,\pi'^m\oplus\pi'^c)$). Furthermore, since the inclusion $\mathfrak{i}:=(i,0,0,\ldots):(V_m,\pi^m)\to(V_m\oplus V_c,\pi^m\oplus\pi^c)$ (resp. the projection $\mathfrak{p}:=(P',0,0,\ldots):(V'_m\oplus V'_c,\pi'^m\oplus\pi'^c)\to(V'_m,\pi'^m)$) is a Lod_∞ quasi-isomorphism, the sequence $h^{\mathfrak{i}}:=h^{-1}\circ\mathfrak{i}$ (resp. $h'^{\mathfrak{p}}:=\mathfrak{p}\circ h'$) is a Lod_∞ quasi-isomorphism from (V_m,π^m) (resp. (V',π')) to (V,π) (resp. (V'_m,π'^m)). Therefore, the map $f^m:=h'^{\mathfrak{p}}\circ f\circ h^{\mathfrak{i}}$ is a Lod_∞ quasi-isomorphism between (V_m,π^m) and (V'_m,π'^m). But, as $H(V_m,\pi_1^m)=V_m$ and $H(V'_m,\pi_1'^m)=V'_m$, the map $(f_1^m)_\sharp=f_1^m:V_m\to V'_m$ is an isomorphism, and so f^m has a Lod_∞ isomorphism inverse $(f^m)^{-1}$, with $(f^m)_1^{-1}=(f_1^m)^{-1}$, see Proposition 11. Consequently, the sequence $g:=h^{\mathfrak{i}}\circ(f^m)^{-1}\circ h'^{\mathfrak{p}}$ is a Lod_∞ quasi-isomorphism from (V',π') to (V,π). Moreover, $g_{1\sharp}=(f_{1\sharp})^{-1}$. Indeed, observe first that $g_1=i\circ(f_1^m)^{-1}\circ P'$ and $f_1^m=P'\circ f_1\circ i$. For any $[v']\in H(V',\pi'_1)$, we thus get $g_{1\sharp}[v']=[(f_1^m)^{-1}P'v']\in H(V,\pi_1)$. On the other hand, $(f_{1\sharp})^{-1}[v']=:[v]\in H(V,\pi_1)$, hence $f_1v=v'+\pi'_1\mathfrak{v}'$, $\mathfrak{v}'\in V'$. It now suffices to show that there is $\mathfrak{v} \in V$, such that $(f_1^m)^{-1}P'v' = v + \pi_1\mathfrak{v}$, i.e.

$$P'v' = f_1^m(v + \pi_1\mathfrak{v}) = P'f_1 v + P'f_1\pi_1\mathfrak{v} = P'(v' + \pi_1'\mathfrak{v}') + P'\pi_1' f_1\mathfrak{v}.$$

This condition is satisfied for any $\mathfrak{v} \in V$, since $P'\pi_1' = 0$. ∎

In view of the preceding proof, we have the following

Corollary 1. *Each* Lod_∞ *algebra is* Lod_∞ *quasi-isomorphic to a minimal one.*

2.7 Graded and strongly homotopy algebra cohomologies

2.7.1 Graded Loday and Chevalley-Eilenberg cohomologies

Let $\pi \in \mathrm{Lod}(V)$ be a $\mathbb{Z}^n$-graded Loday structure on V. As π is canonical for the $\mathbb{Z}^{n+1}$–GLA $(M_r(V), [-,-]^{\otimes})$, see Theorem 7, it is clear that the cohomology of the induced DGLA, with differential $\partial_\pi = [\pi, -]^{\otimes}$, should roughly be the cohomology of the considered Loday algebra.

Proposition 12. *The graded Loday cohomology operator ∂_π of a Loday structure $\pi = \{-,-\}$ on a vector space V, reads, for any $B \in M^{(B,b)}(V)$, $b \geq -1$, and any homogeneous $v_1, \ldots, v_{b+2} \in V$,*

$$\begin{aligned}
&(\partial_\pi B)(v_1, \ldots, v_{b+2}) = (-1)^{b+1}\{B(v_1, \ldots, v_{b+1}), v_{b+2}\} - \\
&\sum_{i=1}^{b+1}(-1)^{i-1}(-1)^{\langle B+v_1+\ldots+v_{i-1}, v_i\rangle}\{v_i, B(v_1, \ldots, \widehat{v_i}, \ldots, v_{b+1}, v_{b+2})\} \\
&+\sum_{i=1}^{b+1}\sum_{j=1}^{i}(-1)^{j+1}(-1)^{\langle v_j, v_{j+1}+\ldots+v_i\rangle} \\
&\qquad B(v_1, \ldots, \widehat{v_j}, \ldots, v_i, \{v_j, v_{i+1}\}, v_{i+2}, \ldots, v_{b+2}).
\end{aligned} \tag{2.57}$$

Proof. For $b \geq 0$, the explicit form of ∂_π is a consequence of Theorem 6. Equation (2.57) suggests extending ∂_π to $M^{-1}(V) = V$ by $(\partial_\pi v)(w) := \pi(v, w) = \{v, w\}$, for any $v, w \in V$. The extended operator ∂_π is a cohomology operator on $M(V)$, since

$$\begin{aligned}
\partial_\pi(\partial_\pi v)(v_1, v_2) \quad = \quad & -\pi(\pi(v, v_1), v_2) \\
& -(-1)^{\langle v, v_1\rangle}\pi(v_1, \pi(v, v_2)) + \pi(v, \pi(v_1, v_2)) = 0,
\end{aligned}$$

in view of the Jacobi identity. ∎

Definition 12. *The* graded Loday cohomology *of a $\mathbb{Z}^n$-graded Loday algebra (V,π) is the cohomology of the complex $(M(V),\partial_\pi)$, where ∂_π is given by Equation (2.57).*

Remark 7. *In the non-graded case, Operator (2.57) coincides with the (non-graded) Loday coboundary operator, see [DT97], and in the antisymmetric situation, it is (the opposite of) the graded Chevalley-Eilenberg differential, see [LMS91].*

2.7.2 Graded Poisson and Jacobi cohomologies

In Sections 2.4, we showed that the bracket $[-,-]^{\otimes}$ is a graded Lie bracket on the (reduced) cochain space of graded Loday and graded Lie structures, which are canonical for $[-,-]^{\otimes}$, so that $[-,-]^{\otimes}$ defines the graded Loday and Lie cohomologies.

In the following we prove that $[-,-]^{\otimes}$ not only restricts to the Nijenhuis-Richardson bracket, but also to the Grabowski-Marmo bracket, see [GM03], and in particular to the Schouten-Jacobi and Schouten brackets. More precisely, we provide evidence that the *stem bracket* $[-,-]^{\otimes}$ is a graded Lie bracket on the cochain spaces of graded Jacobi and graded Poisson structures, that, further, these algebraic structures are canonical with respect to $[-,-]^{\otimes}$, and that the stem bracket leads to the appropriate cohomological concepts for graded Jacobi and Poisson algebras.

The Grabowski-Marmo bracket

Let us first remind that for any commutative unital ring R and any associative commutative R-algebra $\mathscr{A}$ with unit 1, the $\mathscr{A}$-bimodule and associative algebra $\mathrm{Diff}(\mathscr{A}) = \bigoplus_{k\in\mathbb{N}} \mathrm{Diff}_k(\mathscr{A})$ of all differential operators on $\mathscr{A}$, filtered by the "order of differentiation", can be defined algebraically, "à la Vinogradov", see e.g. [Kra99]. Moreover, the splitting

$$\mathrm{Diff}_1(\mathscr{A}) = \mathscr{A} \oplus \mathrm{Der}(\mathscr{A}),$$

given by $D = D(1) + (D - D(1))$, where $\mathrm{Der}(\mathscr{A})$ denotes the $\mathscr{A}$-module and Lie R-algebra of derivations of $\mathscr{A}$, holds true. It is then clear that, for any *first order differential operator* $D \in \mathrm{Diff}_1(\mathscr{A})$ and any "functions" $v,w \in \mathscr{A}$, we have

$$D(vw) = D(v)w + vD(w) - D(1)vw.$$

We now recall the definitions of graded Jacobi and Poisson algebras.

Definition 13. *A* graded Jacobi algebra *of weight $\alpha \in \mathbb{Z}^n$ is a pair $(\mathscr{A}, \{-,-\})$, where $\mathscr{A}$ denotes a $\mathbb{Z}^n$-graded commutative associative algebra with unit* 1*, and where $\{-,-\}$ is a bilinear bracket*

$$\{-,-\} : \mathscr{A} \times \mathscr{A} \to \mathscr{A},$$

which

1. *has* weight α*, i.e.* $\{u,v\} \in \mathscr{A}^{u+v+\alpha}$,
2. *is* graded antisymmetric*, i.e.*

$$\{u,v\} = -(-1)^{\langle u+\alpha, v+\alpha\rangle}\{v,u\}, \tag{2.58}$$

3. *satisfies the* graded Jacobi identity*, i.e.*

$$\{u,\{v,w\}\} = \{\{u,v\},w\} + (-1)^{\langle u+\alpha, v+\alpha\rangle}\{v,\{u,w\}\}, \tag{2.59}$$

   and

4. *verifies the* generalized graded Leibniz rule*, i.e.*

$$\{u,vw\} = \{u,v\}w + (-1)^{\langle u+\alpha, v\rangle} v\{u,w\} - \{u,1\}vw, \tag{2.60}$$

for any $u \in \mathscr{A}^u$, $v \in \mathscr{A}^v$, and $w \in \mathscr{A}^w$.

Similarly, a graded Poisson algebra *of weight $\alpha \in \mathbb{Z}^n$ is a pair $(\mathscr{A}, \{-,-\})$ that verifies the same conditions 1-4, except that the associative algebra $\mathscr{A}$ needs not have a unit and that the last term of Equation (2.60) is omitted.*

Observe that whereas a potential weight $\alpha \in \mathbb{Z}^n$ of a graded Loday or Lie bracket on a $\mathbb{Z}^n$-graded vector space V disappears via an α-shift in the grading of V, weight α reappears for a graded Jacobi or Poisson algebra on a $\mathbb{Z}^n$-graded associative algebra $\mathscr{A}$, after such a shift, in the Leibniz rule,

$$\{u,vw\} = \{u,v\}w + (-1)^{\langle u, v-\alpha\rangle} v\{u,w\}(-\{u,1\}vw).$$

Therefore, the two just mentioned versions of graded Jacobi (resp. Poisson) algebras can be found in literature. Further, we cannot confine ourselves to graded Jacobi (resp. Poisson) structures of weight $\alpha = 0$.

Obviously a graded Jacobi (resp. Poisson) structure $\pi = \{-,-\}$ of weight $\alpha \in \mathbb{Z}^n$ is an α-antisymmetric graded first order bidifferential operator (resp.

graded biderivation) on $\mathscr{A}$ (by α-antisymmetric, we mean that the exchange of v and w does not generate the sign $-(-1)^{\langle v,w\rangle}$, but $-(-1)^{\langle v+\alpha,w+\alpha\rangle}$; we emphasize this difference, which will be of importance below). It follows of course that the appropriate graded Jacobi (resp. graded Poisson) cochains are α-antisymmetric graded first order polydifferential operators (resp. graded polyderivations) of $\mathscr{A}$.

In [GM03], the authors investigated these operators using a variant of Krasil'shchik's calculus, see [Kra91], which is based upon a particular bidegree of the spaces $M(\mathscr{A})$, $A(\mathscr{A})$, or $\mathrm{Diff}_1(\mathscr{A})$ (resp. $\mathrm{Der}(\mathscr{A})$). For instance, we denote by $\mathfrak{M}(\mathscr{A})$ the usual vector space $M(\mathscr{A})$ with the $\mathbb{Z}^{n+1}$-gradation

$$\mathfrak{M}(\mathscr{A}) = \bigoplus_{(A,a)\in\mathbb{Z}^n\times\mathbb{Z}} \mathfrak{M}^{(A+\alpha a,a)}(\mathscr{A}),$$

where $\mathfrak{M}^{(A+\alpha a,a)}(\mathscr{A})$ vanishes for $a<-1$, coincides with $\mathscr{A}^A$ for $a=-1$, see below, and is, for $a\geq 0$, the space of multilinear maps $A:\mathscr{A}^{\times(a+1)}\to\mathscr{A}$ that have weight A, i.e. verify $A(v_1,\ldots,v_{a+1})\in\mathscr{A}^{v_1+\ldots+v_{a+1}+A}$. We denote the corresponding subspaces of α-antisymmetric multilinear mappings by $\mathfrak{A}^{(A+\alpha a,a)}(\mathscr{A})$. The subspaces $\mathfrak{Diff}_1^{(*,a)}(\mathscr{A})$ (resp. $\mathfrak{Der}^{(*,a)}(\mathscr{A})$) of α-antisymmetric graded first order $(a+1)$-differential operators (resp. graded $(a+1)$-derivations) on $\mathscr{A}$, their $\mathscr{A}$-bimodule structures, as well as the associative graded commutative dot-product on these subspaces will be defined inductively. These definitions are obvious, if we keep in mind that we intend of course to construct on the graded Jacobi cochain vector space

$$\mathfrak{Diff}_1(\mathscr{A}) = \bigoplus_{(A,a)\in\mathbb{Z}^n\times\mathbb{Z}} \mathfrak{Diff}_1^{(A+\alpha a,a)}(\mathscr{A}),$$

in addition to the already mentioned dot-product, a graded Lie or even a graded Jacobi structure $[-,-]^{\mathrm{GM}}$, just along the same lines as one defines the Schouten bracket $[-,-]^{\mathrm{SN}}$ on the space of multivector fields $\Gamma(\bigwedge TM)$ of a smooth manifold M, endowed with the wedge product. Thus the bracket $[-,-]^{\mathrm{GM}}$ will for instance verify the conditions $[v,w]^{\mathrm{GM}}=0$ and $[A,v]^{\mathrm{GM}}=A(v)$, with self-explaining notations. Furthermore, the dot-product (resp. the Grabowski-Marmo bracket $[-,-]^{\mathrm{GM}}$) will be of weight $(\alpha,1)\in\mathbb{Z}^n\times\mathbb{Z}$ (resp. $(0,0)\in\mathbb{Z}^n\times\mathbb{Z}$). In order to define $\mathfrak{Diff}_1^{(*,a)}(\mathscr{A})$ (resp. $\mathfrak{Der}^{(*,a)}(\mathscr{A})$), the module structures, the dot-product, and the graded Jacobi (resp. graded Poisson) bracket, it now suffices to proceed by induction and to just write down the properties that necessarily hold true, if all these structures do exist. Afterwards, it then remains to show that the chosen definitions actually fit.

More precisely, fix $\alpha \in \mathbb{Z}^n$ and let $\mathscr{A}$ be an associative $\mathbb{Z}^n$-graded commutative unital algebra with unit 1. Set $\mathfrak{Diff}_1^{(*,a)}(\mathscr{A}) = 0$, for $a < -1$, and $\mathfrak{Diff}_1^{(A-\alpha,-1)}(\mathscr{A}) = \mathscr{A}^A$. Indeed, if there is a dot-product of weight $(\alpha, 1)$ on $\mathfrak{Diff}_1(\mathscr{A})$ that extends the associative product on $\mathscr{A}$, we have

$$\begin{aligned} \mathscr{A}^{v+w} \ni vw = v \cdot w \quad &\in \mathfrak{Diff}_1^{(v-\alpha,-1)}(\mathscr{A}) \cdot \mathfrak{Diff}_1^{(w-\alpha,-1)}(\mathscr{A}) \\ &\subset \mathfrak{Diff}_1^{(v+w-\alpha,-1)}(\mathscr{A}). \end{aligned}$$

The spaces $\mathfrak{Diff}_1^{(*,a)}(\mathscr{A})$, $a \geq 0$, are then defined inductively as the spaces that are made up by those linear maps

$$A : \mathscr{A}^v \ni v \to A(v) \in \mathfrak{Diff}_1^{(A+v+\alpha(a-1),a-1)}(\mathscr{A})$$

that verify

$$\begin{aligned} A(vw) &= A(v) \cdot w + (-1)^{\langle A+\alpha a, v-\alpha+\alpha\rangle + a(-1+1)} v \cdot A(w) - A(1) \cdot vw \\ &= A(v) \cdot w + (-1)^{\langle A+\alpha a, v\rangle} v \cdot A(w) - A(1) \cdot vw \end{aligned}$$

and are α-antisymmetric if viewed as bilinear maps. In this definition $A(v_1)(v_2, \ldots, v_{a+1}) := A(v_1, v_2, \ldots, v_{a+1})$, the left $\mathscr{A}$-module structure is defined inductively by $(v \cdot A)(w) = v \cdot A(w)$ and extends the associative multiplication in $\mathscr{A}$, and the right $\mathscr{A}$-module structure—which will be extended by the dot-product the weight of which will be $(\alpha, 1)$—is of course given by

$$A \cdot v = (-1)^{\langle A+\alpha(a+1), v\rangle} v \cdot A.$$

Observe that the weight of the dot-product also explains the sign in the preceding first order differential operator condition. For any $a \in \mathbb{Z}$, the subspace $\mathfrak{Der}^{(*,a)}(\mathscr{A}) \subset \mathfrak{Diff}_1^{(*,a)}(\mathscr{A})$ of graded $(a+1)$-derivations on $\mathscr{A}$ is defined as the space of those graded first order $(a+1)$-differential operators A that verify $A(1) = 0$.

We eventually extend as aforementioned the associative product on $\mathscr{A}$ and the $\mathscr{A}$-module structures on $\mathfrak{Diff}_1(\mathscr{A})$ to a weight $(\alpha, 1)$ dot-product on $\mathfrak{Diff}_1(\mathscr{A})$. In order to find, for $A \in \mathfrak{Diff}_1^{(A+\alpha a,a)}(\mathscr{A})$ and $B \in \mathfrak{Diff}_1^{(B+\alpha b,b)}(\mathscr{A})$, the inductive definition of this dot-product, we provide between parentheses the upshots that are necessarily valid if a bracket $[-,-]^{\mathrm{GM}}$ with the required properties does exist:

$$\begin{aligned} (A \cdot B)(v) \quad (\quad &= \quad [A \cdot B, v]^{\mathrm{GM}} \\ &= \quad A \cdot [B, v]^{\mathrm{GM}} + (-1)^{\langle B+\alpha(b+1), v-\alpha\rangle + b+1} [A, v]^{\mathrm{GM}} \cdot B \quad) \\ &:= \quad A \cdot B(v) + (-1)^{\langle B+\alpha(b+1), v-\alpha\rangle + b+1} A(v) \cdot B. \end{aligned}$$

Proposition 13. *With respect to the abovedetailed bidegree, the pair* $(\mathfrak{Diff}_1(\mathscr{A}),\cdot)$ *is an associative* $\mathbb{Z}^{n+1}$*-graded commutative unital algebra,* $\cdot$ *has weight* $(\alpha,1)$*, and the pair* $(\mathfrak{Der}(\mathscr{A}),\cdot)$ *is a* $\mathbb{Z}^{n+1}$*-graded subalgebra.*

For the proof of this and the next propositions, we refer the reader to [GM03] and [Kra91].

Let us now come to the construction of the Grabowski-Marmo graded Jacobi (resp. Poisson) bracket $[-,-]^{\mathrm{GM}}$ on the graded Jacobi (resp. Poisson) cochain algebra $(\mathfrak{Diff}_1(\mathscr{A}),\cdot)$ (resp. $(\mathfrak{Der}(\mathscr{A}),\cdot)$). If such a bracket actually exists, we necessarily have

$$[v,w]^{\mathrm{GM}}=0,\ [A,v]^{\mathrm{GM}}=A(v),\ [v,A]^{\mathrm{GM}}=-(-1)^{\langle v-\alpha,A+\alpha a\rangle+a}[A,v]^{\mathrm{GM}},\quad (2.61)$$

and

$$\begin{array}{rcl} [A,B]^{\mathrm{GM}}(v) & = & [[A,B]^{\mathrm{GM}},v]^{\mathrm{GM}} \\ & = & [A,[B,v]^{\mathrm{GM}}]^{\mathrm{GM}}+(-1)^{\langle B+\alpha b,v-\alpha\rangle+b}[[A,v]^{\mathrm{GM}},B]^{\mathrm{GM}} \\ & = & [A,B(v)]^{\mathrm{GM}}+(-1)^{\langle B+\alpha b,v-\alpha\rangle+b}[A(v),B]^{\mathrm{GM}}. \end{array}\quad (2.62)$$

The bracket defined inductively by Equations (2.61) and (2.62) actually fits, see [GM03] and [Kra91]:

Theorem 10. *There is a unique* $\mathbb{Z}^{n+1}$*-graded Jacobi bracket* $[-,-]^{\mathrm{GM}}$ *of degree* $(0,0)\in\mathbb{Z}^n\times\mathbb{Z}$ *on the associative* $\mathbb{Z}^{n+1}$*-graded commutative algebra* $(\mathfrak{Diff}_1(\mathscr{A}),\cdot)$ *that verifies* $[A,v]^{\mathrm{GM}}=A(v)$*. Moreover,* $(\mathfrak{Der}(\mathscr{A}),[-,-]^{\mathrm{GM}},\cdot)$ *is a* $\mathbb{Z}^{n+1}$*-graded Poisson algebra.*

Furthermore, see again [GM03] and [Kra91],

Proposition 14. *The graded Jacobi (resp. Poisson) structures of degree* $\alpha\in\mathbb{Z}^n$ *on the associative* $\mathbb{Z}^n$*-graded commutative unital algebra* $\mathscr{A}$*, are exactly the canonical elements of degree* $(2\alpha,1)$ *of the graded Jacobi (resp. Poisson) algebra* $(\mathfrak{Diff}_1(\mathscr{A}),[-,-]^{\mathrm{GM}},\cdot)$ *(resp.* $(\mathfrak{Der}(\mathscr{A}),[-,-]^{\mathrm{GM}},\cdot)$*).*

Note that we constructed the algebras $\mathfrak{Diff}_1(\mathscr{A})$ and $\mathfrak{Der}(\mathscr{A})$ for α fixed in $\mathbb{Z}^n$, and did not write $\mathfrak{Diff}_{1;\alpha}(\mathscr{A})$ and $\mathfrak{Der}_\alpha(\mathscr{A})$ in order to avoid overcrowded notations. Of course, the graded algebras of Proposition 14 are exactly the just mentioned algebras.

This result immediately leads to the proper concepts of graded Jacobi and graded Poisson cohomologies.

Definition 14. *If π denotes a graded Jacobi (resp. Poisson) structure of weight $\alpha \in \mathbb{Z}^n$ on (the usual) algebra $\mathscr{A}$, then the* graded Jacobi (resp. Poisson) cohomology *of $(\mathscr{A},\pi)$ is the cohomology $H_\pi^{\mathrm{Jac}}(\mathscr{A})$ (resp. $H_\pi^{\mathrm{Poi}}(\mathscr{A})$) of the DGLA*

$$(\mathfrak{D}\mathrm{iff}_1(\mathscr{A}), \partial_\pi = [\pi,-]^{\mathrm{GM}}, [-,-]^{\mathrm{GM}})$$

(*resp.*

$$(\mathfrak{D}\mathrm{er}(\mathscr{A}), \partial_\pi = [\pi,-]^{\mathrm{GM}}, [-,-]^{\mathrm{GM}})).$$

Let us also recall, see [GM01] and [GM03], that in the geometric (ungraded) situation, i.e. in the case $\mathscr{A} = C^\infty(M)$, where $C^\infty(M)$ is the associative commutative unital algebra of smooth functions of a smooth manifold M, we have

$$(\mathrm{Der}(\mathscr{A}), [-,-]^{\mathrm{GM}}, \cdot) = (\Gamma(\bigwedge TM), [-,-]^{\mathrm{SN}}, \tilde{\wedge})),$$

where $[-,-]^{\mathrm{SN}}$ denotes (up to sign) the Schouten-Nijenhuis bracket, and where $\tilde{\wedge}$ is the reversed wedge product. Therefore, the aforedefined graded Poisson cohomology coincides in the geometric case with the usual Poisson cohomology, see [Vai94]. Moreover, it has been shown in [GM03] that the preceding graded Jacobi cohomology coincides for $\mathscr{A} = C^\infty(M)$ with the standard Lichnerowicz-Jacobi cohomology, see [Lic78], i.e. with the cohomology of the subcomplex of 1-differentiable cochains of the Chevalley-Eilenberg complex of the Jacobi-Lie bracket on $C^\infty(M)$ endowed with the natural representation by derivations of $C^\infty(M)$.

Link with the Stem bracket

Equations (2.61) and (2.62) pertaining to uniqueness of $[-,-]^{\mathrm{GM}}$ allow to explicitly compute the Grabowski-Marmo bracket and to provide evidence that this bracket is—in the general graded nongeometric situation—the restriction of the previously depicted stem bracket, see Equations (2.41) and (2.42).

Proposition 15. *For any fixed $\alpha \in \mathbb{Z}^n$, any $A \in \mathfrak{D}\mathrm{iff}_1^{(A+\alpha a,a)}(\mathscr{A})$ and any $B \in \mathfrak{D}\mathrm{iff}_1^{(B+\alpha b,b)}(\mathscr{A})$, the Grabowski-Marmo bracket $[A,B]^{\mathrm{GM}}$ of A by B is given by*

$$[A,B]^{\mathrm{GM}} = A\square B - (-1)^{\langle A+\alpha a, B+\alpha b\rangle + ab} B\square A, \tag{2.63}$$

where $A\square B$ is defined inductively by

$$v\square w := 0, \quad A\square v := A(v),\ a \geq 0, \quad v\square A := 0,\ a \geq 0,$$

$$(A\Box B)(v) := A\Box B(v) + (-1)^{\langle B+\alpha b, v-\alpha\rangle + b} A(v)\Box B,\ a,b \geq 0,$$

for any $v \in \mathscr{A}^v$, $w \in \mathscr{A}^w$. Further, the explicit form of the square-product $\Box$ is

$$\begin{aligned}(A\Box B)\ & (v_1,\ldots,v_{a+b+1}) \\ & = \sum_{\substack{I\cup J = N^{(a+b+1)} \\ |I| = b+1, |J| = a}} (-1)^{(I;J)} \varepsilon_{\downarrow V}(I,J) A(B(V_I),V_J),\end{aligned} \tag{2.64}$$

where $v_i \in \mathscr{A}^{v_i}$, $\downarrow V = \downarrow v_1 \otimes \ldots \otimes \downarrow v_{a+b+1}$, and $\downarrow v_i \in \mathscr{A}^{v_i - \alpha}$.

Proof. The proof of Equation (2.63) is by induction on $a+b$. If $a+b \leq -1$, the claim is obviously true, see Equation (2.61). Assume now that it holds true for $a+b \leq k-1$, $k \geq 0$, and show that it is still valid for $a+b = k$. We have, see Equation (2.62),

$$\begin{aligned}[A,B]^{\mathrm{GM}}(v) &= [A,B(v)]^{\mathrm{GM}} + (-1)^{\langle B+\alpha b, v-\alpha\rangle + b}[A(v),B]^{\mathrm{GM}} \\ &= A\Box B(v) - (-1)^{\langle A+\alpha a, B+v+\alpha(b-1)\rangle + a(b-1)} B(v)\Box A \\ &\quad +(-1)^{\langle B+\alpha b, v-\alpha\rangle + b} A(v)\Box B \\ &\quad -(-1)^{\langle B+\alpha b, v-\alpha\rangle + b + \langle A+v+\alpha(a-1), B+\alpha b\rangle + (a-1)b} B\Box A(v) \\ &= (A\Box B)(v) - (-1)^{\langle A+\alpha a, B+\alpha b\rangle + ab}(B\Box A)(v).\end{aligned}$$

In order to determine the explicit form (2.64) of $\Box$, we proceed again by induction. The announced upshot is clear for $a = -1$ or $b = -1$, hence, in particular for $a+b \leq -1$. Let us now assume that it is valid for $a+b \leq k-1$, $k \geq 0$, and examine the case $a+b = k$. As already pointed out, if $a = -1$ or $b = -1$, our conjecture is verified. Otherwise, $a,b \geq 0$, and, if we set $V'' = v_2 \otimes \ldots \otimes v_{a+b+1}$, we obtain

$$\begin{aligned}
& (A\Box B)(v_1,v_2,\ldots,v_{a+b+1}) \\
= \;& (A\Box B(v_1))(v_2,\ldots,v_{a+b+1}) \\
& +(-1)^{\langle B+\alpha b, v_1-\alpha\rangle + b}(A(v_1)\Box B)(v_2,\ldots,v_{a+b+1}) \\
= \;& \sum_{\substack{I\cup J=\{2,\ldots,a+b+1\} \\ |I|=b,|J|=a}} (-1)^{(I;J)}\varepsilon_{\downarrow V''}(I,J)A(B(v_1,V''_I),V''_J) \\
& +(-1)^{\langle B+\alpha b, v_1-\alpha\rangle + b} \\
& \sum_{\substack{I\cup J=\{2,\ldots,a+b+1\} \\ |I|=b+1,|J|=a-1}} (-1)^{(I;J)}\varepsilon_{\downarrow V''}(I,J)A(v_1,B(V''_I),V''_J) \\
= \;& \sum\ldots \\
& +(-1)^{\langle B+\alpha b, v_1-\alpha\rangle + b+1+\langle B+V''_I-\alpha, v_1-\alpha\rangle} \\
& \sum_{\substack{I\cup J=\{2,\ldots,a+b+1\} \\ |I|=b+1,|J|=a-1}} (-1)^{(I;J)}\varepsilon_{\downarrow V''}(I,J)A(B(V''_I),v_1,V''_J) \\
= \;& \sum\ldots \\
& +(-1)^{b+1+\langle V''_I-\alpha(b+1), v_1-\alpha\rangle} \\
& \sum_{\substack{I\cup J=\{2,\ldots,a+b+1\} \\ |I|=b+1,|J|=a-1}} (-1)^{(I;J)}\varepsilon_{\downarrow V''}(I,J)A(B(V''_I),v_1,V''_J).
\end{aligned}$$

Remark now that in the final result for which we look, $\mathscr{I}$ and $\mathscr{J}$ are two unshuffles that form a partition of $N^{(a+b+1)}$, so that 1 is either the first element of $\mathscr{I}$ or of $\mathscr{J}$. The first term $\sum\ldots$ (resp. second term $(-1)^{\cdots}\sum\ldots$) of the RHS of the last equation, corresponds exactly to the first (resp. second) possibility. Indeed,

$$(-1)^{(I;J)}\varepsilon_{\downarrow V''}(I,J) = (-1)^{(\mathscr{I};\mathscr{J})}\varepsilon_{\downarrow V}(\mathscr{I},\mathscr{J})$$

(resp.

$$(-1)^{b+1+\langle V''_I-\alpha(b+1), v_1-\alpha\rangle}(-1)^{(I;J)}\varepsilon_{\downarrow V''}(I,J) = (-1)^{(\mathscr{I};\mathscr{J})}\varepsilon_{\downarrow V}(\mathscr{I},\mathscr{J})),$$

which completes the proof. ∎

We now explain in which sense the Grabowski-Marmo bracket coincides with the stem bracket, which thus governs not only the graded Loday and Lie cohomologies, but also the graded Poisson and Jacobi cohomologies.

Remark 8. *The bracket* $[-,-]^{\mathscr{GM}}$ *on* $\mathfrak{Diff}_1(\mathscr{A})$*, which is defined for* $A \in \mathfrak{Diff}_1^{(A+\alpha a,a)}(\mathscr{A})$ *and* $B \in \mathfrak{Diff}_1^{(B+\alpha b,b)}(\mathscr{A})$ *by*

$$[A,B]^{\mathscr{GM}} = -(-1)^{ab}[A,B]^{\mathrm{GM}}, \tag{2.65}$$

is clearly a $\mathbb{Z}^{n+1}$-graded Lie bracket on the graded Jacobi and Poisson cochain spaces, which admits these graded algebraic structures as canonical elements. Therefore, the bracket $[-,-]^{\mathscr{G}\mathscr{M}}$ may also be used to define and compute the graded Jacobi and Poisson cohomologies.

In the following, we denote by $\downarrow\mathscr{A}$ the vector space $\mathscr{A}$ endowed with the gradation $(\downarrow\mathscr{A})^{\gamma} = \mathscr{A}^{\gamma+\alpha}$. Set now

$$\mathfrak{A}(\mathscr{A}) = \bigoplus_{(A,a)\in\mathbb{Z}^n\times\mathbb{Z}} \mathfrak{A}^{(A+\alpha a,a)}(\mathscr{A}) \quad \left(\text{resp. } A(\downarrow\mathscr{A}) = \bigoplus_{(A,a)\in\mathbb{Z}^n\times\mathbb{Z}} A^{(A+\alpha a,a)}(\downarrow\mathscr{A})\right), \tag{2.66}$$

where $\mathfrak{A}^{(A+\alpha a,a)}(\mathscr{A})$ (resp. $A^{(A+\alpha a,a)}(\downarrow\mathscr{A})$) denotes as usually the space of multilinear maps $A : \mathscr{A}^{\times(a+1)} \to \mathscr{A}$ (resp. $\tilde{A} : (\downarrow\mathscr{A})^{\times(a+1)} \to \downarrow\mathscr{A}$) that have weight A (resp. $A+\alpha a$), i.e.

$$A : \mathscr{A}^{\beta_1} \times \ldots \times \mathscr{A}^{\beta_{a+1}} \to \mathscr{A}^{A+\beta_1+\ldots+\beta_{a+1}}$$

(resp.

$$\tilde{A} : (\downarrow\mathscr{A})^{\beta_1} \times \ldots \times (\downarrow\mathscr{A})^{\beta_{a+1}} \to (\downarrow\mathscr{A})^{A+\alpha a+\beta_1+\ldots+\beta_{a+1}}),$$

and that are further α-antisymmetric (resp. antisymmetric in the usual sense), i.e.

$$A(\ldots v, w \ldots) = -(-1)^{\langle v+\alpha, w+\alpha\rangle} A(\ldots w, v \ldots)$$

(resp.

$$\tilde{A}(\ldots \tilde{v}, \tilde{w} \ldots) = -(-1)^{\langle \tilde{v}, \tilde{w}\rangle} A(\ldots \tilde{w}, \tilde{v} \ldots)).$$

Obviously, the map $\sim: A \to \tilde{A}$, with $\tilde{A}$ defined from A by

$$\tilde{A}(\tilde{v}_1, \ldots, \tilde{v}_{a+1}) = \downarrow A(\uparrow\tilde{v}_1, \ldots, \uparrow\tilde{v}_{a+1}),$$

is a graded vector space isomorphism, the inverse of which is $\backsim: \tilde{A} \to A$,

$$A(v_1, \ldots, v_{a+1}) = \uparrow\tilde{A}(\downarrow v_1, \ldots, \downarrow v_{a+1}).$$

The graded vector space isomorphism $\sim$ pulls of course the Nijenhuis-Richardson graded Lie bracket $[-,-]^{\mathrm{NR}}$ on the usual space $A(\downarrow\mathscr{A})$, associated with the $\mathbb{Z}^n$-graded vector space $\downarrow\mathscr{A}$, back to a graded Lie bracket

$$[-,-]^{\mathscr{N}\mathscr{R}} = \backsim[\sim -, \sim -]^{\mathrm{NR}} =: \sim^*[-,-]^{\mathrm{NR}} = \sim^*[-,-]^{\otimes}|_{A(\downarrow\mathscr{A})}$$

on $\mathfrak{A}(\mathscr{A})$.

Theorem 11. *If π denotes a graded Jacobi (resp. Poisson) structure of weight $\alpha \in \mathbb{Z}^n$ on (the usual) algebra $\mathscr{A}$, the graded Jacobi (resp. Poisson) cohomology of $(\mathscr{A}, \pi)$ is the cohomology of the DGLA*

$$(\mathfrak{Diff}_1(\mathscr{A}), [-,-]^{\mathscr{G}\mathscr{M}}, \partial_\pi = [\pi, -]^{\mathscr{G}\mathscr{M}})$$

(resp.

$$(\mathfrak{Der}(\mathscr{A}), [-,-]^{\mathscr{G}\mathscr{M}}, \partial_\pi = [\pi, -]^{\mathscr{G}\mathscr{M}})),$$

where $[-,-]^{\mathscr{G}\mathscr{M}}$ is the restriction of $\sim^* [-,-]^{\otimes}|_{A(\downarrow\mathscr{A})}$ to $\mathfrak{Diff}_1(\mathscr{A})$ (resp. $\mathfrak{Der}(\mathscr{A})$), i.e. of the Nijenhuis-Richardson bracket or the (restriction of the) stem bracket read through the canonical isomorphism $\sim$.

Proof. We need only check that on $\mathfrak{Diff}_1(\mathscr{A})$ the bracket $[-,-]^{\mathscr{G}\mathscr{M}}$ is the pullback $[-,-]^{\mathscr{N}\mathscr{R}}$ by $\sim$ of the Nijenhuis-Richardson bracket $[-,-]^{\mathrm{NR}}$ on $A(\downarrow \mathscr{A})$. For $A \in \mathfrak{Diff}_1^{(A+\alpha a, a)}(\mathscr{A})$ and $B \in \mathfrak{Diff}_1^{(B+\alpha b, b)}(\mathscr{A})$, this pullback reads, see Equations (2.17) and (2.66),

$$[A,B]^{\mathscr{N}\mathscr{R}} = \backsim i_{\tilde{A}}\tilde{B} - (-1)^{\langle A+\alpha a, B+\alpha b\rangle + ab} \backsim i_{\tilde{B}}\tilde{A},$$

where

$$\begin{aligned}
&(\backsim i_{\tilde{B}}\tilde{A})(V_{N^{(a+b+1)}}) \\
&= (-1)^{\langle A+\alpha a, B+\alpha b\rangle} \sum_{\substack{I \cup J = N^{(a+b+1)} \\ |I| = b+1, |J| = a}} (-1)^{(I;J)} \varepsilon_{\downarrow V}(I,J) \uparrow \tilde{A}(\downarrow\uparrow \tilde{B}(\downarrow V_I), \downarrow V_J) \\
&= (-1)^{\langle A+\alpha a, B+\alpha b\rangle} \sum_{\substack{I \cup J = N^{(a+b+1)} \\ |I| = b+1, |J| = a}} (-1)^{(I;J)} \varepsilon_{\downarrow V}(I,J) A(B(V_I), V_J) \\
&= (-1)^{\langle A+\alpha a, B+\alpha b\rangle} (A \square B)(V_{N^{(a+b+1)}}).
\end{aligned}$$

It follows that

$$[A,B]^{\mathscr{N}\mathscr{R}} = (-1)^{\langle A+\alpha a, B+\alpha b\rangle} B \square A - (-1)^{ab} A \square B = [A,B]^{\mathscr{G}\mathscr{M}},$$

see Equations (2.65) and (2.63). ∎

Finally:

Corollary 2. *The stem bracket $[-,-]^{\otimes}$ is, up to reading through a canonical isomorphism, a graded Lie bracket on the spaces of graded Loday, graded Lie, graded Poisson, and graded Jacobi cochains, for which the corresponding algebraic structures are canonical elements, and that thus encodes the graded cohomologies of all these structures.*

2.7.3 Strongly homotopy and graded p-ary Loday cohomologies

Since Lod_∞^Q structures on a $\mathbb{Z}^n$-graded vector space V, $Q \in \mathbb{Z}^n$, $\langle Q,Q\rangle$ odd, are the degree Q canonical elements π of the $\mathbb{Z}^n$-graded Lie algebra $(C(V),[-,-]^{\bar{\otimes}})$, we have the natural

Definition 15. *The* cohomology of a Loday infinity algebra (V,π) *is the cohomology of the DGLA* $(C(V),[-,-]^{\bar{\otimes}},[\pi,-]^{\bar{\otimes}})$*, where the coboundary operator is, for any* $\rho \in C^\rho(V)$*, explicitly given by*

$$[\pi,\rho]^{\bar{\otimes}} = \sum_{q\geq 1}\sum_{s+t=q+1}(-1)^{1+(s-1)\langle e_1,\rho\rangle}[\pi_s,\rho_t]^{\otimes}$$

(and Equations (2.41) and (2.42)).

The reader might have noticed that our definition should contain the definition of graded Loday algebra cohomology, which is a $\mathbb{Z}^{n+1}$-GLA, see Theorem 7, whereas the Loday infinity algebra cohomology is only a $\mathbb{Z}^n$-GLA. The following remarks explain inter alia how the $\mathbb{Z}^{n+1}$-gradation appears in the special case of p-ary (and in particular binary) brackets.

1. We first examine the case $\pi = \pi_p$, $p \in \{2,4,6,\ldots\}$, where all but one structure maps vanish. If the odd degree $Q \in \mathbb{Z}^n$ of π is chosen to be equal to $(p-1)e_1$, we have

$$\pi = \pi_p \in M^{(0,p-1)}(V), [\pi_p,\pi_p]^{\otimes} = 0, p \text{ even.} \tag{2.67}$$

Let us recall that essentially two p-ary extensions of the Jacobi identity have been investigated during the last decades by mathematicians and physicists—mainly in the skew-symmetric setting. If $[-,-,\ldots,-]$ denotes an p-linear bracket on V, the first is the generalization, which requires that the adjoint action $[v_1,v_2,\ldots,v_{p-1},-]$ be a derivation for the p-ary bracket $[w_1,w_2,\ldots,w_p]$, see e.g. [Fil85], and leads to *Nambu-Lie* or, in the nonantisymmetric context, to *Nambu-Loday* structures, see [Nam73]. The second has been suggested by P. Michor and A. Vinogradov, see [MV97], and has been further studied in [VV98] and in [VV01]. We refer to this last p-ary extension as *p-ary Lie* or *p-ary Loday* structure. An *p-ary* $\mathbb{Z}^n$*-graded Lie structure* on a $\mathbb{Z}^n$-graded vector space V is a map $P_p \in A^{(0,p-1)}(V)$, such that $i_{P_p}P_p = 0$, where i is the interior product defined in Theorem 2, or, if p is even, equivalently, such that $[P_p,P_p]^{\mathrm{NR}} = 0$. Moreover, the cohomology of an p-ary Lie algebra (V,P_p) is the cohomology of the DGLA $(A(V),[-,-]^{\mathrm{NR}},[P_p,-]^{\mathrm{NR}})$, see [MV97].

Analogously, we call the above structure $\pi = \pi_p$, see Equation (2.67), an *p-ary* $\mathbb{Z}^n$*-graded Loday structure* on V. The cohomology of such a structure should be— and has been defined in the case $p = 2$ (roughly) as—the cohomology of the DGLA $(M_r(V), [-,-]^{\otimes}, [\pi_p, -]^{\otimes})$.

In the following, we show why the cohomology space of $\pi = \pi_p$, viewed as degree $Q = (p-1)e_1$ Loday infinity structure, coincides with the just guessed cohomology space of $\pi = \pi_p$, viewed as p-ary $\mathbb{Z}^n$-graded Loday structure.

Let us first mention that in the case of noninfinity algebras, it is conventional to substitute in cochain space $C(V)$ a direct sum for the direct product, so that

$$C(V) = \bigoplus_{s\in\mathbb{N}^*} \bigoplus_{Q\in\mathbb{Z}^n} M^{(Q-(s-1)e_1, s-1)}(V) = \bigoplus_{s\in\mathbb{N}^*} M^{s-1}(V) = M_r(V).$$

Hence, $C(V)$ coincides with vector space $M_r(V)$; further C(V) gets bigraded, but its bigrading is shifted with respect to the usual bigradation

$$M_r(V) = \bigoplus_{s\in\mathbb{N}^*} \bigoplus_{Q\in\mathbb{Z}^n} M^{(Q,s-1)}(V)$$

of $M_r(V)$.

It is now easy to see that the cohomology spaces of

$$(C(V), [-,-]^{\overline{\otimes}}, [\pi_p, -]^{\overline{\otimes}}) \text{ and } (M_r(V), [-,-]^{\otimes}, [\pi_p, -]^{\otimes})$$

coincide. Indeed, if π is a Lod^Q_∞ structure on V (and in particular in our case $\pi = \pi_p$) and if $\rho_t \in M^{(\rho-(t-1)e_1, t-1)}(V)$, we have

$$[\pi, \rho_t]^{\overline{\otimes}} = \sum_{q\geq 1} (-1)^{1+(q-t)\langle e_1, \rho\rangle} [\pi_{q-t+1}, \rho_t]^{\otimes} \in \prod_{q\geq 1} M^{(Q+\rho-(q-1)e_1, q-1)}(V), \quad (2.68)$$

so that, in the case $\pi = \pi_p$, $Q = (p-1)e_1$, where necessarily $q = t + p - 1$, the weight of cohomology operator $[\pi_p, -]^{\overline{\otimes}}$ with respect to the first mentioned bigrading is $((p-1)e_1, p-1)$. On the other hand, since $\pi_p \in M^{(0,p-1)}(V)$, it is clear that the weight of cohomology operator $[\pi_p, -]^{\otimes}$ with respect to the second bigradation is $(0, p-1)$. It follows that the cohomology spaces of $(C(V), [\pi_p, -]^{\overline{\otimes}})$ and $(M_r(V), [\pi_p, -]^{\otimes})$, say $\overline{H}$ and H, are both $\mathbb{Z}^{n+1}$-graded. Space $\overline{H}^{(\rho+(t-1)e_1, t-1)}$ (resp. $H^{(\rho, t-1)}$) is encoded in the cocycle equation

$$[\pi_p, \rho_t]^{\overline{\otimes}} = 0 \Leftrightarrow [\pi_p, \rho_t]^{\otimes} = 0, \text{ where } \rho_t \in M^{(\rho, t-1)}(V)$$

(resp.

$$[\pi_p, \rho_t]^{\otimes} = 0,\ \rho_t \in M^{(\rho, t-1)}(V)),$$

and in the coboundary equation, which reads, as easily understood,

$$\rho_t = [\pi_p, \tau]^{\overline{\otimes}} = [\pi_p, \pm\tau]^{\otimes},\ \tau \in M^{(\rho,t-p)}(V)$$

(resp.

$$\rho_t = [\pi_p, \tau]^{\otimes},\ \tau \in M^{(\rho,t-p)}(V)),$$

for any cocycle ρ_t. This observation obviously entails that

$$\overline{H}^{(\rho+(t-1)e_1,t-1)} = H^{(\rho,t-1)}.$$

Eventually, as announced, the cohomology spaces $\overline{H}$ and H coincide, and their natural GLA structures, which are induced by $[-,-]^{\overline{\otimes}}$ and $[-,-]^{\otimes}$, are $\mathbb{Z}^n$- and $\mathbb{Z}^{n+1}$-graded respectively.

For $p = 2$, we of course recover the abovedefined cohomology space of a graded Loday algebra.

2. If π is not a sequence of all but one vanishing elements, Equation (2.68) implies that the Loday infinity coboundary operator $[\pi,-]^{\overline{\otimes}}$ maps $M^{(\rho-(t-1)e_1,t-1)}(V)$ into $\prod_{q\geq 1} M^{(Q+\rho-(q-1)e_1,q-1)}(V)$, so that the Loday infinity cohomology is not $\mathbb{Z}$-graded. Nevertheless, if we consider the (decreasing) filtration

$$C_k(V) = \bigoplus_{R\in\mathbb{Z}^n} \prod_{s\geq k} M^{(R-(s-1)e_1,s-1)}(V),\ k \geq 1,$$

and if $\rho = \sum_R \sum_{t\geq k} \rho_{R,t} \in C_k(V)$, we get

$$\begin{aligned}[\pi,\rho]^{\overline{\otimes}} = \quad & \textstyle\sum_R \sum_{q\geq 1} \sum_{s+t=q+1} (-1)^{1+(s-1)\langle e_1,R\rangle} [\pi_s, \rho_{R,t}]^{\otimes} \\ & \in \textstyle\bigoplus_{R\in\mathbb{Z}^n} \prod_{q\geq k} M^{(Q+R-(q-1)e_1,q-1)}(V).\end{aligned}$$

Indeed, as $k \leq t$, we have $q \leq k-1 \Rightarrow q \leq t-1 \Leftrightarrow s = q-t+1 \leq 0$, so that the sum over s,t vanishes for these q. The observation yields

$$[\pi, C_k(V)]^{\overline{\otimes}} \subset C_k(V).$$

Eventually, differential space $(C(V), [\pi,-]^{\overline{\otimes}})$ is a differential filtered module and the theory of spectral sequences may be applied.

Chapter 3

Lie infinity algebras and Deformation Quantization

This Chapter intends to analyze the role of L_∞ algebras in deformation theory and to review Kontsevich's formality Theorem together with his star product formula.

We will make use of the same notations employed previously in Chapter 2.

3.1 Lie infinity algebras and their morphisms

Let V be a $\mathbb{Z}^n$-graded vector space, and consider the $\mathbb{Z}^n$-graded vector subspace

$$\mathscr{C}(V) = \bigoplus_{Q \in \mathbb{Z}^n} \mathscr{C}^Q(V) = \bigoplus_{Q \in \mathbb{Z}^n} \prod_{s \geq 1} A^{(Q-(s-1)e_1, s-1)}(V)$$

of $C(V)$, which is made up by finite sums of sequences of $\mathbb{Z}^n$-graded skew-symmetric multilinear maps on V, see Equation (2.16). Let $[-,-]^{\overline{\mathrm{NR}}}$ be the restriction of the $\mathbb{Z}^n$-graded Lie stem bracket $[-,-]^{\overline{\otimes}}$, introduced in Equation (2.40), to $\mathscr{C}(V)$. The bracket $[-,-]^{\overline{\mathrm{NR}}}$ is obviously a $\mathbb{Z}^n$-graded Lie bracket on $\mathscr{C}(V)$ and, for any graded antisymmetric sequences π, ρ, we have

$$[\pi, \rho]^{\overline{\mathrm{NR}}} \quad = \quad \sum_{q \geq 1} \sum_{s+t=q+1} (-1)^{1+(s-1)\langle e_1, \rho \rangle} [\pi_s, \rho_t]^{\mathrm{NR}},$$

since $[-,-]^{\otimes}$ coincides with $[-,-]^{\mathrm{NR}}$ on $A(V)$.

If the structure maps of a Lod$_\infty$ algebra (V,π) are $\mathbb{Z}^n$-graded skew-symmetric, we recover the notion of Lie infinity (L$_\infty$ for short) algebra. One easily ascertains that the definition of L$_\infty$ algebras given in [LS93] coincides with the following

Definition 16. *A given pair (V,π), made up by a $\mathbb{Z}^n$-graded vector space V and a sequence*

$$\pi \in \mathscr{C}^{e_1}(V) = \prod_{s\geq 1} A^{((2-s)e_1,s-1)}(V),$$

is an L$_\infty$ *algebra if and only if $[\pi,\pi]^{\overline{\mathrm{NR}}} = 0$, i.e. if and only if*

$$\sum_{s+t=p} (-1)^s [\pi_s,\pi_t]^{\overline{\mathrm{NR}}} = 0, \quad \forall p \geq 2. \tag{3.1}$$

We denote by L$_\infty(V)$ the set of L$_\infty$ structures on V.

Example 3. *An* L$_\infty$ *algebra (V,π) reduces to a DGLA, if all structure maps π_s vanish, except π_1 and π_2.*

Since L_∞ structures on V are the degree e_1 canonical elements of the $\mathbb{Z}^n$-graded Lie algebra $(\mathscr{C}(V),[-,-]^{\overline{\mathrm{NR}}})$, the cohomology of an L_∞ structure π on V is the cohomology of the DGLA $(\mathscr{C}(V),[-,-]^{\overline{\mathrm{NR}}},[\pi,-]^{\overline{\mathrm{NR}}})$. This cohomology has been studied in [Pen01] and used in [FP02].

Remark 9. *Replacing in (3.1) bracket $[-,-]^{NR}$ with bracket $[-,-]^{G}$, see Equation (2.15), we recover the concept of* A$_\infty$ *algebra [LS93] which is the homotopy algebra that generalizes associative algebras.*

We have seen that L$_\infty$ structures on V are particular Lod$_\infty$ structures on the nonshifted side. However, on the shifted side, L$_\infty$ structures on V may also be viewed as odd codifferentials, but of a different coalgebra, namely of the coassociative symmetric tensor coalgebra of $\downarrow V$. In the following, we recall this coalgebraic description, but skip most of the proofs.

Remember that the (reduced) symmetric associative tensor algebra $S(V)$ is the quotient of the (reduced) associative tensor algebra $T(V)$ by the ideal generated by the elements of the form $v_1 \otimes v_2 - (-1)^{\langle v_1,v_2\rangle} v_2 \otimes v_1$. The induced product on $S(V)$ will be denoted by $\vee$. The algebra $S(V)$ inherits both, the $\mathbb{Z}$-gradation and the $\mathbb{Z}^n$-gradation. In the sequel, we consider $S(V)$ as a $\mathbb{Z}^n$-graded vector space.

Proposition 16. *The coproduct*

$$\Delta^S : S(V) \to S(V) \bigotimes S(V),$$

defined by

$$\Delta^S(v_1 \vee \ldots \vee v_p) = \sum_{\substack{I\cup J=N^{(p)}\\ I,J\neq\emptyset}} \varepsilon_V(I;J)\, V_I \bigotimes V_J \quad (v_\ell \in V^{v_\ell}, p \geq 1), \tag{3.2}$$

provides a graded coassociative coalgebra structure on $S(V)$*. Here,* $V_I = v_{i_1} \vee \ldots \vee v_{i_{|I|}}$*, for any unshuffle* I*.*

In the following theorem, we characterize the space of coderivations $\mathrm{CoDer}(S(V))$ of the coalgebra $(S(V),\Delta^S)$.

Theorem 12. *A homogenous coderivation* Q^S *of weight* $Q^S \in \mathbb{Z}^n$ *of the coalgebra* $(S(V),\Delta^S)$ *is uniquely determined by its corestriction maps*

$$Q^S_p : S^pV \hookrightarrow S(V) \xrightarrow{Q^S} S(V) \xrightarrow{\mathrm{pr}} V,$$

via the equation

$$Q^S(v_1 \vee \ldots \vee v_p) = \sum_{\substack{I\cup J=N^{(p)}\\ I\neq\emptyset}} \varepsilon_V(I;J) Q^S_{|I|}(V_I) \vee V_J. \tag{3.3}$$

If $\mathscr{S}(V)$ denotes the $\mathbb{Z}^{n+1}$-graded vector subspace of $M(V)$ made up by the $\mathbb{Z}^n$-graded symmetric multilinear maps

$$A(\ldots,v_i,v_{i+1},\ldots) = (-1)^{\langle v_i,v_{i+1}\rangle} A(\ldots,v_{i+1},v_i,\ldots), \tag{3.4}$$

then, clearly, the sequence of corestriction maps $(Q^S_1,Q^S_2,\ldots)$ of a coderivation Q^S of weight $Q^S \in \mathbb{Z}^n$, is an element of the direct product space $\prod_{s\geq 1} \mathscr{S}^{(Q^S,s-1)}(V)$, and the preceding proposition implies that the mapping

$$\psi^S : \mathrm{CoDer}^{Q^S}(S(V)) \ni Q^S \to (Q^S_1,Q^S_2,\ldots) \in \prod_{s\geq 1} \mathscr{S}^{(Q^S,s-1)}(V)$$

is a vector space isomorphism. Furthermore, since, for any $\rho \in \mathbb{Z}^n$, the maps

$$A^{(\rho-(s-1)e_1,s-1)}(V) \ni A \to (-1)^{\frac{s(s-1)}{2}} \downarrow \circ A \circ \uparrow^{\vee s} \in \mathscr{S}^{(\rho,s-1)}(\downarrow V), \tag{3.5}$$

where $s \geq 1$ and where notations are self-explaining, are vector space isomorphisms, their composition with $(\psi^S)^{-1}$ provides the isomorphisms

$$\phi^S_\rho : \mathscr{C}^\rho(V) = \prod_{s\geq 1} A^{(\rho-(s-1)e_1,s-1)}(V) \to \mathrm{CoDer}^\rho(S(\downarrow V)).$$

Proposition 17. *Let $[-,-]^S$ be the $\mathbb{Z}^n$-graded Lie bracket of the space of coderivations of $(S(\downarrow V),\Delta^S)$. Then, the aforementioned $\mathbb{Z}^n$-graded Lie bracket $[-,-]^{\overline{\mathrm{NR}}}$ on $\mathscr{C}(V)$ is given by*

$$[\pi,\rho]^{\overline{\mathrm{NR}}} = (\phi^S_{\pi+\rho})^{-1}[\phi^S_\pi(\pi),\phi^S_\rho(\rho)]^S, \quad \pi\in\mathscr{C}^\pi(V),\rho\in\mathscr{C}^\rho(V).$$

Consequently,

Proposition 18. *A sequence $\pi\in\mathscr{C}^{e_1}(V)$ is an L_∞ structure on the $\mathbb{Z}^n$-graded vector space V if and only if the weight e_1 coderivation $Q^S := \phi^S_{e_1}(\pi)$ of $(S(\downarrow V),\Delta^S)$ is a codifferential.*

We now introduce the notion of L_∞ morphisms.

Definition 17. *Let (V,Q^S) and (V',Q'^S) be two L_∞ algebras. An L_∞* morphism *from (V,Q^S) to (V',Q'^S) is a coalgebra cohomomorphism*

$$\mathscr{F}^S : (S(\downarrow V),\Delta^S) \longrightarrow (S(\downarrow V'),\Delta^S),$$

which intertwines the codifferentials Q^S and Q'^S, i.e. verifies

$$Q'^S\mathscr{F}^S = \mathscr{F}^S Q^S. \tag{3.6}$$

Proposition 19. *Let V and V' be two $\mathbb{Z}^n$-graded vector spaces. A coalgebra cohomomorphism*

$$\mathscr{F}^S : (S(V),\Delta^S) \longrightarrow (S(V'),\Delta^S)$$

is uniquely determined by its corestriction maps

$$\mathscr{F}^S_p : S^pV \hookrightarrow S(V) \xrightarrow{\mathscr{F}^S} S(V') \xrightarrow{\mathrm{pr}} V', \quad p\geq 1,$$

via the equation

$$\mathscr{F}^S(v_1\vee\ldots\vee v_p) = \sum_{s=1}^{p}\frac{1}{s!}\sum_{\substack{I^1\cup\ldots\cup I^s=N^{(p)}\\ I^1,\ldots,I^s\neq\emptyset}} \varepsilon_V(I^1;\ldots;I^s)\mathscr{F}^S_{|I^1|}(V_{I^1})\vee\ldots\vee\mathscr{F}^S_{|I^s|}(V_{I^s}), \tag{3.7}$$

where $v_\ell\in V^{v_\ell}$, for all $\ell\in\{1,\ldots,p\}$.

Since a coalgebra cohomomorphism $\mathscr{F}^S : (S(\downarrow V),\Delta^S)\to(S(\downarrow V'),\Delta^S)$ “is” of course a sequence $f^S=(f^S_1,f^S_2,\ldots)$ of weight $(1-s)e_1$, $\mathbb{Z}^n-$graded skew-symmetric multilinear maps $f^S_s : V^{\times s}\to V'$ that are defined by $f^S_s = \uparrow\circ\mathscr{F}^S_s\circ\downarrow^{\vee s}$, we obtain the following equivalent interpretation of L_∞ morphisms:

Proposition 20. *Let* (V, π^S) *and* (V', π'^S) *be two* L_∞ *algebras. An* L_∞ *morphism* $f^S : (V, \pi^S) \to (V', \pi'^S)$ *is a sequence of* $\mathbb{Z}^n-$*graded skew-symmetric weight* $(1-s)e_1$ *multilinear maps* f^S_s, $s \geq 1$, *which satisfy conditions (2.52) (with* π'^S, f^S, *and* π^S *substituted for* π', f, *and* π *respectively).*

It follows that the category $\mathbf{L}_\infty$ is a subcategory of the category $\mathbf{Lod}_\infty$. Let us recall that a subcategory of a category C is a category whose objects and morphisms are objects and morphisms in C and that has the same composition of morphisms and the same unit morphisms.

In the following, any DGLA $(V, d, \{-,-\})$ we consider has a weight e_1 differential d, so that hence V can be seen as an L_∞ algebra.

3.2 Generalized Maurer Cartan Equation

In this Section, we recall the notion of Maurer Cartan elements of L_∞ algebras and prove that this notion is preserved under the action of L_∞ morphisms.

Definition 18. *Let* $(V, d, \{-,-\})$ *be a DGLA. A* Maurer Cartan element *w of V is a degree* e_1 *element satisfying the* Maurer Cartan equation *(MCE for short)*

$$d\,w - \frac{1}{2}\{w, w\} = 0. \tag{3.8}$$

The MCE can naturally be generalized to an L_∞ algebra $(V, Q^S) \sim (V, \pi)$ by considering the equation

$$\textstyle\sum_{s\geq 1} \frac{1}{s!} Q^S_s(\downarrow w, \ldots, \downarrow w) = 0, \tag{3.9}$$

or, equivalently,

$$\textstyle\sum_{s\geq 1} \frac{1}{s!} (-1)^{\frac{s(s-1)}{2}} \pi_s(w, \ldots, w) = 0 \tag{3.10}$$

for any $w \in V^{e_1}$, because if the L_∞ algebra (V, π) is a DGLA, i.e. $\pi_1 = d$, $\pi_2 = \{-,-\}$ and $\pi_3 = \pi_4 = \ldots = 0$, one recognizes the ordinary MCE (3.8).

The reader may have noticed that in equation (3.9) we encounter a convergence problem, because this equation yields an infinite sum in the case of infinitely many nonzero Q^S_ℓ. We apply the standard solution to this problem:
Let $\mathscr{K}$ be a finite dimensional local $\mathbb{K}-$algebra (commutative with unit), with a nilpotent maximal ideal $\mathbf{m}$, i.e. there exists $N \in \mathbb{N}$ such that $\mathbf{m}^N = 0$. If (V, Q^S) is an L_∞ algebra, then $V_{\mathbf{m}} := V \otimes_{\mathbb{K}} \mathbf{m}$ has an L_∞ structure given by the codifferential

$Q_{\mathbf{m}}$ of corestriction maps $Q^S_{\mathbf{m}\ell}$, $\ell \geq 1$, which are the $\mathscr{K}-$multilinear natural extensions of the corestriction maps Q^S_ℓ. Hence, for an element $w \in V^{e_1} \otimes_{\mathbb{K}} \mathbf{m}$, obviously $Q_\ell(\downarrow w, \ldots, \downarrow w) = 0$ if $\ell \geq N$, and so equation (3.9) makes sense.

From now on, we shall assume that any L$_\infty$ algebra (V, Q^S) we consider, has been tensored with a nilpotent maximal ideal $\mathbf{m}$ of a finite dimensional local ring $\mathscr{K}$, and we denote the result by the same (V, Q^S).

Definition 19. *Let (V, Q^S) be a nilpotent L$_\infty$ algebra. A Maurer Cartan element w of V is a degree e_1 element satisfying the generalized Maurer Cartan equation (GMCE)*

$$Q^S_*(\downarrow w) = \textstyle\sum_{s\geq 1} \frac{1}{s!} Q^S_s(\downarrow w, \ldots, \downarrow w) = 0.$$

Remark 10. *Let us mention that a modification of the Definition 16 of L$_\infty$ algebras via the transformation $\pi_s \mapsto \widetilde{\pi}_s := (-1)^{\frac{s(s-1)}{2}} \pi_s$ allows to write all terms of the equation (3.10) with $+1-$signs rather than the signs $(-1)^{\frac{s(s-1)}{2}}$. With this modification, the L$_\infty$ structure conditions (3.1) become*

$$\textstyle\sum_{s+t=p} (-1)^{s(p+1)} [\widetilde{\pi}_s, \widetilde{\pi}_t]^{\mathrm{NR}} = 0, \quad \forall p \geq 2.$$

Henceforth, we keep the convention of signs fixed in Definition 16.

The upcoming Proposition provides an equivalent formulation for the GMCE. Given an element $v \in V$, let us denote by e^v the exponential

$$e^v = \textstyle\sum_{s\geq 1} \frac{1}{s!} \underbrace{v \vee \ldots \vee v}_{s}.$$

Proposition 21. *A degree e_1 element w of an L$_\infty$ algebra (V, Q^S) is a Maurer Cartan element if and only if*

$$Q^S(e^{\downarrow w}) = 0.$$

Proof. Applying (3.3), for any $v \in V^0$, we have

$$\begin{aligned} Q^S(e^v) &= \textstyle\sum_{i\geq 1} \frac{1}{i!} Q^S(\underbrace{v \vee \ldots \vee v}_{i}) \\ &= \textstyle\sum_{i\geq 1} \frac{1}{i!} \sum_{j=1}^{i} C_i^j Q^S_j(\underbrace{v \vee \ldots \vee v}_{j}) \vee v \ldots \vee v \\ &= Q^S_*(v) \vee (e^v + \mathbf{1}) \end{aligned}$$

where $\mathbf{1}$ is defined so that $V^{\vee k} \vee \mathbf{1} \vee V^{\vee l} = V^{\vee k+l}$ for any $k, l \geq 0$. As $e^v + \mathbf{1}$ is invertible (of inverse $\mathbf{1} + e^{-v}$), then, when replacing in the previous equality v by $\downarrow w$, we get $Q^S(e^{\downarrow w}) = 0$ if and only if $Q^S_*(\downarrow w) = 0$. ∎

We shall denote the set of Maurer Cartan elements of an L$_\infty$ algebra (V, Q^S) by $MC(V)$.

Our next task is to prove that the set $MC(V)$ of an L$_\infty$ algebra (V, Q^S) is preserved under the action of L$_\infty$ morphisms. To achieve this goal, we need the following preliminaries.

Definition 20. *An even degree element p of the coalgebra $(S(\downarrow V), \Delta^S)$ is called* a group-like element *if* $\Delta^S p = p \otimes p$.

Lemma 4. *The set of group-like elements of $S(\downarrow V)$ is made up by elements e^v, where v is any even degree element of $\downarrow V$.*

Proof. When applying (3.2), we can clearly see that $\Delta^S e^v = e^v \otimes e^v$. Hence, e^v is a group like element. Conversely, take $p \in S(\downarrow V)$ as a group-like element. Write p as the sum $\sum_{i \geq 1} p_i$, where each p_i is an even degree element in $S^i(\downarrow V)$. The equation $\Delta^S p = p \otimes p$ implies that

$$\Delta^S p_i = \textstyle\sum_{j+k=i} p_j \otimes p_k \tag{3.11}$$

for any $i \geq 1$. For $i = 2$, we get $\Delta^S p_2 = p_1 \otimes p_1$ and so, in view of the definition of Δ^S, $p_2 = \frac{1}{2!} p_1 \vee p_1$. Proceeding by induction, we easily see that $p_i = \frac{1}{i!}(p_1)^i$. Therefore, we get $p = e^{p_1}$. ∎

Proposition 22. *Let (V, Q^S) and (V', Q'^S) be two* L$_\infty$ *algebras and assume that an* L$_\infty$ *morphism $\mathscr{F}^S : (V, Q^S) \to (V', Q'^S)$ has been defined. If w is a Maurer Cartan element of (V, Q^S), then the element*

$$\mathscr{F}^S_*(\downarrow w) := \textstyle\sum_{s=1}^{\infty} \frac{1}{s!} \uparrow \mathscr{F}^S_s(\downarrow w, \ldots, \downarrow w) \tag{3.12}$$

is a Maurer Cartan element of (V', Q'^S).

Proof. Set $w' = \mathscr{F}^S_*(\downarrow w)$. Observe first that w' is a e_1 degree element of V' since $\downarrow w$ is of degree zero and the maps $\mathscr{F}^S_s$, $s \geq 1$, are of weight zero. We now show that w' satisfies the generalized Maurer Cartan Equation, or, equivalently, that $Q'^S(e^{\downarrow w'}) = 0$ (see Proposition 21).
Since $\mathscr{F}^S$ is an L$_\infty$ morphism, i.e. $\mathscr{F}^S \circ Q^S = Q'^S \circ \mathscr{F}^S$ and since $Q^S(e^{\downarrow w}) = 0$, we

have $Q'^S(\mathscr{F}^S(e^{\downarrow w})) = 0$. By showing that $e^{\downarrow w'} = \mathscr{F}^S(e^{\downarrow w})$, we obtain the proof. Since $\mathscr{F}^S$ is a coalgebra cohomomorphism and since $\Delta^S(e^{\downarrow w}) = e^{\downarrow w} \otimes e^{\downarrow w}$, then

$$\Delta^S \mathscr{F}^S(e^{\downarrow w}) = (\mathscr{F}^S \otimes \mathscr{F}^S)\Delta^S(e^{\downarrow w}) = \mathscr{F}^S(e^{\downarrow w}) \otimes \mathscr{F}^S(e^{\downarrow w})$$

and so $\mathscr{F}^S(e^{\downarrow w})$ is a group-like element. According to Lemma 4, $\mathscr{F}^S(e^{\downarrow w})$ is therefore equal to $e^{v'}$, where v' is the projection of $\mathscr{F}^S(e^{\downarrow w})$ on $\downarrow V'$. Using (3.7), it can be easily checked that

$$v' = \textstyle\sum_{s=1}^{\infty} \frac{1}{s!} \mathscr{F}_s^S(\downarrow w, \dots, \downarrow w) = \downarrow w'. \ \blacksquare$$

3.3 Twisted L$_\infty$ algebras and Twisted L$_\infty$ quasi-isomorphisms

Referring to the papers [Ye06] and [Dol05], we recall in this Section the twisting procedure of L$_\infty$ algebras and L$_\infty$ quasi-isomorphisms by Maurer Cartan elements.

Given an L$_\infty$ algebra (V, Q^S) endowed with a Maurer Cartan element $w \in V^{e_1}$, a new L$_\infty$ structure on V is obtained as follows:

Proposition 23. *Let (V, Q^S) be an L$_\infty$ algebra equipped with a Maurer Cartan element w. Define a coderivation Q^{Sw} on $(S(\downarrow V), \Delta^S)$ with corestriction maps*

$$Q_p^{Sw}(\downarrow v_1, \dots, \downarrow v_p) := \textstyle\sum_{i \geq 0} \frac{1}{i!} Q_{p+i}^S(\downarrow w, \dots, \downarrow w, \downarrow v_1, \dots, \downarrow v_p)$$

for any $v_1, \dots, v_p \in V$ and any $p \geq 1$. The pair (V, Q^{Sw}) is an L$_\infty$ algebra, called the twisted L$_\infty$ algebra of (V, Q^S) by the Maurer Cartan element w.

Proof. Consider the coalgebra map

$$\overline{Q} : S(\downarrow V) \to S(\downarrow V)$$

defined by

$$\overline{Q} := \varphi_w^{-1} Q^S \varphi_w$$

where φ_w is the invertible coalgebra map

$$\varphi_w : S(\downarrow V) \to S(\downarrow V), \quad X \mapsto (\mathbf{1} + e^{\downarrow w}) \vee X.$$

It is easily seen that the map $\overline{Q}$ satisfies $\overline{Q}^2 = 0$ (because Q is a codifferential) and that its corestriction maps coincide with Q_p^{Sw} for any $p \geq 1$. Hence, in view

of Theorem 12, if we prove that $\overline{Q}$ is a coderivation then $Q^{Sw} = \overline{Q}$ and we can conclude that the pair (V, Q^{Sw}) is an L_∞ algebra.
Let us then show that $\overline{Q}$ is a coderivation. By application of equation (3.2), a direct computation shows that for any $X \in S(\downarrow V)$ and any $s \in \mathbb{Z}$

$$\Delta^S \varphi_{sw}(X) = (\varphi_{tw} \otimes \varphi_{sw}) \Delta^S(X) + e^{\downarrow tw} \otimes \varphi_{sw}(X) + \varphi_{sw}(X) \otimes e^{\downarrow sw}. \tag{3.13}$$

Hence, it follows that

$$\begin{aligned}
&\Delta^S\left(\overline{Q}(X)\right) = \Delta^S\left(\varphi_w^{-1}\left(Q^S\varphi_w(X)\right)\right)\\
&= \left(\varphi_w^{-1} \otimes \varphi_w^{-1}\right)\left(\Delta^S\left(Q^S\varphi_w(X)\right)\right)\\
&\qquad +e^{\downarrow -w} \otimes \varphi_w^{-1}\left(Q^S\varphi_w(X)\right) + \varphi_w^{-1}\left(Q^S\varphi_w(X)\right) \otimes e^{\downarrow -w}\\
&= \left(\varphi_w^{-1} \otimes \varphi_w^{-1}\right)\left(\left(Q^S \otimes \mathrm{id} + \mathrm{id} \otimes Q^S\right)\left(\Delta^S\left(\varphi_w(X)\right)\right)\right)\\
&\qquad +e^{\downarrow -w} \otimes \varphi_w^{-1}\left(Q^S\varphi_w(X)\right) + \varphi_w^{-1}\left(Q^S\varphi_w(X)\right) \otimes e^{\downarrow -w}\\
&= \left(\varphi_w^{-1} \otimes \varphi_w^{-1}\right)\\
&\qquad \left(\left(Q^S \otimes \mathrm{id} + \mathrm{id} \otimes Q^S\right)\left((\varphi_w \otimes \varphi_w)\Delta^S(X) + e^{\downarrow w} \otimes \varphi_w(X) + \varphi_w(X) \otimes e^{\downarrow w}\right)\right)\\
&\qquad +e^{\downarrow -w} \otimes \varphi_w^{-1}\left(Q^S\varphi_w(X)\right) + \varphi_w^{-1}\left(Q^S\varphi_w(X)\right) \otimes e^{\downarrow -w}\\
&= \left(\overline{Q}^S \otimes \mathrm{id} + \mathrm{id} \otimes \overline{Q}^S\right)\left(\Delta^S(X)\right),
\end{aligned}$$

where in the last equality we use the propriety $Q^S(e^{\downarrow w}) = 0$ (w is a Maurer Cartan element) as well as the following obvious identity

$$\varphi_w^{-1}(e^{\downarrow w}) = -e^{\downarrow -w}. \ \blacksquare$$

Example 4. *Let $(V, Q^S) \sim (V, \pi)$ be an L_∞ algebra equipped with a Maurer Cartan element w. If (V, π) is a DGLA $(V, d, \{-,-\})$, then the twisted DGLA of (V, π) by w is the DGLA $(V, d - \{w, -\}, \{-,-\})$.*

Proposition 24. *Let (V, Q^S) and (V', Q'^S) be two L_∞ algebras and assume that an L_∞ quasi-isomorphism $\mathscr{F}^S : (V, Q^S) \to (V', Q'^S)$ has been defined. Set an element $w \in MC(V)$ and let $w' = \mathscr{F}^S_*(\downarrow w)$ be the element of $MC(V')$ as defined in Proposition 22. Let (V, Q^{Sw}) and $(V', Q'^{Sw'})$ be respectively the twisted L_∞ algebra of*

(V,Q^S) and (V',Q'^S) by the Maurer Cartan elements w and w'. Then the sequence of maps

$$\mathscr{F}_s^{Sw}(\downarrow v_1,\ldots,\downarrow v_p) := \textstyle\sum_{i\geq 0}\frac{1}{i!}\mathscr{F}_{s+i}^{S}(\downarrow w,\ldots,\downarrow w,\downarrow v_1,\ldots,\downarrow v_s), \quad s\geq 1 \tag{3.14}$$

defines an L$_\infty$ *quasi-morphism $\mathscr{F}^{Sw}$ from (V,Q^{Sw}) to $(V',Q'^{Sw'})$, called* the twisted L$_\infty$ quasi-morphism of the L$_\infty$ quasi-morphism $\mathscr{F}^S$ by w.

Proof. Set $\overline{\mathscr{F}} := \varphi_{w'}^{-1}\mathscr{F}^S\varphi_w$. As $\mathscr{F}^S$ is an L$_\infty$ morphism, then $\overline{\mathscr{F}}Q^{Sw} = Q'^{Sw'}\overline{\mathscr{F}}$. By application of equation (3.13), and by using similar reasoning developed in the previous proof, we see that

$$\Delta^S\overline{\mathscr{F}} = (\overline{\mathscr{F}}\otimes\overline{\mathscr{F}})\Delta^S.$$

Hence, $\overline{\mathscr{F}}$ is an L$_\infty$ morphism from (V,Q^{Sw}) to $(V',Q'^{Sw'})$. Moreover, as the corestriction maps of $\overline{\mathscr{F}}$ are exactly $\mathscr{F}_p^{Sw}$, $p\geq 1$, then $\mathscr{F}^{Sw} = \overline{\mathscr{F}}$ is an L$_\infty$ morphism. By application of spectral sequence arguments it can be shown that $\mathscr{F}^{Sw}$ is an L$_\infty$ quasi-isomorphism (i.e. $(\mathscr{F}_1^{Sw})_\sharp$ is an isomorphism between the corresponding cohomology). We refer the reader to [Dol05]. ∎

Dolgushev [Dol05] used this twisting technique in his globalization procedure for the local Kontsevich's formality on $\mathbb{R}^d$. Chapter 4 will review this construction. We will see next that this technique is also useful for investigating some proprieties of the Moduli space of L$_\infty$ algebras.

3.4 Moduli space of L$_\infty$ algebras

Let $\mathbb{K}[[t]]$ be the space of polynomials in the parameter t with coefficients in $\mathbb{K}$.

Definition 21. *Given an* L$_\infty$ *algebra (V,Q^S), two elements $w_0\in MC(V)$ and $w_1\in MC(V)$ are called* gauge equivalent *if and only if there exists an element $u(t)\in V^0\otimes\mathbb{K}[[t]]$ such that*

$$\frac{d}{dt}\downarrow w(t) = Q_1^{Sw(t)}(\downarrow u(t)) = \textstyle\sum_{i\geq 0}\frac{1}{i!}Q_{1+i}^S(\underbrace{\downarrow w(t),\ldots,\downarrow w(t)}_{i},\downarrow u(t)) \tag{3.15}$$

where $w(t)$ lies in $MC(V\otimes\mathbb{K}[[t]])$ satisfying $w(0)=w_0$ and $w(1)=w_1$.

Here, $V\otimes\mathbb{K}[[t]]$ inherits the L$_\infty$ structure of V with obvious manner.

Clearly, this gauge transformation is an equivalence relation and we can therefore define

Definition 22. *Given an* L_∞ *algebra* (V, Q^S)*, the set of gauge equivalence classes of* $\mathrm{MC}(V)$ *is the moduli space*

$$\mathscr{M}(V) := MC(V)/_\sim,$$

where $\sim$ *is the gauge equivalence in Definition 21.*

A geometric meaning of the gauge equivalence was given in terms of the language of $Q-$manifold, see [Kon03]; we will not discuss this point further. However, let us explain why the moduli space $\mathscr{M}(V)$ in the sense of L_∞ algebras coincides with the "ordinary" well known moduli space $\mathrm{Def}(V)$ in the sense of DGLAs.

Let us first recall the definition of the space $\mathrm{Def}(V)$ of any DGLA $(V, d, \{-,-\})$. Just as for the space $\mathscr{M}(V)$, $\mathrm{Def}(V)$ of a given DGLA V is composed by the set of Maurer Cartan elements, but modulo the gauge group action. The gauge group of the DGLA V is the group $G(V) = \exp(V^0)$, where the multiplication is given by the Baker-Campbell-Hausdorff formula:

$$\exp(u)\exp(v) = \exp(H(u,v)),$$

where

$$H(u,v) = u + v + \frac{1}{2}\{u,v\} + \dots$$

for any $u, v \in V^0$. Note that this product is well defined since V is actually $V \otimes_{\mathbb{K}} \mathbf{m}$ and $\mathbf{m}$ is nilpotent. The group $G(V)$ acts on V^{e_1} by

$$\exp(u).w = \exp(\mathrm{ad}\,u)w + \textstyle\sum_{i\geq 0} \frac{(\mathrm{ad}\,u)^i}{(i+1)!} du \tag{3.16}$$

for any $u \in V^0$ and $w \in V^{e_1}$. It is quite easy to show that this action preserves the set $MC(V)$. Indeed, if we assume that $\{d,d\} = 0$ and $\{x,d\} = \{d,x\} = (-1)^{\langle e_1, x\rangle} dx$ for any x in V^0 or in V^1, then the MCE (3.8) reads

$$\{d+w, d+w\} = 0$$

and the gauge group action (3.16) can be written as

$$\exp(u).w + d = \exp(\mathrm{ad}\,u)(w+d).$$

Moreover, since $\exp(\mathrm{ad}\,u)$ is a Lie algebra morphism (see Lemma 1), then for any $w \in MC(V)$

$$\begin{aligned} \{d + \exp(u).w, d + \exp(u).w\} &= \{\exp(\mathrm{ad}\,u)(w+d), \exp(\mathrm{ad}\,u)(w+d)\} \\ &= \exp(\mathrm{ad}\,u)(\{w+d, w+d\}) = 0. \end{aligned}$$

Thus, we define the moduli space of the DGLA V as the quotient $\mathrm{Def}(V) = MC(V)/G(V)$.

Proposition 25. *If $(V,Q^S) \sim (V,d,\{-,-\})$ is a DGLA, then $\mathscr{M}(V) = \mathrm{Def}(V)$.*

Proof. Let w_0 and w_1 be two elements of $MC(V)$ that are gauge equivalent by means of the gauge group action, i.e. there is $u \in V^0$ so that $w_1 = \exp(u).w_0$. We want to show that w_0 and w_1 are gauge equivalent in the sense of L$_\infty$ algebras as in Definition 21. When taking $u = u(t)$ and $w(t) = \exp(tu).w_0$, we get

$$\begin{aligned}
\frac{d}{dt} \downarrow w(t) &= \frac{d}{dt} \downarrow (\exp(\mathrm{ad}\, tu)(w_0 + d) - d) \\
&= \downarrow \{u, \exp(\mathrm{ad}\, tu)(w_0 + d)\} \\
&= \downarrow du + \downarrow \{u, w(t)\} = \downarrow du - \downarrow \{w_t, u\} \\
&= Q_1^S(\downarrow u) + Q_2^S(\downarrow w(t), \downarrow u)
\end{aligned}$$

where in the last equality we use $d = \uparrow Q_1 \downarrow$ and $\{-,-\} = \uparrow Q_2 \downarrow^{\vee 2}$—see Proposition 18. This proves that a gauge equivalence in the sense of DGLAs implies a gauge equivalence in the sense of L$_\infty$ algebras.
Conversely, suppose now that w_0 and w_1 are equivalent in the sense of L$_\infty$ algebras, i.e. there is $u(t) \in V^0 \otimes \mathbb{K}[[t]]$ so that

$$\frac{d}{dt} \downarrow w(t) = Q_1^S(\downarrow u(t)) + Q_2^S(\downarrow w(t), \downarrow u(t)),$$

where $w(t) \in MC(V \otimes \mathbb{K}[[t]])$ satisfies $w(0) = w_0$ and $w(1) = w_1$. As previously shown,

$$\frac{d}{dt} w(t) = d\,u(t) + \{u(t), w(t)\} = \{u(t), w(t) + d\}$$

and so

$$\frac{d}{dt}(w(t) + d) = \{u(t), w(t) + d\}.$$

It follows that $w(t) = \exp(\mathrm{ad}\, u(t))(w_0) - d = \exp(u(t)).w_0$ is a solution for the above differential equation. Hence $w_1 = \exp(u(1)).w_0$ and so w_0 and w_1 are equivalent by means of the gauge group action (3.16). We conclude therefore that $\mathrm{Def}(V) = \mathscr{M}(V)$. ∎

Example 5. *The moduli space of a contractible* L$_\infty$ *algebra (V, Q^S) is trivial. Indeed, since the set of Maurer Cartan elements is*

$$MC(V) = \{w \in V^{e_1};\ Q_1^S(\downarrow w) = 0\}$$

and because the cohomology space $H(V,Q_1^S)$ is trivial, there exists $u \in V^0$ so that $Q_1^S(\downarrow u) = \downarrow w$. As the element $w(t) = tw$ lies in $MC(V \otimes \mathbb{K}[[t]])$, then equation (3.15) implies that the class of $w = w(1)$ in $\mathscr{M}(V)$ is that of $w(0) = 0$.

Example 6. *Let (V,Q^S) and (V',Q'^S) be two* L$_\infty$ *algebras. Consider $(V \oplus V', Q^S \oplus Q'^S)$ the* direct sum L$_\infty$ algebra *of (V,Q^S) and (V',Q'^S), where $Q^S \oplus Q'^S$ is the codifferential defined by*

$$Q^S \oplus Q'^S{}_{|S(\downarrow V)} = Q^S,\ Q^S \oplus Q'^S{}_{|S(\downarrow V')} = Q'^S \text{ and } Q^S \oplus Q'^S{}_{|S(\downarrow V) \vee S(\downarrow V')} = 0.$$

Then

$$\mathscr{M}(V \oplus V') \cong \mathscr{M}(V) \times \mathscr{M}(V').$$

Indeed, by definition of the direct sum L$_\infty$ *algebra, it is clear that $MC(V \oplus V') \cong MC(V) \times MC(V')$. Moreover, this factorization is preserved by the gauge equivalence because the twisted* L$_\infty$ *algebra of the* L$_\infty$ *algebra $(V \oplus V', Q^S \oplus Q'^S)$ by an element $w + w' \in MC(V \oplus V')$ is nothing else than the* direct sum L$_\infty$ algebra *$(V \oplus V', Q^{Sw} \oplus Q'^{Sw'})$ of the twisted* L$_\infty$ *algebras (V,Q^{Sw}) and $(V',Q'^{Sw'})$. This proves the claim.*

Subsequently, we shall show that the gauge equivalence is preserved under the action of L$_\infty$ morphisms.

According to Proposition 22, any L$_\infty$ morphism $\mathscr{F}^S : (V,Q^S) \to (V',Q'^S)$ induces a well defined map $\mathscr{F}^S_* : MC(V) \to MC(V')$ which assigns the element $\mathscr{F}^S_*(\downarrow w) \in MC(V')$ given in (3.12) to any element $w \in MC(V)$. Moreover

Proposition 26. *The map $\mathscr{F}^S_*$ descends to the quotients by the gauge equivalence, providing a well defined map $\mathscr{M}\mathscr{F}^S_* : \mathscr{M}(V) \to \mathscr{M}(V')$ between the corresponding moduli spaces.*

This result raises from

Lemma 5. *Set $w_0, w_1 \in MC(V)$. If $w_0 \sim w_1$, then $\mathscr{F}^S_*(\downarrow w_0) \sim \mathscr{F}^S_*(\downarrow w_1)$.*

Proof. Assume that $w_1 \sim w_2$ and consider an element $w(t) \in MC(V \otimes \mathbb{K}[[t]])$

and an element $u(t) \in V^0 \otimes \mathbb{K}[[t]]$ as in Definition 21. It is easily seen that

$$\begin{aligned}
\frac{d}{dt} \downarrow \mathscr{F}^S_*(\downarrow w(t)) &= \sum_{s\geq 1} \frac{1}{s!} \frac{d}{dt} \mathscr{F}^S_s(\downarrow w(t), \dots, \downarrow w(t)) \\
&= \sum_{s\geq 1} \frac{1}{(s-1)!} \mathscr{F}^S_s \left(\frac{d}{dt} \downarrow w(t), \downarrow w(t) \dots, \downarrow w(t) \right) \\
&= \sum_{s\geq 0} \frac{1}{s!} \mathscr{F}^S_{1+s} \left(Q_1^{Sw(t)}(\downarrow u(t)), \downarrow w(t), \dots, \downarrow w(t) \right) \\
&\overset{(*)}{=} \mathscr{F}_1^{Sw(t)} \left(Q_1^{Sw(t)}(\downarrow u(t)) \right),
\end{aligned}$$

where at $(*)$ we use equation (3.14). Moreover, from Proposition 24, we know that $\mathscr{F}^{Sw(t)}$ is an L$_\infty$ morphism from the twisted L$_\infty$ algebra $(V \otimes \mathbb{K}[[t]], Q^{Sw(t)})$ to the twisted L$_\infty$ algebra $(V' \otimes \mathbb{K}[[t]], Q'^{Sw'(t)})$, where $w'(t) = \mathscr{F}^S_*(\downarrow w(t))$. Thus, it follows that $\mathscr{F}_1^{Sw(t)} Q_1^{Sw(t)} = Q_1^{Sw'(t)} \mathscr{F}_1^{Sw(t)}$ and so

$$\frac{d}{dt} \downarrow \mathscr{F}^S_*(\downarrow w(t)) = Q_1^{Sw'(t)} \left(\mathscr{F}_1^{Sw(t)}(\downarrow u(t)) \right).$$

This entails that $\mathscr{F}^S_*(\downarrow w_0) \sim \mathscr{F}^S_*(\downarrow w_1)$ and proves the Proposition. ∎

With this premise, we are finally prepared to prove the main Theorem and shall also later explain its crucial role for deformation theory.

Theorem 13. *Let (V, Q^S) and (V', Q'^S) be two L$_\infty$ algebras and assume that an L$_\infty$ quasi-morphism $\mathscr{F}^S : (V, Q^S) \to (V', Q'^S)$ has been defined. The mapping $\mathscr{M}\mathscr{F}^S_*$: $\mathscr{M}(V) \to \mathscr{M}(V')$ is an isomorphism.*

Proof. By application of the minimal model Theorem, V (resp. V') is L$_\infty$ isomorphic to the direct sum $V_m \oplus V_c$ (resp. $V'_m \oplus V'_c$) of a minimal L$_\infty$ algebra V_m (resp. V'_m) and a contractible L$_\infty$ algebra V_c (resp. V'_c). Composing the aforementioned two L$_\infty$ isomorphisms with the considered L$_\infty$ quasi-morphism $\mathscr{F}^S : (V, Q^S) \to (V', Q'^S)$, we get an L$_\infty$ quasi-morphism from $V_m \oplus V_c$ to $V'_m \oplus V'_c$, which will also be denoted by $\mathscr{F}^S$. The composition of the following L$_\infty$ quasi-morphisms

$$V_m \overset{\iota_m}{\hookrightarrow} V_m \oplus V_c \overset{\mathscr{F}^S}{\to} V'_m \oplus V'_c \overset{P'_m}{\twoheadrightarrow} V'_m,$$

provides an L$_\infty$ quasi-morphism $\mathscr{F}^{Sm} : V_m \to V'_m$, which is actually an L$_\infty$ isomorphism (see the proof of Theorem 8). Thus, the induced composition

$$\mathscr{M}(V_m) \overset{\mathscr{M}\iota_{m*}}{\hookrightarrow} \mathscr{M}(V_m) \times \mathscr{M}(V_c) \overset{\mathscr{M}\mathscr{F}^S_*}{\to} \mathscr{M}(V'_m) \times \mathscr{M}(V'_c) \overset{\mathscr{M}P_{m*}}{\twoheadrightarrow} \mathscr{M}(V'_m)$$

is the induced isomorphism $\mathcal{MF}^{Sm} : \mathcal{M}(V_m) \to \mathcal{M}(V'_m)$. Moreover, as seen in Example 5, both $\mathcal{M}(V_c)$ and $\mathcal{M}(V'_c)$ are trivial. Thus, $\mathcal{M}\iota_{m*}$ and $\mathcal{M}P_{m*}$ are two isomorphisms. This entails that $\mathcal{MF}^S_*$ is an isomorphism and completes the proof of the Theorem. ∎

3.4.1 Application: Moduli space of a canonical element

Let us consider a GLA $(\mathfrak{g}, \{-,-\})$ endowed with a e_1 degree canonical element π. Remember that the triple $(\mathfrak{g}, \partial_\pi, \{-,-\})$ is a DGLA with ∂_π being the Hamiltonian differential $\{\pi, -\}$, and that this DGLA structure induces a formal DGLA $(\mathfrak{g} \otimes \nu\mathbb{K}[[\nu]], \partial_\pi, \{-,-\})$ in an obvious manner. Remember also that a formal series

$$\pi_\nu = \pi + \sum_{i\geq 1} \nu^i \pi_i \in \mathfrak{g}^{e_1} \otimes \mathbb{K}[[\nu]]$$

is a formal deformation of π if and only if

$$\partial_\pi(\pi_p) + \tfrac{1}{2} \sum_{i+j=p} \{\pi_i, \pi_j\} = 0, \quad \forall p \geq 1,$$

i.e. if and only if

$$\partial_\pi C_\pi + \tfrac{1}{2}\{C_\pi, C_\pi\} = 0, \tag{3.17}$$

where C_π is the formal series

$$C_\pi := \sum_{i\geq 1} \nu^i \pi_i \in \mathfrak{g}^{e_1} \otimes \nu\mathbb{K}[[\nu]].$$

When modifying the formal DGLA structure of $\mathfrak{g} \otimes \nu\mathbb{K}[[\nu]]$ by setting

$$\{-,-\}^- := -1 \times \{-,-\}, \tag{3.18}$$

then, in view of equation (3.17), C_π is a Maurer Cartan element of the formal DGLA $\mathfrak{g}^- := (\mathfrak{g} \otimes \nu\mathbb{K}[[\nu]], \partial_\pi, \{-,-\}^-)$.

This allows concluding that the set of Maurer Cartan elements $MC(\mathfrak{g}^-)$ of the formal DGLA $\mathfrak{g}^-$ encodes all formal deformations of π.

Let us now consider two equivalent formal deformations π'_ν and π_ν of π in the sense of Definition 5, i.e. there is a formal series $\chi_\nu \in \mathfrak{g}^0 \otimes \mathbb{K}[[\nu]]$ so that

$$\pi'_\nu = \exp(\mathrm{ad}\,\chi_\nu)\,\pi_\nu = \exp(\mathrm{ad}^-\,\xi_\nu)\,\pi_\nu, \tag{3.19}$$

where $\xi_\nu = -\chi_\nu$ and $\mathrm{ad}^-\,\xi_\nu = \{\xi_\nu, -\}^-$. When writing $\pi'_\nu = \pi + C'_\pi$, it follows, as seen previously, that $C'_\pi \in \mathrm{MC}(\mathfrak{g}^-)$ and, moreover, C'_π satisfies

$$C'_\pi = \exp(\mathrm{ad}^-\,\xi_\nu)C_\pi + \sum_{i\geq 0} \tfrac{(\mathrm{ad}^-\,\xi_\nu)^i}{(i+1)!} \partial_\pi \xi_\nu.$$

This defines an equivalence between Maurer Cartan elements of $\mathrm{MC}(\mathfrak{g}^-)$. This equivalence is clearly similar to the gauge equivalence defined in equation (3.16) with the only difference being that the Lie algebra $\nu\mathfrak{g}^0[[\nu]]$ is not nilpotent. Nevertheless, we can formally define the gauge group as the set $G(\mathfrak{g}^-) := \exp(\mathfrak{g}^0 \otimes \nu\mathbb{K}[[\nu]])$ and introduce a well defined product taking (formally) the Baker-Campbell-Hausdorff formula. Therefore, the moduli space of the formal DGLA $\mathfrak{g}^-$ is the quotient $\mathrm{Def}(\mathfrak{g}^-) := \mathrm{MC}(\mathfrak{g}^-)/G(\mathfrak{g}^-)$ made up by the set of all classes of formal deformations of the canonical element π.

If $\mathfrak{g}$ is the GLA $(M_r(V), [-,-]^{\otimes})$ (resp. $(M(V), [-,-]^G)$, $(A(V), [-,-]^{NR})$, $(\mathrm{Diff}_1(V), [-,-]^{NR})$, $(\mathrm{Der}(V), [-,-]^{SN})$) and π is a graded Loday structure (resp. a graded associative structure, a graded Lie structure, a $0-$weight graded Jacobi structure, a $0-$weight graded Poisson structure) on V, then $\mathrm{Def}(\mathfrak{g}^-)$ represents the classes of all formal deformations of π, where $\mathfrak{g}^-$ is the formal DGLA $(\mathfrak{g} \otimes \nu\mathbb{K}[[\nu]], \partial_\pi, [-,-]^{\otimes -})$ (resp. $(M(V), \partial_\pi, [-,-]^{G-}), \ldots)$ with $[-,-]^{\otimes -} = -1 \times [-,-]^{\otimes}$ (resp. $[-,-]^{G-} = -1 \times [-,-]^G, \ldots$).
As it is easily understood from the above discussion, we modify the original GLA by multiplying the bracket by -1 in order to get the right sign in the Maurer Cartan Equation (3.8).

We have now seen two points of view on how to control formal deformations: firstly, the cohomological approach given in Section 2.2, and secondly, in terms of the corresponding modulo space. The second point of view has been revealed as being more efficient for building equivalence between two different deformation problems; more particularly, if the corresponding formal DGLAs of the two deformation problems are quasi-isomorphic as L_∞ algebras then, by application of Theorem 13, the corresponding moduli spaces of the two (formal) induced DGLAs are isomorphic. This was one of the main ingredients used by Kontsevich to build a one-to-one correspondence between formal Poisson structures and star products on Poisson manifolds, as we will discuss in the next Section.

3.5 Deformation quantization of Poisson manifolds

This Section aims to review the relationship between the above studied algebraic tools and Kontsevich's proof of the existence of deformation quantization on Poisson manifolds and their classifications.

Let us recall some necessary notions and preliminaries. Let M be a smooth manifold. Let $C^\infty(M)$ be the commutative associative algebra of $\mathbb{K}$-valued smooth functions on M and denote μ the pointwise multiplication of functions.

3.5.1 Formal Poisson structure

Definition 23. *A smooth manifold M is called a* Poisson manifold *if $(C^\infty(M), \{-,-\})$ is a $(0-$graded$)$ Poisson algebra.*

Proposition 27. *Poisson structures $\{-,-\}$ on $C^\infty(M)$ are in one-to-one correspondence with* Poisson bivectors fields*, i.e. smooth sections of $\Lambda \in \Gamma(M, \wedge^2 TM)$ satisfying $[\Lambda,\Lambda]^{SN} = 0$.*

Before proving this Proposition, let us remember the explicit formula of the Schouten-Nijenhuis bracket $[-,-]^{SN}$. For $X_i, Y_j \in \Gamma(M, \wedge^1 TM)$, the bracket of the two polyvectorfields $X_1 \wedge \ldots \wedge X_p \in \Gamma(M, \wedge^p TM)$ and $Y_1 \wedge \ldots \wedge Y_q \in \Gamma(M, \wedge^q TM)$, $p, q \geq 1$, is defined by

$$[X_1 \wedge \ldots \wedge X_p, Y_1 \wedge \ldots \wedge Y_q]^{SN}$$

$$= \sum_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge X_p \wedge Y_1 \wedge \ldots \wedge \widehat{Y_j} \wedge \ldots \wedge Y_q,$$

and for any $u \in C^\infty(M)$

$$[X_1 \wedge \ldots \wedge X_p, u]^{SN} = \sum_{1 \leq i \leq p} (-1)^{i+1} [X_i, u] \wedge X_1 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge X_p$$

where $[-,-]$ is the Lie bracket on vectorfields.

Proof. Any bivector $\Lambda \in T^1_{poly}$ defines a bracket $\{-,-\}$ on $C^\infty(M)$ via the pairing between exterior powers of the tangent and the cotangent space

$$\{u, v\} := \langle \Lambda, du \wedge dv \rangle, \quad u, v \in C^\infty(M),$$

where d denotes the exterior derivative. Clearly, bracket $\{-,-\}$ satisfies all the axioms of a Poisson algebra except for the Jacobi identity. But, if P is equal to $\Lambda = \sum_{i,j} \Lambda^{ij} \partial_i \wedge \partial_j$ in local coordinates x^i of M, where $\partial_i = \partial / \partial_{x^i}$, then

$$\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0$$

$$\Leftrightarrow \Lambda^{ir} \partial_r (\Lambda^{jk}) \left(\partial_i(u) \partial_j(v) \partial_k(w) + \partial_i(v) \partial_j(w) \partial_k(u) + \partial_i(w) \partial_j(u) \partial_k(v) \right) = 0$$

$$\Leftrightarrow \Lambda^{ir} \partial_r (\Lambda^{jk}) \partial_i \wedge \partial_j \wedge \partial_k = 0$$

$$\Leftrightarrow [\Lambda, \Lambda]^{SN} = 0. \blacksquare$$

We pointed out in Section 2.7.2 that the $\mathbb{Z}-$graded vector space of polyvector-fields

$$T_{poly}(M) := \bigoplus_{k\geq -1} T^k_{poly}, \quad T^k_{poly} = \Gamma(M, \wedge^{k+1}TM),$$

endowed with the Schouten-Nijenhuis bracket is a GLA. The GLA $T_{poly}(M)$ is then turned into a DGLA with the differential $d = 0$.

Definition 24. *A formal Poisson bivector Λ_ν is a formal series $\Lambda_\nu := \sum_{i\geq 1}\nu^i\Lambda_i \in T^1_{poly}\otimes \nu\mathbb{K}[[\nu]]$ satisfying*

$$[\Lambda_\nu, \Lambda_\nu]^{SN} = 0.$$

Mark that the first term Λ_1 of a formal Poisson bivector Λ_ν has to be a Poisson bivector field because if one expands the previous equality in terms of powers of ν, the lowest identity reads $[\Lambda_1, \Lambda_1]^{SN} = 0$.

As previously, let us denote by $T^-_{poly}(M)$ the DGLA $(T_{poly}(M)\otimes \nu\mathbb{K}[[\nu]], 0, [-,-]^{SN-})$ where $[-,-]^{SN-} = -1\times[-,-]^{SN}$.

Proposition 28. *The moduli space $\mathscr{M}(T^-_{poly}(M))$ is made up by classes of formal Poisson bivectors.*

Proof. Obviously, the set $MC(T^-_{poly}(M))$ is made up by formal Poisson bivectors. The gauge group in this case is $G(T^-_{poly}(M)) = \exp(T^0_{poly}\otimes\nu\mathbb{K}[[\nu]])$ and the action is given by formal vector fields $X_\nu \in T^0_{poly}\otimes\nu\mathbb{K}[[\nu]]$, where the action is as defined in (3.16), with the only difference being that the second term in the RHS vanishes because $d = 0$. ∎

The reader may have noticed that here, the multiplication of $[-,-]^{SN}$ by -1 was not applied in order to obtain the right sign in the Maurer Cartan equation, as $d = 0$, but rather so to end up with homogenous notations.

3.5.2 Star products on Poisson manifold

Definition 25. *[BFFLS78] Let (M, Λ) be a Poisson manifold. A* differential star product *(also called* deformation quantization*) on M is a bilinear map*

$$*: C^\infty(M)\times C^\infty(M) \to C^\infty(M)[[\nu]] \qquad (u,v)\to u*v := \mu(u,v)+\textstyle\sum_{r\geq 1}\nu^r C_r(u,v),$$

such that

1. *its $\mathbb{K}[[\nu]]$-bilinear extension is an associative product $(u*v)*w = u*(v*w)$;*

2. $C_1(u,v) - C_1(v,u) = 2\Lambda(du,dv)$;

3. *each C_r is a bidifferential operator vanishing on constants.*

Example 7. *The basic example of a star product is the Moyal product for a constant Poisson structure Λ on $\mathbb{R}^d$.*
Let

$$\Lambda = \Lambda^{ij}\partial_i \wedge \partial_j$$

where we are using the Einstein summation convention, $\partial_i = \partial/\partial_{x^i}$, and the x^i are coordinates on $\mathbb{R}^d$, $i = 1,\dots,d$.
The Poisson bivector Λ can be viewed as a bidifferential operator given by

$$\widetilde{\Lambda}(u,v) = 2\Lambda^{ij}(\overleftarrow{\partial}_i \overrightarrow{\partial}_j)(u,v)$$

where the operator $\overleftarrow{\partial}_i$ (resp. $\overrightarrow{\partial}_j$) acts on u (resp. on v). The Moyal product is then given by exponentiating $\frac{1}{2}\widetilde{\Lambda}$:

$$\begin{aligned} u *_M v &= \exp(\nu\Lambda^{ij}\overleftarrow{\partial}_i \overrightarrow{\partial}_j)(u,v) \\ &= \mu(u,v) + \nu\Lambda^{i_1 i_2}\overleftarrow{\partial}_{i_1}(u)\overrightarrow{\partial}_{i_2}(v) + \frac{\nu^2}{2!}\Lambda^{i_1 i_2}\Lambda^{i_3 i_4}\overleftarrow{\partial}_{i_1}\overleftarrow{\partial}_{i_3}(u)\overrightarrow{\partial}_{i_2}\overrightarrow{\partial}_{i_4}(v) + \dots. \end{aligned}$$

Definition 26. *Two differential star products $*$ and $*'$ on a Poisson manifold M are said to be equivalent if there exists a formal series*

$$T_\nu = \mathrm{id} + \textstyle\sum_{r\geq 1} \nu^r T_r,$$

where each T_r is a differential operator, such that

$$u *' v = T_\nu^{-1}(T_\nu(u) * T_\nu(v)) \tag{3.20}$$

for any $u,v \in C^\infty(M)$. Here, T_ν^{-1} stands for the inverse of the formal series T_ν.

Equivalence classes of a star product on a Poisson manifold can also be encoded in the moduli space of the DGLA of polydifferential operators. Recall that the $\mathbb{Z}-$graded vector space of *polydifferential operators* is

$$D_{poly}(M) = \bigoplus_{i\in\mathbb{Z}} D^i_{poly}$$

where

$$D^i_{poly} \subset M^{(0,i+1)}(\mathscr{A}), \quad \mathscr{A} := C^\infty(M)$$

consists of polydifferential operators acting on smooth functions of M. Be aware that $M^{(0,i+1)}$ denotes the space of $0-$weight $(i+1)-$multilinear maps and M stands for a smooth manifold.

Proposition 29. *The triple $(D_{poly}(M), \partial_\mu, [-,-]^G)$ is a differential graded Lie subalgebra of the DGLA $(M(\mathscr{A}), \partial_\mu, [-,-]^G)$, recalled in Section 2.2.2. We call $(D_{poly}(M), \partial_\mu, [-,-]^G)$ the DGLA of polydifferential operators.*

Proof. It is clear that $D_{poly}(M)$ is closed under the Gerstenhaber bracket $[-,-]^G$ and also under the Hochschild differential $\partial_\mu = [\mu,-]^G$, because μ is viewed as a differential operator of order 0. ∎

Let $D^-_{poly}(M)$ be the DGLA $(D_{poly}(M) \otimes \nu\mathbb{K}[[\nu]], \partial_\mu, [-,-]^{G-})$ where $[-,-]^{G-} := -1 \times [-,-]^G$.

Proposition 30. *The moduli space $\mathscr{M}(D^-_{poly}(M))$ represents equivalence classes of star products on M.*

Proof. Since μ is a canonical element of the GLA $D_{poly}(M)$, any star product on a Poisson M is simply a formal deformation of the pointwise product μ (in the direction of the Poisson bracket $\{-,-\}$). Hence, in view of subsection 3.4.1, the set $MC(D^-_{poly}(M))$ contains star products on the Poisson manifold M.
Observe now that for any formal series $T_\nu = \mathrm{id} + \sum_{r\geq 1} \nu^r T_r$ of differential operators, there exists a formal series $\xi_\nu = \sum_{r\geq 1} \nu^r \xi_r$ of differential operators so that $T_\nu = \exp(\xi_\nu)$. Hence, two star products $*$ and $*'$ are equivalent via the formal series T_ν if

$$u *' v = T_\nu^{-1}(T_\nu(u) * T_\nu(v)) = \exp(-\xi_\nu)(\exp(\xi_\nu)(u) * \exp(\xi_\nu)(v)).$$

But, as it easily checked, the previous equality entails

$$*' := \exp(\mathrm{ad}^- \xi_\nu) * .$$

Because we find that this result puts us in the exact situation of subsection 3.4.1, we can conclude the end of the Proof. ∎

If we now assume the existence of an L_∞ morphism

$$f^S \sim \mathscr{F}^S : (T_{poly}(M), 0, [-,-]^{SN-}) \sim (T_{poly}(M), Q^S_T)$$
$$\longrightarrow (D_{poly}(M), \partial_\mu, [-,-]^{G-}) \sim (D_{poly}(M), Q^S_D)$$

(where the notations are those of Section 3.1), then there is a canonical way to prove the existence of a star product of a given Poisson manifold (M, Λ).
Indeed, observe first that the L_∞ morphism $f^S \sim \mathscr{F}^S$ can be naturally extended to

the formal DGLA $T^{-}_{poly}(M) \sim (T_{poly}(M) \otimes \nu\mathbb{K}[[\nu]], Q^{S}_{T})$ by $\nu-$ linearity. Moreover, as $\nu\Lambda \in MC(T^{-}_{poly}(M))$, it follows from Proposition 22 that the element

$$\mathscr{F}^{S}_{*}(\nu \downarrow \Lambda) = \sum_{s=1}^{\infty} \uparrow \tfrac{\nu^s}{s!} \mathscr{F}^{S}_{s}(\downarrow \Lambda, \dots, \downarrow \Lambda)$$

is a Maurer Cartan element of the set $MC(D^{-}_{poly}(M))$; remember that this set encodes the existence of star products. Thus, we obtain a star product $*$ on the Poisson manifold (M, Λ), given by

$$* := \sum_{s=0}^{\infty} \tfrac{\nu^s}{s!} (-1)^{\frac{s(s-1)}{2}} f^{S}_{s}(\Lambda, \dots, \Lambda)$$

where f_0 is the pointwise product μ; remember that $\mathscr{F}^{S}_{s} = (-1)^{\frac{s(s-1)}{2}} \uparrow f^{S}_{s} \downarrow^{\vee s}$. Moreover, if $f^S \sim \mathscr{F}^S$ is an L_∞ quasi-isomorphism, it follows from Proposition 28, Proposition 30 and Theorem 13, that there is a one-to-one correspondence between formal Poisson structures and star products.

Therefore, after having constructed an L_∞ quasi-isomorphism from the DGLA $(T_{poly}(M), 0, [-,-]^{SN-})$ to the DGLA $(D_{poly}(M), \partial_\mu, [-,-]^{G-})$, the problem of existence and classification of star products on a Poisson manifold M is solved. Such an L_∞ quasi-isomorphism was constructed by Kontsevich [Kon03] through the so called formality Theorem.

Theorem 14. *There exists an* L_∞ *quasi-isomorphism from the DGLA* $(T_{poly}(M), 0, [-,-]^{SN-})$ *to the DGLA* $(D_{poly}(M), \partial_\mu, [-,-]^{G-})$.

Kontsevich proved this Theorem in two steps. First, he gave an explicit formula for an L_∞ quasi-isomorphism from the DGLA $(T_{poly}(M), 0, [-,-]^{SN-})$ to the DGLA $(D_{poly}(M), \partial_\mu, [-,-]^{G-})$ assuming that $M = \mathbb{R}^d$. He then globalized this formula on a general manifold using abstract arguments. The upcoming subsection will be concerned with outlining the first of Kontsevich's steps as well as looking at his explicit formula of a star product on $\mathbb{R}^d$. The second step will be elaborated in Chapter 4 through the angle of a more direct construction provided in [Dol05].

3.5.3 On the formality Theorem on $\mathbb{R}^d$

Kontsevich's star product on $\mathbb{R}^d$

Kontsevich's first main idea for constructing a star product on $\mathbb{R}^d$ was to provide a graphical representation for bidifferential operators. For the better comprehension of this procedure we go back to the basic example of the Moyal product introduced in Example 7. Its graphical illustration reveals the following figure

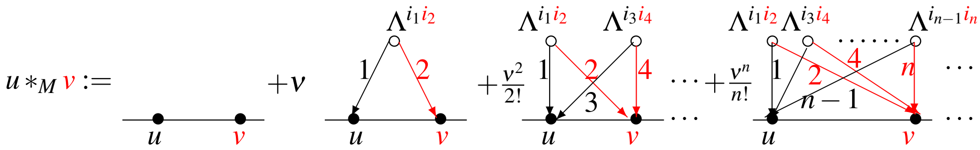

Figure 1: A graphical representation of the Moyal star product.

where $\Lambda^{i_\ell i_{\ell+1}}$ are the components of the Poisson bivector field Λ and where the left (resp. right) arrow $\swarrow$ (resp. $\searrow$) emerging from the vertex $\Lambda^{i_\ell i_{\ell+1}}$ represents $\overleftarrow{\partial}_{i_\ell}$ (resp. $\overrightarrow{\partial}_{i_{\ell+1}}$) acting on u (resp. on v).

When writing the Moyal product with a non-constant Poisson bivector, we loose the associativity already at the power 2 in the formal parameter ν. Nevertheless, it can be shown that a modification of the term in ν^2 of $*_M$ yields a star product on $(\mathbb{R}^d, \Lambda)$ up to order 2 given by

$$u *_2 v = \mu(u,v) + \nu \Lambda^{i_1 i_2} \partial_i(u) \partial_j(v) +$$
$$\frac{\nu^2}{2} \Lambda^{i_1 i_2} \Lambda^{i_3 i_4} \partial^2_{i_1 i_3}(u) \partial^2_{i_2 i_4}(v) + \frac{\nu^2}{3} \underbrace{\Lambda^{i_1 i_2} \partial_{i_2} \Lambda^{i_3 i_4} (\partial^2_{i_1 i_3} u \partial_{i_4} v + \partial_{i_4} u \partial^2_{i_1 i_3} v)}_{\mathfrak{D}}, \tag{3.21}$$

see [Kon03]. Hereafter, we omit the arrow in ∂.

The function (bidifferential operator) $\mathfrak{D}$ in the previous equation produces another type of graphs in which the arrows also end in vertices of type $\circ$:

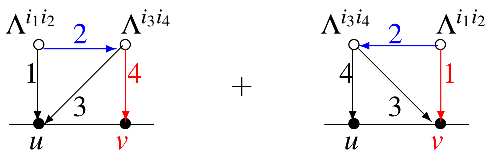

Figure 2

When trying to extend $*_2$ to the order 3 in ν by using the "classical" method, namely the cohomological obstruction method, one can easily see that any bidifferential operator that may appear in the formula of a star product, is represented by special graphs:

(a) All edges start from a vertex of the type $\circ$.

(b) There are no loops, i.e. edges start and end at the same vertex

(c) There are 8 vertices and 6 edges; (mark that the $p-$th term (graph) of the Moyal product contain $p+2$ vertices and $2p$ edges).

However, particular care has to be devoted to the numerical coefficients assigned to these graphs so that the associativity holds up to order 3. One has to take into account the Jacobi identity and symmetries; e.g. the Jacobi identity creates the factor $\frac{1}{3}$ in (3.21). Computations of these numerical coefficients are already quite long and delicate for the order 3; these coefficients can be found in Theorem 1 of Chapter 4, where we explicitly constructed a star product up to order 3 on a general Poisson manifold.
In view of the above discussion, a natural question arises:
Can we introduce an appropriate set of (suitable) graphs, and is there a natural way of assigning numerical coefficients or weights to graphs so that a weighted sum for all (suitable) graphs will yield a star product? The following Theorem by Kontsevich [Kon03] shows that the answer is yes.

Theorem 15. *Let Λ be a Poisson bivector field on $\mathbb{R}^d$. The formula*

$$u *_K v = \mu(u,v) + \sum_{p=1}^{\infty} \nu^p \sum_{\Gamma \in \mathscr{G}_{p,2}} W_\Gamma \mathfrak{f}_\Gamma(\underbrace{\Lambda, \dots, \Lambda}_{p})(u,v)$$

defines a star product on $(\mathbb{R}^d, \Lambda)$ for any $u, v \in C^\infty(\mathbb{R}^d)$.

In this formula, $\mathscr{G}_{p,2}$ denotes a subset of the set of graphs, the so called admissible graph, with $n+2$ vertices and $2n$ edges. The bidifferential operators $\mathfrak{f}_\Gamma(\Lambda, \dots, \Lambda)$ are constructed from the graph Γ, and W_Γ are numerical coefficients obtained as integrals over certain configuration spaces of a differential form depending on Γ.

Our focus turns to introducing the set of graphs $\mathscr{G}_{p,2}$ to a more general setting, namely to the context of the formality Theorem. We shall also look closely at the operators $\mathfrak{f}_\Gamma$ since we will require their descriptions in Chapter 4. For a detailed explanation of the configuration spaces and the coefficients W_Γ, we refer the reader to [AMM02]. The proof of the Theorem 15 was given in [Kon03], [AMM02] and many references therein.

Admissible graphs

Definition 27. *An* oriented graph Γ *is a pair $(V(\Gamma), E(\Gamma))$ of two finite sets such that $E(\Gamma)$ is a subset of $V(\Gamma) \times V(\Gamma)$.*

Elements of $V(\Gamma)$ are the vertices of Γ, and elements of $E(\Gamma)$ are its edges. For $e=(v_1,v_2)\in E(\Gamma)$ we say that e starts at v_1 and ends at v_2. The set of edges starting (resp. ending) at a given vertex v will be denoted by $\mathrm{Star}(v)$ (resp. $\mathrm{End}(v)$).

Definition 28. *The set $G_{p,q}$, $p,q\geq 0$ of* admissible graphs *consists of oriented graphs satisfying the following proprieties:*

- *The set of vertices $V(\Gamma)$ is decomposed into two ordered subsets $V_1(\Gamma)=\{a_1,a_2,\dots,a_p\}$ and $V_2(\Gamma)=\{b_1,b_2,\dots,b_q\}$ whose elements are called respectively vertices of the first type (aerial vertices) and vertices of the second type (ground vertices).*
- *The number of vertices of the two types verifies $2p+q-2\geq 0$.*
- *All edges in $E(\Gamma)$ start from a vertex of the first type.*
- *The edges emanating from a vertex a_ℓ can land on any vertex other than a_ℓ itself, i.e. for every $a_\ell\in V_1(\Gamma)$ the pair $(a_\ell,a_\ell)\notin E(\Gamma)$ (no loops).*
- *The set of edges E_Γ is endowed with a total order compatible with the order of vertices*
$$a_1<a_2<\ldots<a_p,\quad b_1<b_2<\ldots<b_q,$$
namely if $(\sharp(\mathrm{Star}(a_\ell))=t_\ell)$ for any $1\leq\ell\leq p$, then the starting edges at a_ℓ are labeled by numbers
$$t_1+t_2+\ldots+t_{\ell-1}+1,\dots,t_1+t_2+\ldots+t_\ell.$$

Example 8. *The first and the second graphs of Fig. 3 are admissible, while the others are not.*

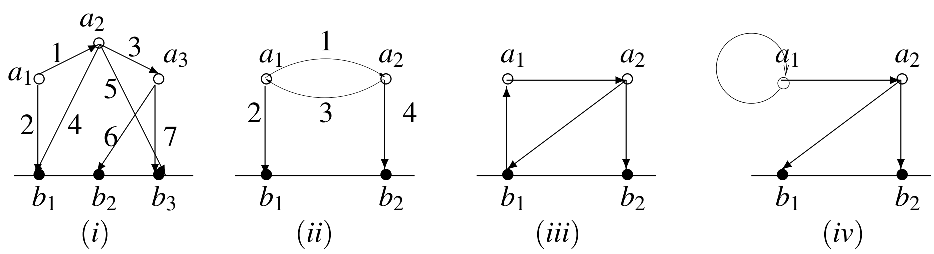

Figure 3: Some examples of admissible and non-admissible graphs.

Definition 29. *We denote by $\mathscr{G}_{p,q}$ the subclass of the set of admissible graphs with $2p+q-2$ edges, where $p\geq 1$ and $q\geq 0$ (and automatically $2p+q-2\geq 0$).*

Polydifferential operators associated to admissible graphs

Let $\Gamma \in \mathscr{G}_{p,q}$ be an admissible graph. Consider a collection of polydifferential operators $\Lambda_1, \ldots, \Lambda_p$ on $\mathbb{R}^d$ such that $\Lambda_\ell \in T^{k_\ell}_{poly}$ and $k_\ell = \sharp(\mathrm{Star}(a_\ell)) - 1$ for any $1 \leq \ell \leq p$, where $a_\ell \in V_1(\Gamma)$. For such data, we associate a polydifferential operator $\mathfrak{f}_\Gamma(\Lambda_1, \ldots, \Lambda_p) \in D^{q-1}_{poly}$ acting on q smooth functions $u_1, \ldots, u_q$ on $\mathbb{R}^d$ as follows:

1. We associate to each vertex $a_\ell \in V_1(\Gamma)$ the skew-symmetric tensor $\Lambda_\ell^{i_1 \ldots i_{k_\ell+1}}$ corresponding to the tensor Λ_ℓ via the natural identification.
2. We place a function u_ℓ at each vertex of the second type.
3. We label the edges of $\mathrm{Star}(\Lambda_\ell)$ for any $1 \leq \ell \leq p$ by the numbers $k_1 + \ldots + k_{\ell-1} + \ell, \ldots, k_1 + \ldots + k_\ell + \ell$, in other words, we identify $E(\Gamma)$ with
$$\{1, \ldots, k_1+1, k_1+2, \ldots, k_1+k_2+2, \ldots, k_1+\ldots+k_{p-1}+p, \ldots, k_1+\ldots+k_p+p\}$$
4. We associate to any vertex c_ℓ of the first type (i.e. Λ_ℓ) or of the second type (i.e. u_ℓ), the operator
$$\partial_{\mathrm{End}(c_\ell)} = \frac{\partial^s}{\partial_{i_{t_1}} \ldots \partial_{i_{t_s}}}$$
when $\mathrm{End}(c_\ell)$ is made up by the edges $t_s \in E(\Gamma)$, for any $1 \leq s \leq 2p+q-2$.
5. Finally, we define the function $\mathfrak{f}_\Gamma(\Lambda_1, \ldots, \Lambda_p)(u_1, \ldots, u_q)$ by
$$\begin{aligned} &\mathfrak{f}_\Gamma(\Lambda_1, \ldots, \Lambda_p)(u_1, \ldots, u_q) \\ &= \sum_{\substack{1 \leq i_1, \ldots, i_k \leq d \\ 1 \leq k \leq d}} \prod_{\ell=1}^{p} \partial_{\mathrm{End}(\Lambda_\ell)} \Lambda_\ell^{i_{k_1+\ldots+k_{\ell-1}+\ell}, \ldots, {}_{k_1+\ldots+k_\ell+\ell}} \prod_{s=1}^{q} \partial_{\mathrm{End}(u_j)} u_j. \end{aligned}$$

As an example, the polydifferential operator corresponding to the first graph (resp. to the second graph) of Fig. 3 is given by
$$\Lambda^{i_1 i_2} \partial_{i_1}(\Lambda^{i_3 i_4 i_5}) \partial_{i_3}(\Lambda^{i_6 i_7}) \partial^2_{i_2 i_4}(u_1) \partial_{i_6}(u_2) \partial^2_{i_5 i_7}(u_3)$$
(resp.
$$\partial_{i_3}(\Lambda^{i_1 i_2}) \partial_{i_1}(\Lambda^{i_3 i_4}) \partial_{i_2}(u_1) \partial_{i_4}(u_2).)$$

Thus, given any admissible graph $\Gamma \in \mathscr{G}_{p,q}$, the above procedure provides a multilinear map
$$\mathfrak{f}_\Gamma : \underbrace{T_{poly}(M) \times \ldots \times T_{poly}(M)}_{p} \to D^{q-1}_{poly}$$

that associates any collection of polyvectorfields $\Lambda_1, \ldots, \Lambda_p$, $\Lambda_\ell \in T^{k_\ell}_{poly}$ with a poly-differential operator $\mathfrak{f}_\Gamma(\Lambda_1, \ldots, \Lambda_p) \in D^{q-1}_{poly}$, which is given by the above detailed procedure if $k_\ell = \sharp(\mathrm{Star}(a_\ell)) - 1$ for any $1 \leq \ell \leq p$ or equal to zero otherwise. Moreover, it is easily seen that $\mathfrak{f}_\Gamma$ is a $(1-p)$ weight $\mathbb{Z}-$graded skew-symmetric map. Indeed, by definition we have

$$\textstyle\sum_{\ell=1}^p k_\ell = \sum_{\ell=1}^p (\sharp(\mathrm{Star}(a_\ell)) - 1) = 2p + q - 2 - p = p + q - 2;$$

so $\mathfrak{f}_\Gamma$ is a $(1-p)$ weight multilinear map because of the construction $\mathfrak{f}_\Gamma(\Lambda_1, \ldots, \Lambda_p) \in D^{q-1}_{poly}$. The map $\mathfrak{f}_\Gamma$ is a $\mathbb{Z}-$graded skew-symmetric map because when permuting the order in which we label the edges, we get a sign equal to the signature of the permutation; the polyvectorfields are of course skew-symmetric and the partial derivatives are "symmetric".

We are now prepared to state the Kontsevich formula for an L$_\infty$ quasi-isomorphism on $\mathbb{R}^d$.

Kontsevich's L$_\infty$ quasi-isomorphism on $\mathbb{R}^d$

Theorem 16. *For any $p \geq 1$, the $(1-p)$ weight $\mathbb{Z}-$graded skew-symmetric maps*

$$f_p^K = \sum_{q \geq 0} \sum_{\Gamma \in \mathscr{G}_{p,q}} W_\Gamma \mathfrak{f}_\Gamma \tag{3.22}$$

define an L$_\infty$ quasi-isomorphism

$$f^K = (f_1^K, f_2^K, \ldots) : (T_{poly}(\mathbb{R}^d), 0, [-,-]^{SN-}) \to (D_{poly}(\mathbb{R}^d), \partial_\mu, [-,-]^{G-}).$$

The proof of this theorem is based on the special choice of the numerical coefficients W_Γ, which are obtained as integrals over certain configuration spaces of a differential form depending on Γ. We will not describe these coefficients W_Γ and prove this Theorem. We refer again to [Kon03],[AMM02]. Let us just investigate the first structure map f_1^K.
By equation (3.22),

$$f_1^K = \sum_{q \geq 0} \sum_{\Gamma \in \mathscr{G}_{1,q}} W_\Gamma \mathfrak{f}_\Gamma.$$

Now, the set $\mathscr{G}_{1,q}$ contains only the graph Γ_q with one vertex of the first type, with $2 \times 1 + q - 2 = q$ edges and q ground vertices. Thus, for any polyvectorfield $\Lambda \in T^{q-1}_{poly}$, we associate the polydifferential operator given by

$$W_{\Gamma_q} \mathfrak{f}_{\Gamma_q}(\Lambda)(u_1, \ldots, u_q) = W_{\Gamma_q} \Lambda^{i_1 \ldots i_q} \partial_{i_1}(u_1) \ldots \partial_{i_q}(u_q).$$

Observe then that f_1^K is a 0 weight linear map from T_{poly} to D_{poly} since each $\mathfrak{f}_\Gamma$ is too for any $\Gamma \in \mathscr{G}_{1,q}$. It remains to prove that f_1^K induces an isomorphism between the corresponding cohomology, i.e.

$$f_{1\sharp}^K : T_{poly} = H(T_{poly}(\mathbb{R}^d), 0) \to H(D_{poly}(\mathbb{R}^d), \partial_\mu)$$

is an isomorphism. This is due to a previous result by Hochschild, Kostant and Rosenberg [HKR62] which establishes an isomorphism between $T_{poly}(\mathbb{R}^d)$ and $H(D_{poly}(\mathbb{R}^d), \partial_\mu)$.
For any $q \geq 0$ this isomorphism is given by the map

$$h_q(X_1 \wedge \ldots \wedge X_q)(u_1, \ldots, u_q) = \tfrac{1}{q!} \sum_{\sigma \in \mathbb{S}_q} (-1)^\sigma X_1(u_{\sigma(1)}) \ldots X_q(u_{\sigma(q)})$$

where $X_\ell \in T^0_{poly}$, $u_\ell \in C^\infty(\mathbb{R}^d)$ and where $\mathbb{S}_q$ stands for the group of permutation and $(-1)^\sigma$ denotes the signature of the permutation σ. Since it can be shown that $W_{\Gamma_q} = \frac{1}{q!}$, we conclude that $W_{\Gamma_q} \mathfrak{f}_{\Gamma_q} = h_q$.

Let us mention that this result was first proven for smooth affine algebraic varieties in [HKR62]. A smooth version of it was outlined in [Vey75] and a detailed proof was given in [GR99]. Independently, Kontsevich gave also another proof for smooth manifold in [Kon03].

Remark 11. *Note that the "formality" in the name of the Theorem stems from homotopy theory where a DGLA is called formal if it is quasi-isomorphic to its cohomology, regarded as a DGLA with zero differential and the induced bracket. In other words, the formality Theorem states that $D_{poly}(M)$ is formal for any smooth manifold M.*

In Chapter 4, we shall analyze Dolgushev's globalization procedure [Dol05] for the Kontsevich formality quasi-isomorphism f^K. This construction needs certain particular proprieties of the Kontsevich quasi-isomorphism f^K, which are presented in the following Theorem.

Theorem 17. *Kontsevich's quasi-isomorphism*

$$f^K : (T_{poly}(\mathbb{R}^d), 0, [-,-]^{SN-}) \to (D_{poly}(\mathbb{R}^d), \partial_\mu, [-,-]^{G-})$$

satisfies the following proprieties

1. *f^K is also an L_∞ quasi-isomorphism in the formal setting, i.e. one can replace $\mathbb{R}^d$ by the space $\mathbb{R}[[x^1,\dots,x^d]]$ of formal power series in $x = (x^1,\dots,x^d) \in \mathbb{R}^d$ with real coefficients, because the coefficients of the polydifferential operators f^K_ℓ are polynomial functions of the derivatives of the coordinates of the polyvectorfields.*

2. *f^K is equivariant with respect to linear transformations of the coordinates $x^1,\dots,x^d$.*

3. *If $p \geq 2$, then*
$$f^K_p(X_1, X_2, \dots, X_p) = 0 \tag{3.23}$$
for any set of vector fields $X_1, X_2, \dots, X_p \in T^0_{poly}$.

4. *If $n \geq 2$ and $X \in T^0_{poly}$ is linear in the coordinates $x^1,\dots,x^d$, then for any set of polyvectorfields $\Lambda_2,\dots,\Lambda_n$*
$$f^K_p(X, \Lambda_2, \dots, \Lambda_p) = 0. \tag{3.24}$$

Points (3) and (4) result from an explicit calculation of certain integrals over configuration spaces, see [Kon03].
As a concluding act, let us observe that the coefficients of the polydifferential operators $f^K_p(\Lambda_1,\dots,\Lambda_p)$ for any $p \geq 1$ are given by universal (admissible) polynomials in the polyvectorfields $\Lambda_1,\dots,\Lambda_p$ and their partial derivatives, where the concatenations are given by the subset $\mathscr{G}_{p,q}$, $q \geq 0$, of the admissible graph $G_{p,q}$; the concatenations only arise between different polyvectorfields because there are no loops, see Example 8.

Chapter 4

Universal Star Products

Throughout this Chapter, the notations employed are the same as those used in Chapter 3.

4.1 Introduction

Using Kontsevich's formality on $\mathbb{R}^d$, a construction of a star product on a d-dimensional Poisson manifold (M,Λ) was given by Cattaneo, Felder and Tomassini in [CFT02]. Given a torsionfree connection ∇ on (M,Λ) one builds an identification of the commutative algebra $C^\infty(M)$ of smooth functions on M with the algebra of flat sections of the jet bundle $E \to M$, for the Grothendieck connection D^G. The next point is to "quantize" this situation: a deformed algebra structure on $\Gamma(M,E)[[\nu]]$ is obtained through fiberwize quantization of the jet bundle using Kontsevich star product on $\mathbb{R}^d$, and a deformed flat connection D which is a derivation of this deformed algebra structure is constructed "à la Fedosov". Then one constructs an identification between the formal series of functions on M and the algebra of flat sections of this quantized bundle of algebras; this identification defines the star product on M. Later, Dolgushev [Dol05] gave in a similar spirit a construction for a Kontsevich formality quasi-isomorphism for a general smooth manifold. The construction starts again with a torsionfree linear connection ∇ on M and the identification of the commutative algebra $C^\infty(M)$ of smooth functions on M with the algebra of flat sections of the jet bundle $E \to M$, for a connection D^F constructed "à la Fedosov". This is extended to a resolution of the space $T_{poly}(M)$ of polyvectors on the manifold using the complexes of forms on M with values in the bundle of formal fiberwize polyvectorfields on E and a resolution of the space $D_{poly}(M)$ of polydifferential operators on the manifold using the complexes of forms on M with values in the bundle of formal fiberwize polydifferential operators on E. The fiberwize Kontsevich L_∞ quasi-morphism is then twisted and contracted to yield an L_∞ quasi-morphism from $T_{poly}(M)$ to $D_{poly}(M)$.

In the following, we introduce the notion of *universal formality L_∞ quasi-isomorphism* and *the notion of universal deformation quantization.*
Given a torsionfree linear connection ∇ on a manifold M, any polydifferential operator $\mathrm{Op}: C^\infty(M)^{\times k} \to C^\infty(M)$ writes in a unique way as

$$\mathrm{Op}(u_1,\ldots,u_k) = \sum_{I_1,\ldots,I_k} \mathrm{Op}^{I_1,\ldots,I_k} \nabla^{sym}_{I_1} u_1 \ldots \nabla^{sym}_{I_k} u_k \tag{4.1}$$

where the $I_1,\ldots,I_k$ are multiindices and $\nabla^{sym}_I u$ is the symmetrized covariant derivative of order $|I|$ of u:

$$\nabla^{sym}_I u = \sum_{\sigma\in S_m} \frac{1}{m!} \nabla^m_{i_{\sigma(1)}\cdots i_{\sigma(m)}} u \qquad \text{for } \mathrm{I} = (\mathrm{i}_1,\ldots,\mathrm{i}_\mathrm{m}),$$

where $\nabla^m_{i_1\ldots i_m} u := \nabla^m u(\partial_{i_1},\ldots,\partial_{i_m})$ with $\nabla^m u$ defined inductively by $\nabla u := d\,u$ and $\nabla^m u(X_1,\ldots,X_m) = (\nabla_{X_1}(\nabla^{m-1}u))(X_2,\ldots,X_m)$.
The tensors $\mathrm{Op}^{I_1,\ldots,I_k}$ are covariant tensors of order $|I_1|+\ldots+|I_k|$ which are symmetric within each block of I_r indices; they are called *the tensors associated* to Op for the given connection.

Definition 30. *For any integer $k \geq 1$, a* universal $k-$polyvectorfields-related polydifferential operator *will be the association to any manifold M, any torsionfree connection ∇ on M and any collection of polyvectorfields $\Lambda_1,\ldots,\Lambda_k \in T_{poly}(M)$, of a polydifferential operator $Op^{(M,\nabla,\Lambda_1,\ldots,\Lambda_k)} : C^\infty(M)^{\times j} \to C^\infty(M)$, so that, the tensors associated to $Op^{(M,\nabla,\Lambda_1,\ldots,\Lambda_k)}$ for ∇ are given by universal polynomials in $\Lambda_1,\ldots,\Lambda_k$, the curvature tensor R and their covariant multiderivatives, involving concatenations and the association being linear in each Λ_i.*
We shall say that a universal $k-$polyvectorfields-related polydifferential operator is of no-loop type *if the concatenations only arise between different terms, not within a given term (i.e.*

$$(\nabla_r\Lambda_\ell)^{\cdot\cdot i\cdot\cdot s\cdot\cdot}(\nabla_s\Lambda_{\ell'})^{\cdot\cdot j\cdot\cdot r\cdot\cdot}\nabla^{sym}_{\cdot\cdot i\cdot\cdot j\cdot\cdot}$$

is of no-loop type but

$$(\nabla_{\cdot\cdot t\cdot\cdot}\Lambda_\ell)^{\cdot\cdot t\cdot\cdot i\cdot\cdot}\nabla^{sym}_{\cdot\cdot i\cdot\cdot} \quad or \quad R^r_{str}\Lambda_\ell^{\cdot\cdot s\cdot\cdot i\cdot\cdot}\Lambda_{\ell'}^{\cdot\cdot t\cdot\cdot j\cdot\cdot}\nabla^{sym}_{\cdot\cdot i\cdot\cdot j\cdot\cdot}$$

are not).

Definition 31. *A* universal Poisson-related polydifferential operator *will be the association to any manifold M, any torsionfree connection ∇ on M, any Poisson tensor Λ on M and any integer $k \geq 1$, of a universal $k-$polyvectorfields-related polydifferential operator $Op^{(M,\nabla,\Lambda,\ldots,\Lambda)}$, which will be denoted by $Op^{(M,\nabla,\Lambda)}$.*
We shall say that a universal Poisson-related polydifferential operator is of no-loop type if $Op^{(M,\nabla,\Lambda)}$ is too.
We shall say that the universal Poisson-related polydifferential operator $Op^{(X,\nabla,\Lambda)}$ is a polynomial of degree r in the Poisson structure *if $r = k$.*

Definition 32. *A* universal formality L_∞ quasi-isomorphism *will be the association to any given manifold M and any torsionfree linear connection ∇ on M, of an L_∞ quasi-morphism $f = (f_1, f_2, \ldots)$ from the DGLA $(T_{poly}(M), 0, [-,-]^{SN-})$ to the DGLA $(D_{poly}(M), \partial_\mu, [-,-]^{G-})$, where, for any $k \geq 1$, each structure map f_k is a $k-$polyvectorfields-related polydifferential operator of no-loop type.*

Definition 33. *A* universal star product $* = \mu + \sum_{r\geq 1} \nu^r C_r$ *will be the association to any manifold M, any torsionfree connection ∇ on M and any Poisson tensor Λ on M, of a differential star product $*^{(M,\nabla,\Lambda)} := \mu + \sum_{r\geq 1} \nu^r C_r^{(M,\nabla,\Lambda)}$ where each C_r is a universal Poisson-related bidifferential operator of no-loop type, which is a polynomial of degree r in the Poisson structure.*

An example of a universal star product at order 3 is given in section 4.2. Unicity at order 3 is studied in section 4.3 using universal Poisson cohomology. This, we compute for universal Poisson-related bidifferential operators of order 1 in each argument defined by low order polynomials in the Poisson structure.
The existence of a universal formality L_∞ quasi-isomorphism and a universal star product are implied by the Dolgushev's globalization procedure [Dol05]. The globalization proof of Cattaneo, Felder and Tomassini [CFT02, CF01], using the exponential map of a torsionfree linear connection, gives also the existence of a universal star product. We show these existences in sections 4.5 and 4.6, stressing first the relations between the resolutions involved in the two constructions in section 4.4.

4.2 An example at order 3

Theorem 1. *There exists a universal star product up to order three, which associates to a Poisson manifold (M,Λ) and a torsionfree linear connection ∇ on M,* *the star product at order three defined by*

$$u\tilde{*}_3^{(M,\nabla,\Lambda)}v = \mu(u,v) + \nu\{u,v\} + \nu^2\tilde{C}_2^{(M,\nabla,\Lambda)}(u,v) + \nu^3\tilde{C}_3^{(M,\nabla,\Lambda)}(u,v), \quad u,v \in C^\infty(M) \tag{4.2}$$

$$\begin{aligned}&\tilde{C}_2^{(M,\nabla,\Lambda)}(u,v)\\ &= \tfrac{1}{2}\Lambda^{kr}\Lambda^{ls}\nabla^2_{kl}u\nabla^2_{rs}v + \tfrac{1}{3}\Lambda^{kr}\nabla_r\Lambda^{ls}(\nabla^2_{kl}u\nabla_s v + \nabla_s u\nabla^2_{kl}v) + \tfrac{1}{6}\nabla_l\Lambda^{kr}\nabla_k\Lambda^{ls}\nabla_r u\nabla_s v,\end{aligned} \tag{4.3}$$

and

$$\tilde{C}_3^{(M,\nabla,\Lambda)}(u,v) = \frac{1}{6}S_\nabla^{(M,\Lambda)\ 3}(u,v) = -\frac{1}{6}\Lambda^{ls}(\mathscr{L}_{X_u}\nabla)^j_{kl}(\mathscr{L}_{X_v}\nabla)^k_{js} \qquad \text{with } X_u = i(du)\Lambda, \tag{4.4}$$

where $\mathscr{L}_{X_u}\nabla$ is the tensor defined by the Lie derivative of the connection ∇ in the direction of the Hamiltonian vector field X_u

$$(\mathscr{L}_{X_u}\nabla)^j_{kl} = \Lambda^{ij}\nabla^3_{kli}u + \nabla_k\Lambda^{ij}\nabla^2_{li}u + \nabla_l\Lambda^{ij}\nabla^2_{ki}u + \nabla^2_{kl}\Lambda^{ij}\nabla_i u + R^j_{ikl}\Lambda^{si}\nabla_s u.$$

This can be seen by direct computation.

Remark 12. *The operator $S_\nabla^{(M,\Lambda)\ 3}$ was introduced by Flato, Lichnerowicz and Sternheimer [FLS76]; it is a Chevalley-cocycle on (M,Λ), i.e.*

$$\underset{u,v,w}{\circlearrowleft\!\!\!\!\bigoplus}\left\{S_\nabla^{(M,\Lambda)\ 3}(u,v),w\right\} + S_\nabla^{(M,\Lambda)\ 3}(\{u,v\},w) = 0,$$

where $\underset{u,v,w}{\circlearrowleft\!\!\!\!\bigoplus}$ denotes the sum over cyclic permutations of u,v,w.

For this universal star product at order 3, there exists a universal Poisson-related-differential-operator-valued 1-form D defined as follows:

Proposition 31. *Given any Poisson manifold (M,Λ), any torsionfree linear connection ∇ on M, and any vector field X on M, the differential operator $D_X^{(M,\nabla,\Lambda)}$ defined by*

$$D_X^{(M,\nabla,\Lambda)}v = Xv - \nu^2\frac{1}{6}\Lambda^{ls}(\mathscr{L}_X\nabla)^j_{kl}(\mathscr{L}_{X_v}\nabla)^k_{js}, \quad v \in C^\infty(M)$$

verifies at order 3 in ν

$$\begin{aligned}&D_X^{(M,\nabla,\Lambda)}(u\tilde{*}_3^{(M,\nabla,\Lambda)}v) - (D_X^{(M,\nabla,\Lambda)}u)\tilde{*}_3^{(M,\nabla,\Lambda)}v - u\tilde{*}_3^{(M,\nabla,\Lambda)}(D_X^{(M,\nabla,\Lambda)}v)\\ &= \tfrac{d}{dt}\big|_{t=0}u\tilde{*}_3^{(M,\nabla,\phi^X_{t*}\Lambda)}v + O(\nu^4).\end{aligned}$$

where ϕ_t^X denotes the flow of the vectorfield X.
If X is a Hamiltonian vector field corresponding to a function $u \in C^\infty(M)$, then $D_{X_u}^{(M,\nabla,\Lambda)}$ coincides with the inner derivation at order 3 of $\tilde{*}_3$ defined by the function u, i.e.

$$D_{X_u}^{(M,\nabla,\Lambda)} v = \frac{1}{2\nu}(u \tilde{*}_3^{(M,\nabla,\Lambda)} v - v \tilde{*}_3^{(M,\nabla,\Lambda)} u).$$

4.3 Equivalence of universal star products – Universal Poisson cohomology

Lemma 1.

- *Any universal star product $* = \mu + \sum_{r\geq 1} \nu^r C_r$ is a natural star product, i.e. each bidifferential operator C_r is of order at most r in each argument. Indeed C_r is a universal $r-$Poisson-related bidifferential operator; this implies, in view of the Bianchi's identities for the curvature tensor, that C_r is of order at most r in each argument.*
- *The universal Poisson-related bidifferential operator C_1 of any universal star product is necessarily the Poisson bracket $C_1^{(M,\nabla,\Lambda)} = \Lambda^{ij}\nabla_i \wedge \nabla_j$.*
- *The Gerstenhaber bracket $[-,-]^G$ of two universal Poisson-related polydifferential operator of degree k and l in Λ, is a universal Poisson-related polydifferential operator of degree $k+l$ in Λ.*
- *If a universal Poisson-related p-differential operator C is a Hochschild p-cocycle (where $\partial_\mu = [\mu,-]^G$ denotes the Hochschild differential) then $C = A + \partial_\mu B$ where A a universal Poisson-related p-differential operator which is of order 1 in each argument and is the totally skew-symmetric part of C, and where B is a universal Poisson-related $(p-1)$-differential operator.*

The last point comes from the explicit formulas [GR99] for the tensors associated to B in terms of those associated to C when one is given a connection.

Definition 34. *A* universal Poisson p-cocycle *is a universal Poisson-related p-differential skew-symmetric operator C of order 1 in each argument which is a cocycle for the Chevalley cohomology for the adjoint representation of $(C^\infty(M), \{-,-\})$, i.e. with the coboundary defined by*

$$\begin{aligned}\delta_\Lambda C(u_1,\dots,u_{m+1}) \quad &= \quad \sum_{i=1}^{m+1}(-1)^i\{u_i, C(u_1,\dots\hat{u}_i\dots,u_{m+1})\} \\ &\qquad + \sum_{i<j}(-1)^{i+j}C(\{u_i,u_j\},u_1\dots\hat{u}_i\dots\hat{u}_j\dots,u_{m+1}).\end{aligned}$$

which can be written as a multiple of

$$[\Lambda, C^{(M,\nabla,\Lambda)}]^{NR}$$

where skew indicates the skew-symmetrization in all its arguments of an operator. Equivalently, a universal Poisson p-cocycle C is defined by a universal Poisson related skew-symmetric p-tensor c (with $C(u_1,\ldots,u_p)=c(du_1,\ldots,du_p)$) so that

$$[\Lambda, c^{(M,\nabla,\Lambda)}]^{SN}=0$$

A universal Poisson p-cocycle C is a universal Poisson coboundary *if there exists a universal Poisson-related skew-symmetric $(p-1)$-differential operator C of order* 1 *in each argument so that*

$$C^{(M,\nabla,\Lambda)}=\delta_\Lambda B^{(M,\nabla,\Lambda)}(=[\Lambda,B^{(M,\nabla,\Lambda)}]^{NR});$$

(equivalently, if there exists a universal Poisson related tensor b so that $c^{(M,\nabla,\Lambda)}=[\Lambda,b^{(M,\nabla,\Lambda)}]^{SN}$).
The universal Poisson cohomology *H^p is the quotient of the space of universal Poisson p-cocycles by the space of universal Poisson p-coboundaries.*
We can restrict ourselves to the space of universal Poisson p-cocycles defined by polynomials of degree k in the Poisson structures and make the quotient by the space of universal Poisson coboundaries defined by polynomials of degree $k-1$. We speak then of the universal Poisson p-cohomology of degree k in the Poisson structure and we denote it by H^p_{polk}. We can further restrict ourselves to universal Poisson related tensors (or operators of order 1 *in each argument) of no-loop type.*

Definition 35. *If $*=\mu+\sum_{r\geq 1}\nu^r C_r$ is a universal star product and if $E=\sum_{r=2}^{\infty}\nu E_r$ is a formal series of universal differential operators vanishing on constants, of no-loop type, with each E_r a polynomial of degree r in the Poisson structure, then the series $*'$ defined by*

$$*'=(\exp \operatorname{ad} E)*$$

is an equivalent universal star product. We say that $*$ and $*'$ are universally equivalent.

Lemma 2. *If $*$ and $*'$ are universal star products which coincide at order k in the deformation parameter ν, then, by the associativity relation at order k, C'_k-C_k is a universal Hochschild* 2*-cocycle of no-loop type which is a polynomial of degree k in the Poisson structure. Furthermore, associativity at order $k+1$ implies that its skew-symmetric part p_2 is a universal Poisson* 2*-cocycle :*

$$\underset{u,v,w}{\circlearrowleft\!\!\!\!\!\bigoplus}\left\{p_2^{M,\nabla,\Lambda}(u,v),w\right\}+p_2^{M,\nabla,\Lambda}(\{u,v\},w)=0,$$

where $\underset{u,v,w}{\circlearrowleft\!\!\!\!\bigoplus}$ *denotes the sum over cyclic permutations of* u,v,w.

If it is a universal Poisson 2*-coboundary of no-loop type, then there is a a formal series* E *of universal differential operators vanishing on constants such that* $(\exp \operatorname{ad} E)*$ *and* $*'$ *coincide at order* $k+1$.

In particular, two universal star products are universally equivalent if $H^2_{(no-loop,pol)} = \{0\}$. *They are always equivalent at order* k *in the deformation parameter* ν *if* $H^2_{(no-loop)pol\,j} = \{0\}\ \forall 1 \le j \le k$.

Consider now any universal star product $* = \mu + \sum_{r\geq 1} \nu^r C_r$. We automatically have that C_1 is the Poisson bracket. Associativity at order 2 yields $\partial C_2 = \partial \tilde{C}_2^{(M,\nabla,\Lambda)}$ so

$$C_2(u,v) = \tilde{C}_2^{(M,\nabla,\Lambda)}(u,v) + p_2(u,v) + \partial E_2(u,v)$$

and the skew-symmetric part of associativity at order 3 yields that p_2 is a universal Poisson 2-cocycle (which is a polynomial of degree 2 in the Poisson structure).

Proposition 32. *The spaces* $H^2_{pol2}(\Lambda)$ *and* $H^2_{(no-loop)pol2}(\Lambda)$ *of universal Poisson* 2*-cohomology of degree* 2 *in the Poisson structure vanish.*

Proof. The universal skew-symmetric 2-tensors of degree 2 in Λ are combinations of

$$\begin{aligned}
&\nabla_s\Lambda^{ir}\nabla_r\Lambda^{js}\nabla_i\wedge\nabla_j,\\
&(\nabla^2_{rs}\Lambda^{ir}\Lambda^{js} - \nabla^2_{rs}\Lambda^{jr}\Lambda^{is})\nabla_i\wedge\nabla_j,\\
&(\Lambda^{ir}\Lambda^{st}R^j_{rst} - \Lambda^{jr}\Lambda^{st}R^i_{rst})\nabla_i\wedge\nabla_j,\\
&\Lambda^{ri}\Lambda^{sj}R^t_{rst}\nabla_i\wedge\nabla_j.
\end{aligned}$$

The only universal cocycles are the multiples of

$$\Lambda^{ri}\Lambda^{sj}R^t_{rst}\nabla_i\wedge\nabla_j$$

and those are the boundaries of the multiples of $\nabla_r\Lambda^{ir}\partial_i$. Remark that there are no cocycles of no-loop type. ∎

Thus, universal star product at order 2 are unique modulo equivalence and one can assume that $C_2 = \tilde{C}_2^{(M,\nabla,\Lambda)}$. Then the skew-symmetric part of the Hochschild 2-cocycle $C_3 - \tilde{C}_3^{(M,\nabla,\Lambda)}$ is a universal Poisson 2-cocycle of no-loop type which is a polynomial of degree 3 in the Poisson structure.

Proposition 33. *The space $H^2_{(no-loop)pol3}(\Lambda)$ of universal Poisson 2-cohomology of no-loop type and of degree 3 in the Poisson structure vanishes.*

Proof. We consider all possible universal Poisson 2-cochains of no loop type which are polynomials of degree 3 in the Poisson structure. They are defined by universal skew-symmetric 2-tensors of degree 3 in Λ which are combinations with constant coefficients of the different concatenations (with no loops) of

$$\begin{array}{lll}
\Lambda^{\cdot\cdot}\Lambda^{\cdot\cdot}\Lambda^{\cdot\cdot}(\nabla^2_{\cdot\cdot}R)^{\cdot}_{\cdots} & \Lambda^{\cdot\cdot}\Lambda^{\cdot\cdot}\Lambda^{\cdot\cdot}R^{\cdot}_{\cdots}R^{\cdot}_{\cdots}, & \\
\Lambda^{\cdot\cdot}(\nabla_{\cdot}\Lambda)^{\cdot\cdot}(\nabla_{\cdot}\Lambda)^{\cdot\cdot}R^{\cdot}_{\cdots} & \Lambda^{\cdot\cdot}\Lambda^{\cdot\cdot}(\nabla_{\cdot}\Lambda)^{\cdot\cdot}(\nabla_{\cdot}R)^{\cdot}_{\cdots} & \\
\Lambda^{\cdot\cdot}\Lambda^{\cdot\cdot}(\nabla^2_{\cdot\cdot}\Lambda)^{\cdot\cdot}R^{\cdot}_{\cdots} & \Lambda^{\cdot\cdot}(\nabla^2_{\cdot\cdot}\Lambda)^{\cdot\cdot}(\nabla^2_{\cdot\cdot}\Lambda)^{\cdot\cdot} & (\nabla^2_{\cdot\cdot}\Lambda)^{\cdot\cdot}(\nabla_{\cdot}\Lambda)^{\cdot\cdot}(\nabla_{\cdot}\Lambda)^{\cdot\cdot}.
\end{array}$$

Using the symmetry properties of R, the Bianchi's identities and the fact that Λ is a Poisson tensor, one is left with a combination with constant coefficients of 49 independent terms.
Universal 2-coboundaries come from the boundaries of universal 1-tensors of degree 2 in Λ; such 1-tensors are given by combinations with constant coefficients of concatenations of

$$\Lambda^{\cdot\cdot}\Lambda^{\cdot\cdot}(\nabla_{\cdot}R)^{\cdot}_{\cdots} \qquad \Lambda^{\cdot\cdot}(\nabla_{\cdot}\Lambda)^{\cdot\cdot}R^{\cdot}_{\cdots}.$$

Hence, modulo universal coboundaries, one can assume that the coefficients of 4 of the 49 terms in a universal cochain are zero.
The cohomology that we are looking for is then given by the combinations with constant coefficients of the remaining 45 terms which are cocycles (for all possible choices of manifold, Poisson structure Λ and connection ∇.)
Let C be a combination of those 45 terms. The cocycle condition is $[\Lambda, C]^{SN} = 0$. We plug in examples of Poisson structures and connections and impose this cocycle condition. This shows that all 45 coefficients must vanish.
It is enough, for instance, to consider the example on $\mathbb{R}^4$, with the non vanishing coefficients of the connection defined by

$$\begin{array}{l}
\Gamma^1_{12} = x_1^3, \quad \Gamma^1_{14} = x_4 \quad \Gamma^2_{11} = x_1^2, \quad \Gamma^2_{13} = 1, \quad \Gamma^2_{22} = 1, \quad \Gamma^2_{14} = x_3, \quad \Gamma^3_{13} = -x_4, \\
\Gamma^3_{33} = 1, \quad \Gamma^3_{44} = -x_2x_3x_4, \quad \Gamma^4_{11} = 1, \quad \Gamma^4_{13} = 1, \quad \Gamma^4_{22} = x_1, \quad \Gamma^3_{44} = -3.
\end{array}$$

and the quadratic Poisson structure defined by

$$\Lambda = \sum_{1=i<j=4} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

From this example, one gets that 41 of the 45 coefficients have to vanish. One is left with a combination with constant coefficients of four terms and an example with constant Poisson structure in dimension 7 shows that all those coefficients must vanish. The non vanishing coefficients of this example are:

$$\begin{array}{lllllll} \Lambda^{12}=1 & \Lambda^{15}=1 & \Lambda^{17}=1 & \Lambda^{25}=1 & \Lambda^{26}=2 & \Lambda^{27}=2 & \\ \Lambda^{34}=1 & \Lambda^{37}=3 & \Lambda^{46}=1 & \Lambda^{47}=4 & \Lambda^{56}=1 & \Lambda^{57}=5 & \Lambda^{67}=6. \end{array}$$

$$\begin{array}{llllll} \Gamma^1_{12}=1 & \Gamma^1_{62}=x_7 & \Gamma^1_{77}=-1 & \Gamma^2_{17}=x_1 & \Gamma^3_{13}=x_6 & \Gamma^4_{11}=1 \\ \Gamma^4_{22}=1 & \Gamma^4_{33}=2\Gamma^4_{77}=3 & \Gamma^5_{12}=x_1x_5 & \Gamma^5_{33}=x_2 & \Gamma^6_{11}=1 & \\ \Gamma^6_{44}=x_2 & \Gamma^6_{44}=x_5 & \Gamma^7_{11}=x_7 & \Gamma^7_{44}=x_3 & \Gamma^7_{17}=x_1.\ \blacksquare & \end{array}$$

Corollary 3. *Any universal star product is universally equivalent to one whose expression at order* 3 *is given by formula (4.2).*

4.4 Grothendieck- and Dolgushev-resolution of the space of functions

Our purpose in this section is to prove that the Fedosov-resolution of the algebra of smooth functions constructed in Dolgushev [Dol05] coincides with its resolution given by Cattaneo, Felder and Tomassini in [CFT02]. We also give explicitly the identification of smooth functions, polyvectorfields and polydifferential operators on M with flat sections in the corresponding bundles.

Let M be a d-dimensional manifold and consider the *jet bundle* $E \to M$ (the bundle of infinite jet of functions) with fibers $\mathbb{R}[[y^1,\dots,y^d]]$ (i.e. formal power series in $y \in \mathbb{R}^d$ with real coefficients) and transition functions induced from the transition functions of the tangent bundle TM. Thus

$$E = F(M) \times_{\mathrm{Gl}(d,\mathbb{R})} \mathbb{R}[[y^1,\dots,y^d]] \tag{4.5}$$

where $F(M)$ is the frame bundle. Remark that E can be seen as the formally completed symmetric algebra of the cotangent bundle T^*M; a section $s \in \Gamma(M,E)$ can be written in the form

$$s = s(x;y) = \sum_{p=0}^{\infty} s_{i_1\dots i_p}(x) y^{i_1}\cdots y^{i_p}$$

with repeated indices varying from 1 to d, and where the $s_{i_1\dots i_p}$ are components of symmetric covariant tensors on M. This bundle E is denoted $\mathscr{S}M$ by Dolgushev.

The construction of a star product on a d-dimensional Poisson manifold (M,Λ) given by Cattaneo, Felder and Tomassini in [CFT02], using a linear torsionfree connection ∇ on the manifold M, starts with the identification of the commutative algebra $C^\infty(M)$ of smooth functions on M with the algebra $\mathscr{Z}^0(\Gamma(M,E),D^G)$ of flat sections of the jet bundle $E \to M$, for the Grothendieck connection D^G (which is constructed using ∇). Let us recall this construction.

The exponential map for the connection ∇ gives an identification

$$\exp_x : U \cap T_xM \to M \quad y \mapsto \exp_x(y) \tag{4.6}$$

at each point x, of the intersection of the tangent space T_xM with a neighborhood U of the zero section of the tangent bundle TM with a neighborhood of x in M.

To a function $u \in C^\infty(M)$, one associates the section u_ϕ of the jet bundle $E \to M$ given, for any $x \in M$ by the Taylor expansion at $0 \in T_xM$ of the pullback $u \circ \exp_x$.

Lemma 3. *The section u_ϕ is given by:*

$$u_\phi(x;y) = u(x) + \sum_{n>0} \frac{1}{n!} \nabla^n_{i_1\dots i_n} u(x)\, y^{i_1}\dots y^{i_n} = u(x) + \sum_{n>0} \frac{1}{n!} \nabla^{n,sym}_{i_1\dots i_n} u(x)\, y^{i_1}\dots y^{i_n}. \tag{4.7}$$

Proof. In local coordinates x^i's one has

$$\frac{d}{dt} u(\exp_x ty) = \sum_{k=1}^{d} (\partial_{x^k} u)(\exp_x ty)\, \frac{d}{dt}(\exp_x ty)^k$$

and

$\frac{d^2}{dt^2} u(\exp_x ty)$

$= \sum_{k,l} (\partial^2_{x^k x^l} u)(\exp_x ty)\, \frac{d}{dt}(\exp_x ty)^k\, \frac{d}{dt}(\exp_x ty)^l + \sum_{k=1}^d (\partial_{x^k} u)(\exp_x ty) \frac{d^2}{dt^2}(\exp_x ty)^k.$

The definition of the exponential map imply that

$$\frac{d^2}{dt^2}(\exp_x ty)^k = -\sum_{r,s} \Gamma^k_{rs}(\exp_x ty)\, \frac{d}{dt}(\exp_x ty)^r\, \frac{d}{dt}(\exp_x ty)^s \tag{4.8}$$

hence

$$\frac{d^2}{dt^2} u(\exp_x ty) = \sum_{k,l} (\nabla^2_{kl} u)(\exp_x ty)\, \frac{d}{dt}(\exp_x ty)^k\, \frac{d}{dt}(\exp_x ty)^l.$$

By induction, one gets

$$\frac{d^n}{dt^n}u(\exp_x ty) = \sum_{k_1,\dots,k_n} (\nabla^n_{k_1\dots k_n}u)(\exp_x ty)\,\frac{d}{dt}(\exp_x ty)^{k_1}\dots\frac{d}{dt}(\exp_x ty)^{k_n}$$

and the result follows at $t=0$. ∎

Definition 36. *[CF01] The Grothendieck connection D^G on E is defined by:*

$$D^G_X s(x;y) := \frac{d}{dt}_{|t=0} s(x(t);\exp^{-1}_{x(t)}(\exp_x(y))) \tag{4.9}$$

for any curve $t \to x(t) \in M$ representing $X \in T_xM$ and for any $s \in \Gamma(M,E)$. It is locally given by

$$D^G_X = \sum_{i=1}^d X^i \left(\partial_{x^i} + \sum_k \sum_j \left(\frac{\partial \phi_x}{\partial y}^{-1}\right)^k_j \frac{\partial \phi^j}{\partial x^i}\partial_{y^k}\right) \tag{4.10}$$

where $\phi_x(y) = \phi(x,y)$ is the Taylor expansion of $\exp_x y$ *at $y=0$:*

$$\phi(x,y)^k = x^k + y^k$$

$$-\tfrac{1}{2}\textstyle\sum_{rs}\Gamma^k_{rs}(x)y^r y^s + \tfrac{1}{3!}\sum_{rst}\left(-(\partial_{x^r}\Gamma^k_{st})(x) + 2\sum_u \Gamma^u_{rs}(x)\Gamma^k_{ut}(x)\right) y^r y^s y^t + O(y^4).$$

Remark 13. *• From the definition (4.9) it is clear that D^G is flat ($D^G_X \circ D^G_Y - D^G_Y \circ D^G_X = D^G_{[X,Y]}$).*
• It is also obvious that $D^G(u_\phi) = 0\ \forall u \in C^\infty(M)$.

Lemma 4. *[CF01] Introducing the operator on E-valued forms on M*

$$\delta = \sum_i dx^i \frac{\partial}{\partial y^i}, \tag{4.11}$$

one can write

$$D^G = -\delta + \nabla' + A, \tag{4.12}$$

where

$$\nabla' = \sum_i dx^i \left(\partial_{x^i} - \sum_{jk}\Gamma^k_{ij}y^j\,\partial_{y^k}\right) \tag{4.13}$$

is the covariant derivative on E associated to ∇ and where A is a 1-form on M with values in the fiberwize vectorfields on E,

$$A(x;y) =: \sum_{ik} dx^i A^k_i(x;y)\,\partial_{y^k} = \sum_{ik} dx^i\left(-\frac{1}{3}\sum_{rs}R^k_{ris}(x)y^r y^s + 0(y^3)\right)\partial_{y^k}. \tag{4.14}$$

One extends as usual the operator D^G to the space $\Omega(M,E)$ of E-valued forms on M:

$$D^G = -\delta + \nabla' + A \ \text{ with } \nabla' = d - \sum_{ijk} dx^i \Gamma^k_{ij} y^j \partial_{y^k}. \tag{4.15}$$

One introduces the operator $\delta^* = \sum_j y^j i(\frac{\partial}{\partial x^j})$ on $\Omega(M,E)$. Clearly $(\delta^*)^2 = 0, \delta^2 = 0$ and for any $\omega \in \Omega^q(M,E_p)$, i.e. a q-form of degree p in y, we have $(\delta\delta^* + \delta^*\delta)\omega = (p+q)\omega$.
Defining, for any $\omega \in \Omega^q(M,E_p)$

$$\begin{aligned} \delta^{-1}\omega &= \frac{1}{p+q}\delta^*\omega && \text{when } p+q \neq 0 \\ &= 0 && \text{when } p = q = 0 \end{aligned}$$

we see that any δ-closed q-form ω of degree p in y, when $p+q>0$, writes uniquely as $\omega = \delta\sigma$ with $\delta^*\sigma = 0$; σ is given by $\sigma = \delta^{-1}\omega$.
One proceeds by induction on the degree in y to see that the cohomology of D^G is concentrated in degree 0 and that any flat section of E is determined by its part of degree 0 in y. Indeed a q-form ω is D^G-closed if and only if $\delta\omega = (\nabla' + A)\omega$; this implies that $\delta\omega_p = 0$ for ω_p the terms of lowest order (p) in y. When $p+q>0$ we can write $\omega_p = \delta(\delta^{-1}\omega_p)$ and $\omega - D^G(\delta^{-1}\omega_p)$ has terms of lowest order at least $p+1$ in y. Remark that given any section s of E then $s(x; y=0)$ determines a smooth function u on M. If $D^G s = 0$, then $s - u_\phi$ is still D^G closed. By the above, its terms of lowest order in y must be of the form $\delta\sigma$ hence must vanish since we have a 0-form. Hence we have:

Lemma 5. *[CF01] Any section of the jet bundle $s \in \Gamma(E)$ is the Taylor expansion of the pullback of a smooth function u on M via the exponential map of the connection ∇ if and only if it is horizontal for the Grothendieck-connection D^G:*

$$s = u_\phi \text{ for a } u \in C^\infty(M) \Leftrightarrow s \in \Gamma_{hor}(E) := \{\, s' \in \Gamma(E) \,|\, D^G s' = 0 \,\}. \tag{4.16}$$

Furthermore, the cohomology of D^G is concentrated in degree 0. In other word, one obtains a "Grothendieck-resolution" of the algebra of smooth functions, i.e.

$$H^\bullet(\Omega(M,E), D^G) = H^0(\Omega(M,E), D^G) = \Gamma_{hor}(E) \cong C^\infty(M).$$

Remark 14. *When a q-form ω is D^G exact, we have written $\omega = D^G\sigma$ where the tensors defining σ are given by universal polynomials (with no-loop concatenations) in the tensors defining ω, the tensors defining A, the curvature of the connection, and their iterated covariant derivatives.*

Lemma 6. *The* 1*-form A on M with values in the fiberwize vectorfields on E is given by $A(x;y) =: \sum_{ik} dx^i A_i^k(x;y)\,\partial_{y_k}$ where the A_i^k are universal polynomials given by (no-loop) concatenations of iterative covariant derivatives of the curvature; they are of the form*

$$\sum (\nabla_{\dots} R)_{i\cdot\cdot}^{j_1} (\nabla_{\dots} R)_{j_1\cdot\cdot}^{j_2} \dots (\nabla_{\dots} R)_{j_{s-1}\cdot\cdot}^{k}\, y^{\cdot} \dots y^{\cdot}. \tag{4.17}$$

In particular $\delta^{-1}A = 0$ since the curvature is skew-symmetric in its first two lower arguments. The 1*-form A is uniquely characterized by the fact that $\delta^{-1}A = 0$ and the fact that $D^G = -\delta + \nabla' + A$ is flat, i.e; $\left(D^G\right)^2 = 0$ which is equivalent to*

$$\delta A = R^{\nabla'} + \nabla' A + \frac{1}{2}[A,A] \tag{4.18}$$

for

$$\frac{1}{2}[A,A](X,Y) := [A(X),A(Y)] \quad \textit{and} \quad R^{\nabla'} = -\frac{1}{2} R_{ijk}^{l} dx^i \wedge dx^j y^k \frac{\partial}{\partial y^l}.$$

Proof. Any section $s \in \Gamma(M,E)$ writes $\sum_{p=0}^{\infty} s_{i_1 \dots i_p}^{p}(x) y^{i_1} \cdots y^{i_p}$ with symmetric p-covariant tensors $s_{i_1 \dots i_p}^{p}$. Write

$$A(x;y) = \sum_{r \geq 2} dx^i (A^{(r)}(x))_{i,j_1 \dots j_r}^{k} y^{j_1} \dots y^{j_r} \partial_{y^k}$$

with $(A^{(2)}(x))_{i,rs}^{k} = -\frac{1}{3}\sum_{rs} R_{ris}^{k}(x) y^r y^s$. Then the covariant tensors of $D_X^G s$ are given by the symmetrization of

$$\left(D_X^G s\right)^p = -i(X) s^{p+1} + \nabla_X s^p + \sum_{r=0}^{p-2} (A^{(p-r)}(X))^k\, \partial_{y^k} s^{r+1}.$$

The fact that $D^G(u_\phi) = 0\ \forall u \in C^\infty(M)$ implies the expression given in the lemma for A. Indeed, the symmetric tensors defining u_ϕ are given by $\frac{1}{p!}\nabla^{p,sym} u$ and we must have

$$\begin{aligned} 0 \;=\; \left(D_{\partial_{x_i}}^G u_\phi\right)_{j_1 \dots j_p}^{p} &= -\frac{1}{(p+1)!} (\nabla^{p+1,sym} u)_{ij_1 \dots j_p} + \frac{1}{p!} \left(\nabla(\nabla^{p,sym} u)\right)_{ij_1 \dots j_p} \\ &\quad + \sum_{r=0}^{p-2} \left(A^{(p-r)}(x)\right)_{i,j_1 \dots j_{p-r}}^{k} \frac{1}{r!} (\nabla^{r+1,sym} u)_{kj_{p-r+1} \dots j_p} \end{aligned}$$

with the last terms symmetrized in the $j's$. The commutation of covariant derivatives of a q-form ω gives

$$(\nabla^{p+2}u)_{klj_1\dots j_p} - (\nabla^{p+2}u)_{lkj_1\dots j_p} = -\sum_{r=0}^{p} R^s_{klj_r}(\nabla^p u)_{j_1\dots j_{r-1}sj_{r+1}\dots j_p}$$

and implies by induction that $(\nabla(\nabla^{p,sym}u))_{ij_1\dots j_p} - (\nabla^{p+1,sym}u)_{ij_1\dots j_p}$ is a universal expression contracting covariant derivatives of the curvature tensor with lower covariant derivatives of u of the form

$$(\nabla_{\dots}R)^{t_1}_{i\cdot\cdot}(\nabla_{\dots}R)^{t_2}_{t_1\cdot\cdot}\dots(\nabla_{\dots}R)^{s}_{t_{s-1}\cdot\cdot}(\nabla^{r+1,sym}u)_{s\dots}$$

with the $j's$ put in a symmetrized way at the $\cdot$'s, and for $0 \le r \le p-2$. Hence the expression for A.

Observe that $d\delta + \delta d = 0$ and also $\delta\nabla' + \nabla'\delta = 0$ since ∇ is torsionfree. Hence $\left(D^G\right)^2 = 0$ if and only if $-\delta A + R^{\nabla'} + \nabla' A + \frac{1}{2}[A,A]$ vanishes on all sections of E; since it is a 2-form on M with values in the fiberwize vectorfields on E, this must vanish. ∎

Dolgushev [Dol05] gave in a similar spirit a construction for a Kontsevich's formality quasi-isomorphism for a general smooth manifold. The construction starts again with a torsionfree linear connection ∇ on M. A resolution (called Fedosov's resolution in Dolgushev's paper) of the algebra of functions is given using the complex of algebras $(\Omega(M,E), D_F)$ for a flat connection (differential) D_F defined by

$$D_F := \nabla' - \delta + A \tag{4.19}$$

where A is a 1-form on M with values in the fiberwize vectorfields on E, obtained by induction on the order in y by the equation

$$A = \delta^{-1}R^{\nabla'} + \delta^{-1}(\nabla' A + \frac{1}{2}[A,A]). \tag{4.20}$$

This implies that $\delta^{-1}A = 0$ and $\delta A = R^{\nabla'} + \nabla' A + \frac{1}{2}[A,A]$ so that A coincides with the 1-form already considered. Hence

Lemma 7. *The differential D^G and D_F coincide.*

Similarly, Dolgushev defined a resolution of polydifferential operators and polyvectorfields on M using the complexes $(\Omega(M,\mathscr{D}_{poly}), D_F^{\mathscr{D}_{poly}})$ and

$(\Omega(M,\mathscr{T}_{poly}), D_F^{\mathscr{T}_{poly}})$ where $\mathscr{T}_{poly}$ is the bundle of formal fiberwize polyvector-fields on E and $\mathscr{D}_{poly}$ is the bundle of formal fiberwize polydifferential operators on E. A section of $\mathscr{T}^k_{poly}$ is of the form

$$\mathscr{P}(x;y) = \sum_{n=0}^{\infty} \mathscr{P}^{j_1\dots j_{k+1}}_{i_1\dots i_n}(x) y^{i_1}\dots y^{i_n} \frac{\partial}{\partial y^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_{k+1}}}, \tag{4.21}$$

where $\mathscr{P}^{j_1\dots j_{k+1}}_{i_1\dots i_n}(x)$ are coefficients of tensors, symmetric in the covariant indices $i_1,\dots,i_n$ and antisymmetric in the contravariant indices $j_1,\dots,j_{k+1}$. A section of $\mathscr{D}^k_{poly}$ is of the form

$$\mathscr{O}(x;y) = \sum_{n=0}^{\infty} \mathscr{O}^{\alpha_1\dots\alpha_{k+1}}_{i_1\dots i_n}(x) y^{i_1}\dots y^{i_n} \frac{\partial^{|\alpha_1|}}{\partial y^{\alpha_1}} \otimes \dots \otimes \frac{\partial^{|\alpha_{k+1}|}}{\partial y^{\alpha_{k+1}}}, \tag{4.22}$$

where the α_l are multi-indices and $\mathscr{O}^{\alpha_1\dots\alpha_{k+1}}_{i_1\dots i_n}(x)$ are coefficients of tensors symmetric in the covariant indices $i_1,\dots,i_n$ and symmetric in each block of α_i contravariant indices.

The spaces $\Omega(M,\mathscr{T}_{poly})$ and $\Omega(M,\mathscr{D}_{poly})$ have a formal fiberwize DGLA structure. Namely, the degree of an element in $\Omega(M,\mathscr{T}_{poly})$ (resp. $\Omega(M,\mathscr{D}_{poly})$) is defined by the sum of the degree of the exterior form and the degree of the polyvector field (resp. the polydifferential operator), the bracket on $\Omega(M,\mathscr{T}_{poly})$ is defined by $[\omega_1\otimes\mathscr{P}_1, \omega_2\otimes\mathscr{P}_2]^{SN} := (-1)^{k_1 q_2}\omega_1\wedge\omega_2\otimes[\mathscr{P}_1,\mathscr{P}_2]^{SN}$ for ω_i a q_i form and $\mathscr{P}_i$ a section in $\mathscr{T}^{k_i}_{poly}$ and similarly for $\Omega(M,\mathscr{D}_{poly})$ using the Gerstenhaber bracket. The differential on $\Omega(M,\mathscr{T}_{poly})$ is 0 and the differential on $\Omega(M,\mathscr{D}_{poly})$ is defined by $\partial := [\mu_{pf}, -]^G$ where μ_{pf} is the fiberwize multiplication of formal power series in y of E.

Definition 37. *[Dol05] The differential $D_F^{\mathscr{T}_{poly}}$ is defined on $\Omega(M,\mathscr{T}_{poly})$ by*

$$D_F^{\mathscr{T}_{poly}}\mathscr{P} := \nabla^{\mathscr{T}_{poly}}\mathscr{P} - \delta^{\mathscr{T}_{poly}}\mathscr{P} + [A,\mathscr{P}]^{SN} \tag{4.23}$$

where

$$\nabla^{\mathscr{T}_{poly}}\mathscr{P} = d\mathscr{P} - \left[\sum_{ijk} dx^i \Gamma^k_{ij} y^j\, \partial_{y_k}, \mathscr{P}\right]^{SN}$$

and where

$$\delta\mathscr{P} = \left[\sum_i dx^i \frac{\partial}{\partial y^i}, \mathscr{P}\right]^{SN}.$$

Similarly $D_F^{\mathscr{D}_{poly}}$ is defined on $\Omega(M,\mathscr{D}_{poly})$ by

$$D_F^{\mathscr{D}_{poly}}\mathscr{O} := \nabla^{\mathscr{D}_{poly}}\mathscr{O} - \delta^{\mathscr{D}_{poly}}\mathscr{O} + [A,\mathscr{O}]^G \tag{4.24}$$

with $\nabla^{\mathscr{D}_{poly}}$ and $\delta^{\mathscr{D}_{poly}}$ defined as above with the Gerstenhaber bracket.
Again the cohomology is concentrated in degree 0 and a flat section $\mathscr{P} \in \mathscr{T}_{poly}$ or $\mathscr{O} \in \mathscr{D}_{poly}$ is determined by its terms $\mathscr{P}_0$ or $\mathscr{O}_0$ of order 0 in y; it is defined inductively by

$$\mathscr{P} = \mathscr{P}_0 + \delta^{-1}\left(\nabla^{\mathscr{T}_{poly}}\mathscr{P} + [A,\mathscr{P}]^{SN}\right) \qquad \mathscr{O} = \mathscr{O}_0 + \delta^{-1}\left(\nabla^{\mathscr{D}_{poly}}\mathscr{O} + [A,\mathscr{O}]^{G}\right).$$

On the other hand, if $s_1 \dots s_{k+1}$ are sections of E, we have for a $\mathscr{P} \in \Gamma(M,\mathscr{T}_{poly})$:

$$\begin{aligned} & D_F\left(\mathscr{P}(s_1,\dots,s_{k+1})\right) \\ &= (D_F^{\mathscr{T}_{poly}}\mathscr{P})(s_1,\dots,s_{k+1}) + \mathscr{P}(D_F s_1,\dots,s_{k+1}) + \cdots + \mathscr{P}(s_1,\dots,D_F s_{k+1}) \end{aligned} \tag{4.25}$$

and similarly for a $\mathscr{O} \in \Gamma(M,\mathscr{D}_{poly})$.

Definition 38. *[CFT02] As in Cattaneo et al. we associate to a polyvector field $F \in T^k_{poly}$ a section $P_\phi \in \Gamma(M,\mathscr{T}_{poly})$: for a point $x \in M$ one considers the Taylor expansion (infinite jet) $P_\phi(x;y)$ at $y=0$ of the push-forward $(\exp_x)_*^{-1}P(\exp_x y)$. Clearly this definition implies that $X_\phi(u_\phi) = (Xu)_\phi$ so that P_ϕ is uniquely determined by the fact that*

$$P_\phi(u^1_\phi,\dots,u^{k+1}_\phi) = \Big(P(u^1,\dots,u^{k+1})\Big)_\phi \qquad \forall u^j \in C^\infty(M). \tag{4.26}$$

Similarly we associate to a differential operator $O \in D^k_{poly}$ a section $O_\phi \in \Gamma(M,\mathscr{D}_{poly})$ determined by the fact that

$$O_\phi(u^1_\phi,\dots,u^{k+1}_\phi) = \Big(O(u^1,\dots,u^{k+1})\Big)_\phi \qquad \forall u^j \in C^\infty(M). \tag{4.27}$$

Observe that $D_F^{\mathscr{T}_{poly}}P_\phi = 0$ by 4.25 and 4.26 and similarly $D_F^{\mathscr{D}_{poly}}O_\phi = 0$, hence we have

Proposition 34. *A section of $\mathscr{T}_{poly}$ is $D_F^{\mathscr{T}_{poly}}-$horizontal if and only if is a Taylor expansion of a polyvectorfield on M, i.e. if and only if it is of the form P_ϕ for some $P \in T^k_{poly}$.*
Similarly a section of $\mathscr{D}_{poly}$ is $D_F^{\mathscr{D}_{poly}}-$horizontal if and only if is of the form O_ϕ for

some $O \in D^k_{poly}$.
The terms in such a flat section are defined by tensors which are universal polynomials (involving concatenations of no-loop type) in the tensors defining the polyvectorfield (or differential operator) on M, the curvature tensor and their iterated covariant derivatives.

Observe also that a D_F closed section $s \in \Omega^q(M,E)$ or $\mathscr{P} \in \Omega^q(M,\mathscr{T}_{poly})$ or $\mathscr{O} \in \Omega^q(M,\mathscr{D}_{poly})$ for $q \geq 1$ is the boundary of a section defined by tensors which are given by universal polynomials (involving concatenations of no-loop type) in the tensors defining the section, the curvature tensor and their iterated covariant derivatives.

The isomorphisms obtained are isomorphisms of differential graded Lie algebras.

4.5 Construction of a universal star product

The Cattaneo, Felder and Tomassini construction of a star product on any Poisson manifold consists of quantizing the identification of the commutative algebra of smooth functions on M with the algebra of flat sections of E in the following way. A deformed algebra structure on $\Gamma(M,E)[[\nu]]$ is obtained through fiberwize quantization of the jet bundle using Kontsevich's star product on $\mathbb{R}^d$. Precisely, one considers the fiberwize Poisson structure on E defined by Λ_ϕ and, in view of point (1) of Theorem 17, the fiberwize Kontsevich star product on $\Gamma(M,E)[[\nu]]$:

$$\sigma *_K^{\Lambda_\phi} \tau = \mu_{pf}(\sigma,\tau) + \sum_{p=1}^{\infty} \frac{\nu^n}{p!}(-1)^{\frac{p(p-1)}{2}} f_p^K(\Lambda_\phi,\ldots,\Lambda_\phi)(\sigma,\tau).$$

The operator D_X^G is not a derivation of this deformed product; one constructs a flat connection D which is a derivation of $*_K^{\Lambda_\phi}$. One defines first

$$D_X^1 = X + \sum_{j=0}^{\infty} \frac{\nu^j}{j!}(-1)^{\frac{j(j-1)}{2}} f_{j+1}^K(\hat{X},\Lambda_\phi,\ldots,\Lambda_\phi) \tag{4.28}$$

where $\hat{X} := D_X^G - X$ is a vertical vectorfield on E. The formality equations (L_∞ morphisms conditions (2.52)) imply that D_X^1 is a derivation of the star product. Using the fact that $f_1(\xi) = \xi$ for any vector field ξ and that, for $n \geq 2$, the maps $f_n(\xi,\alpha_2,\ldots,\alpha_n) = 0$ if ξ is a linear vector field (see point (4) of Theorem 17), we see that

$$D_X^1 = D_X^G + \sum_{j=1}^{\infty} \frac{\nu^j}{j!}(-1)^{\frac{j(j-1)}{2}} f_{j+1}^K(\hat{\hat{X}},\Lambda_\phi,\ldots,\Lambda_\phi) \tag{4.29}$$

where $\hat{\hat{X}} = \sum_i X^i(-\partial_{y^i} + \sum_k A_i^k(x;y)\,\partial_{y^k})$ as defined in equation 4.14, so that it is given by universal polynomials (with no-loop type concatenations) in the tensors defining X, the curvature tensor and their iterated derivatives. The connection D^1 is not flat so one deforms it by

$$D := D^1 + [\gamma, \cdot]_{\overset{\wedge\phi}{*_K}}$$

so that D is flat. The 1-form γ is constructed inductively using the fact that the cohomology of D^G vanishes.
The next point is to identify series of functions on M with the algebra of flat sections of this quantized bundle of algebras to define the star product on M.
This is done by building a map $\rho : \Gamma(M,E)[[\nu]] \to \Gamma(M,E)[[\nu]]$ so that $\rho \circ D^G = D \circ \rho$. This map is again constructed by induction using the vanishing of the cohomology.
All these points show that the star product constructed in this way is universal.

4.6 Construction of a universal formality L_∞ quasi-isomorphism

Dolgushev's formality L_∞ quasi-isomorphism was obtained in two steps from the fiberwize Kontsevich formality from $\Omega(M, \mathscr{T}_{poly})$ to $\Omega(M, \mathscr{D}_{poly})$ building first a twist that depends only on the curvature and its covariant derivatives, then building a contraction by using the vanishing of the D_F cohomology. Below, we give more details of this construction and show that the Dolgushev L_∞ quasi-isomorphism is a universal formality L_∞ quasi-isomorphism.

In view of point (1) of Theorem 17, the local Kontsevich L_∞ quasi isomorphism f^K on $\mathbb{R}^d$ gives rise to a fiberwize quasi-isomorphism

$$\widetilde{f}^K : (\Omega(M, \mathscr{T}_{poly}), 0, [-,-]^{SN-}) \to (\Omega(M, \mathscr{D}_{poly}), \mu_{pf}, [-,-]^{G-})$$

such that the structure maps $\widetilde{f}_n^K$ are given by

$$\widetilde{f}_n(\mathscr{P}_1 \otimes \zeta_1, \ldots, \mathscr{P}_n \otimes \zeta_n) = (-1)^{\sum_{i=1}^{n-1} |\zeta_i|(|\mathscr{P}_{i+1}|+\ldots+|\mathscr{P}_n|-n+i)} f_n^K(\mathscr{P}_1, \ldots, \mathscr{P}_n)\zeta_1 \wedge \ldots \wedge \zeta_n$$

for any $\mathscr{P}_i \otimes \zeta_i \in \Omega^{|\zeta_i|}(M, \mathscr{T}_{poly}^{|\mathscr{P}_i|})$. Here, $[-,-]^{SN-} := -1 \times [-,-]^{SN}$ and $[-,-]^{G-} := -1 \times [-,-]^{G}$; remember that we change the DGLA structure in order to get the right sign in MCE (3.8).

Take a contractible coordinates neighborhood W of M. It is easily seen that d commutes with both the fiberwize DGLA structures of $\Omega(W,\mathscr{T}_{poly})$ and $\Omega(W,\mathscr{D}_{poly})$. Hence, both $(\Omega(W,\mathscr{T}_{poly}),d,[-,-]^{SN-})$ and $(\Omega(W,\mathscr{D}_{poly}),d+\partial_{\mu_{pf}},[-,-]^{G^-})$ are fiberwize DGLAs. Hence, $\widetilde{f}^K$ obviously extends to an L_∞ morphism

$$\widetilde{f}^W:(\Omega(W,\mathscr{T}_{poly}),d,[-,-]^{SN-})\to(\Omega(W,\mathscr{D}_{poly}),d+\partial_{\mu_{pf}},[-,-]^{G^-}).$$

However, as W is contractible, then the L_∞ morphism $\widetilde{f}^W$ is actually an L_∞ quasi-isomorphism.
We now apply the twisting procedure described in Proposition 24 to the L_∞ quasi-isomorphism $\widetilde{f}^W$. This goal obviously requires to have a Maurer Cartan of the DGLA $(\Omega(W,\mathscr{T}_{poly}),d,[-,-]^{SN-})$. Write

$$D_F^{\mathscr{T}_{poly}}=d-[B,-]^{SN-},$$

where $B\in\Omega^1(M,\mathscr{T}^0_{poly})$ and

$$B=-\textstyle\sum_i dx^i\frac{\partial}{\partial y^i}-\sum_{ijk}dx^i\Gamma^k_{ij}y^j\,\partial_{y^k}+\sum_{r\geq 2}dx^iA^k_{i,j_1\dots j_r}y^{j_1}\dots y^{j_r}\partial_{y^k}. \tag{4.30}$$

As the connection $D_F^{\mathscr{T}_{poly}}$ is flat ($(D_F^{\mathscr{T}_{poly}})^2=0$), then B is a Maurer Cartan element (as vector field) of the DGLA $(\Omega(W,\mathscr{T}_{poly}),d,[-,-]^{SN-})$. Thus, in view of Proposition 22, we obtain a Maurer Cartan element of the DGLA $(\Omega(W,\mathscr{D}_{poly}),d+\partial_{\mu_{pf}},[-,-]^{G^-})$ defined by

$$B'=\sum_{n\geq 1}\frac{1}{p!}(-1)^{\frac{p(p-1)}{2}}\widetilde{f}^W_p(B,\dots,B).$$

But, as B is a valued 1-form vectorfield, point (3) of Theorem 17 implies that $B'=B$. Using now the twisted procedure described in Proposition 24 and Example 4, the twisted quasi-isomorphism of $\widetilde{f}^W$ by the Maurer Cartan element B is the L_∞ quasi-isomorphism

$$\widetilde{f}^{WB}:(\Omega(W,\mathscr{T}_{poly}),D_F^{\mathscr{T}_{poly}},[-,-]^{SN-})\to(\Omega(W,\mathscr{D}_{poly}),D_F^{\mathscr{D}_{poly}}+\partial_{\mu_{pf}},[-,-]^{G-}),$$

with structure maps

$$\widetilde{f}^{WB}_p(\mathscr{P}_1,\dots,\mathscr{P}_n)=\widetilde{f}^W_p(\mathscr{P}_1,\dots,\mathscr{P}_p)+\sum_{s\geq 1}\frac{1}{s!}\widetilde{f}^W_{p+s}(\underbrace{B,\dots,B}_{s},\mathscr{P}_1,\dots,\mathscr{P}_p)$$

for any $\mathscr{P}_\ell\in\Omega(W,\mathscr{T}_{poly})$.
Moreover, observe that B transforms upon a change of coordinates by adding

a $1-$form valued in linear vector fields, because all the coefficients in its expression are tensorial except Γ_{ij}^k. This, together with point (4) of Theorem 17, entails that $\widetilde{f}^{WB}$ is independent of the choice of the coordinates neighborhood W and also that, for any $p \geq 1$, the polydifferential operators $\widetilde{f}_p^{WB}(\mathscr{P}_1, \ldots, \mathscr{P}_p)$ are $p-$polyvetorfields-related universal polydifferential operators of no-loop type. Eventually, using the Fedosov resolution of $T_{poly}(M)$, we therefore obtain an L_∞ quasi-isomorphism

$$f : (T_{poly}(M), 0, [-,-]^{SN-}) \to (\Omega(M, \mathscr{D}_{poly}), D_F^{\mathscr{D}_{poly}} + \partial_{\mu_{pf}}, [-,-]^{G-})$$

such that $f_p(\Lambda_1, \ldots, \Lambda_p)$ are $p-$polyvetorfields-related universal polydifferential operators of no-loop type.

The second step of Dolgushev's construction consists in contracting f to an L_∞ quasi-isomorphism taking values in the space of $D_F^{\mathscr{D}_{poly}}-$closed sections $H^0(\Omega(M, \mathscr{D}_{poly}), D_F^{\mathscr{D}_{poly}})$. Observe that once we succeed with this contraction, we automatically obtain the desired L_∞ quasi-isomorphism

$$f^D : (T_{poly}(M), 0, [-,-]^{SN-}) \to (D_{poly}(M), \partial_\mu, [-,-]^{G-})$$

because $H^0(\Omega(M, \mathscr{D}_{poly}), D_F^{\mathscr{D}^{poly}})$ is isomorphic via the Fedosov resolution to $D_{poly}(M)$.
The contraction of f is based on the following general result.

Lemma 6. *[Dol05] Let (V, π) and (V', π') be two ($\mathbb{Z}-$graded) L_∞ algebras and assume that an L_∞ quasi-morphism $f = (f_1, f_2, \ldots) : (V, \pi) \to (V, \pi')$ has been defined. For any $p \geq 1$ and any $(-p)$ weight $\mathbb{Z}-$graded skew-symmetric map*

$$h : V^{\times p} \to V'$$

one can construct an L_∞ quasi-morphism

$$\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2, \ldots) : (V, \pi) \to (V, \pi')$$

such that $\widetilde{f}_\ell = h_\ell$ for any $\ell < p$,

$$\widetilde{f}_p(v_1, \ldots, v_p) = f_p(v_1, \ldots, v_p)$$

$$+\pi_1 h(v_1, \ldots, v_p) - \sum_{\ell=1}^{p} (-1)^{p+v_1+\ldots+v_{\ell-1}} h(v_1, \ldots v_{\ell-1}, \pi_1(v_\ell), v_{\ell+1}, \ldots, v_p)$$

and

$$\widetilde{f}_\ell = f_\ell + g_\ell$$

for any $\ell > p$, where $g_\ell : V^{\times p} \to V'$ are $(1-\ell)$ weight $\mathbb{Z}-$graded skew-symmetric maps given in terms of h, f_s, π_s and π'_s for $s \leq \ell$.

This result can easily be obtained from the L_∞ morphism conditions (2.52).

We now come back our main task and modify the L_∞ quasi-isomorphism

$$f : (T_{poly}(M), 0, [-,-]^{SN-}) \to (\Omega(M, \mathscr{D}_{poly}), D_F^{\mathscr{D}_{poly}} + \partial_{\mu_{pf}}, [-,-]^{G-})$$

to an L_∞ quasi-isomorphism taking values in the space $H^0(\Omega(M, \mathscr{D}_{poly}), D)$. Firstly, we modify the first structure map f_1 and then proceed by induction on the order p of the structure maps f_p.
For $p = 1$, the L_∞ morphism conditions (2.52) implies that

$$(D_F^{\mathscr{D}_{poly}} + \partial_{\mu_{pf}}) f_1(\Lambda) = 0 \tag{4.31}$$

for any $\Lambda \in T_{poly}(M)$. When decomposing f_1 w.r.t. the exterior degrees as

$$f_1 = \sum_{\ell=1}^{d} (f_1)^\ell,$$

equation (4.31) entails that $D_F^{\mathscr{D}_{poly}}(f_1^d(\Lambda)) = 0$ since the manifold M is $d-$dimensional. But, as the cohomology of $D_F^{\mathscr{D}_{poly}}$ is concentrated in degree zero, there exists a map $h_1^d : T_{poly} \to \Omega^{d-1}(M, \mathscr{D}_{poly})$ such that

$$f_1(\Lambda) + (D_F^{\mathscr{D}_{poly}} + \partial_{\mu_{pf}}) h_1^d(\Lambda)$$

is of maximal exterior degree $q_{max} < d$. Proceeding in this way, we can construct a $(-1)-$weight map $h : T_{poly} \to \Omega(M, \mathscr{D}_{poly})$ such that

$$f_1(\Lambda) + (D_F^{\mathscr{D}_{poly}} + \partial_{\mu_{pf}})(h_1(\Lambda))$$

is of exterior degree zero. Applying now Lemma 6, we can modify the L_∞ quasi-isomorphism f_1 to an L_∞ quasi-isomorphism $\widetilde{f}$ such that the first structure map is given by

$$\widetilde{f}_1(\Lambda) := f_1(\Lambda) + (D_F^{\mathscr{D}_{poly}} + \partial_{\mu_{pf}}) h_1(\Lambda).$$

Finally, as it is easily seen that $D_F^{\mathscr{D}_{poly}}(\widetilde{f}_1(\Lambda)) = 0$, we therefore obtain an L_∞ quasi-isomorphism $\widetilde{f}$ with $D_F^{\mathscr{D}_{poly}}-$closed first structure map $\widetilde{f}_1$.

Suppose now that for any $1 \leq \ell < p$ the structure maps $\widetilde{f}_\ell$ of the L_∞ quasi-isomorphism $\widetilde{f}$ are $D_F^{\mathscr{D}_{poly}}-$closed. It follows from the L_∞ morphism conditions (2.52) that

$$(D_F^{\mathscr{D}_{poly}} + \partial_{\mu_{pf}})(\widetilde{f}_p(\Lambda_1 \ldots, \Lambda_p)) =$$

$$\frac{1}{2} \sum_{\substack{r+s=p \\ r,s\geq 1}} \sum_{\sigma \in \mathbb{S}_{r,p}} \tfrac{(-1)^\theta}{r!s!} (-1)^\sigma \varepsilon_\Lambda(\sigma) \left[\widetilde{f}_r(\Lambda_{\sigma(1)}, \ldots, \Lambda_{\sigma(r)}), \widetilde{f}_s(\Lambda_{\sigma(r+1)}, \ldots, \Lambda_{\sigma(p)}) \right]^{G-}$$

$$+ \sum_{1 \leq r < s \leq p} (-1)^{s-r} (-1)^\eta \widetilde{f}_{p-1}([\Lambda_r, \Lambda_s]^{SN-}, \Lambda_1, \ldots, \hat{\Lambda}_r, \ldots, \hat{\Lambda}_s, \ldots, \Lambda_p),$$

where $\mathbb{S}_{r,p}$ is the set of unshuffles $\sigma \in \mathbb{S}_p$ with $\sigma(1) < \ldots < \sigma(r)$, $\sigma(r+1) < \ldots < \sigma(p)$ and where

$$\theta = r + (s+1)(|\Lambda_1| + \ldots + |\Lambda_r|),$$

$$\eta = (|\Lambda_r| + |\Lambda_s|)(|\Lambda_1| + \ldots + |\Lambda_{r-1}|) + |\Lambda_s|(|\Lambda_{r+1}| + \ldots + |\Lambda_{s-1}|).$$

By the hypothesis of induction, the RHS of the previous equality has a zero exterior degree. Thus, by using similar arguments as applied above, one can construct a $(-p)$ weight $\mathbb{Z}-$graded skew-symmetric map

$$h_p : T_{poly}(M)^{\times p} \to \Omega(M, \mathscr{D}_{poly})$$

so that

$$\widetilde{f}_p(\Lambda_1, \ldots, \Lambda_p) + (D_F^{\mathscr{D}_{poly}} + \partial_{\mu_{pf}}) h_p(\Lambda_1 \ldots, \Lambda_p)$$

is of exterior degree zero. Using again Lemma 6, we can modify the L_∞ quasi-isomorphism $\widetilde{f}$ to an L_∞ quasi-isomorphism $\widetilde{\widetilde{f}}$ such that the $p-$th structure map is $D_F^{\mathscr{D}_{poly}}-$closed. This completes the induction, and thus we obtain the desired contraction of the L_∞ quasi-isomorphism f.

We conclude therefore that there exists an L_∞ quasi-isomorphism f^D from the DGLA $(T_{poly}(M), 0, [-,-]^{SN-})$ to the DGLA $(D_{poly}(M), \partial_\mu, [-,-]^{G-})$.

Finally, as the second step (the contraction) of Dolgushev's proof involves only the vanishing of the cohomology of $D_F^{\mathscr{D}_{poly}}$, the L_∞ quasi-isomorphism f^D is a universal formality L_∞ quasi-isomorphism. This implies in particular the existence of a universal star product on a given Poisson manifold (M,Λ) defined by

$$u *_D v = \mu(u,v) + \sum_{p=1}^{\infty} \frac{\nu^p}{p!} f_p^D(\Lambda,\ldots,\Lambda)(u,v)$$

for any $u,v \in C^\infty(M)$.

Chapter 5

Formal Poisson Cohomology of Twisted r-matrix Induced Structures

5.1 Introduction

It is easily seen that any quadratic Poisson tensor of the Dufour-Haraki classification (DHC), [DH91], reads

$$\Lambda = \Lambda_I + \Lambda_{II} = aY_{23} + bY_{31} + cY_{12} + \Lambda_{II}, \tag{5.1}$$

where $a, b, c \in \mathbb{R}$, where the Y_i are linear, mutually commuting vector fields ($Y_{ij} = Y_i \wedge Y_j$), and where Λ_{II} is—as Λ_I—a quadratic Poisson structure. This entails of course that Λ_I and Λ_{II} are compatible, i.e. that $[\Lambda_I, \Lambda_{II}] = 0$, where $[.,.]$ is the Schouten bracket. Except for structure 10 of the DHC, where $\Lambda_{II} = (3b+1)(y^2 - 2xz)\partial_{23}$ ($\partial_{23} = \partial_{x_2}\partial_{x_3} = \partial_y\partial_z$), the second Poisson structure is always Koszul-exact, i.e.

$$\Lambda_{II} = \Pi_\phi := (\partial_1\phi)\partial_{23} + (\partial_2\phi)\partial_{31} + (\partial_3\phi)\partial_{12}, \quad \phi \in \mathscr{S}^3\mathbb{R}^{3*}.$$

In [Xu92], P. Xu has proved that any quadratic Poisson tensor of $\mathbb{R}^3$ reads

$$\Lambda = \frac{1}{3} K \wedge \mathscr{E} + \Pi_f, \tag{5.2}$$

where K is the curl of Λ, $\mathscr{E}$ the Euler field, and $f \in \mathscr{S}^3\mathbb{R}^{3*}$.

In most cases (only cases 9 and 10 of the DHC are exceptional), term Λ_I of Equation (5.1), which is twisted by the exact term Λ_{II} and is—as easily seen—implemented by an r-matrix in the stabilizer $\mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$, $\mathfrak{g}_\Lambda = \{A \in \mathrm{gl}(3,\mathbb{R}) : [A,\Lambda] = 0\}$, is given by

$$\Lambda_I = \frac{1}{3} K \wedge \mathscr{E} + \Pi_{\lambda D},$$

where $\lambda \in \mathbb{R}^*$ and $D = \det(Y_1, Y_2, Y_3)$, whereas

$$\Lambda_{II} = \Pi_\phi = \Pi_{f-\lambda D}.$$

Hence, the difference between decompositions (5.1) and (5.2) is that in (5.1) the biggest possible part of Λ is incorporated into the r-matrix induced structure, whereas in (5.2) it is incorporated into the exact structure.

We privilege decomposition (5.1), since a general computing technique allows to deal with the cohomology of Λ_I, [MP06], and Λ_{II} vanishes in many cases. In most of the cases where the small exact tensor Λ_{II} does not vanish, the decomposition

$$\partial_\Lambda := [\Lambda, .] = [\Lambda_I, .] + [\Lambda_{II}, .] =: \partial_{\Lambda_I} + \partial_{\Lambda_{II}}, \quad \partial_{\Lambda_I}^2 = \partial_{\Lambda_{II}}^2 = \partial_{\Lambda_I}\partial_{\Lambda_{II}} + \partial_{\Lambda_{II}}\partial_{\Lambda_I} = 0$$

leads to a vertically positive double complex and the corresponding spectral sequence allows to deduce bit by bit the cohomology of Λ from that of Λ_I.

In Section 2, we show how twisted r-matrix induced tensors generate vertically positive double complexes. As richness of Poisson cohomology entails computation through the whole associated spectral sequence, we detail a complete model of the sequence in Section 3. Section 4 contains the computation of the cohomology of tensor Λ_4 of the Dufour-Haraki classification. More precisely, Subsection 4.1 provides the second term of the spectral sequence, i.e. the cohomology of the r-matrix induced part $\Lambda_{4,I}$ of Λ_4, which is accessible to the general cohomological technique developed in [MP06]. After some preliminary work in Subsections 4.2 and 4.3, we are prepared to compute, in Subsection 4.4, through the entire spectral sequence, see Theorem 18. As we aim at the extraction of "true results", we are obliged to detail all the isomorphisms involved in the theory of spectral sequences and to read our upshots through these isomorphisms. Hence, in particular, a study of the limiting process in the sequence and of the reconstruction of the cohomology, precedes, in Subsection 4.5.1, the concrete description of the cohomology of twisted structure Λ_4, see Theorem 19 in Subsection 4.5.2, and of twisted tensor Λ_8, Theorem 20 in Subsection 5.

The description of the main features of the cohomology of r-matrix induced Poisson structures has been given in [MP06]. The tight relation between Casimir functions and Koszul-exactness of these Poisson tensors is recalled in Subsection 4.1, see Equation (5.10) (a generalization can be found in Subsection 4.3, see Equation(5.11)). Since our r-matrix induced Poisson structures are built with infinitesimal Poisson automorphisms Y_i, see Equation (5.1), the wedge products of the Y_i constitute a priori "privileged" cocycles. The associative graded commutative algebra structure of the Poisson cohomology space now explains part of the cohomology classes. The second and third term of this cohomology space contain, in addition to the just mentioned wedge products of Casimir functions and infinitesimal automorphisms Y_i, non-bounding cocycles the coefficients of which are—in a broad sense—polynomials on the singular locus of the considered Poisson tensor. The "weight in cohomology" of the singularities increases with closeness of the Poisson structure to Koszul-exactness. The appearance of some "accidental Casimir-like" non bounding cocycles completes the depiction of the main characteristics of the cohomology.

If the r-matrix induced structure is twisted by an exact quadratic tensor, the aforementioned spectral sequence constructs little by little the cohomology of Λ from that of Λ_I. In the examined cases, the basic Casimir C_I of Λ_I is the first term of the expansion by Newton's binomial theorem of the basic Casimir C of Λ. Beyond the emergence of systematic conditions on the coefficients of the powers C^i, $i \in \mathbb{N}$, and the methodic disappearance of monomials on the singular locus of Λ_I, the main impact on Poisson cohomology of twist Λ_{II} is the (partial) passage from first term C_I to complete expansion C, a change that takes place gradually for all powers of these Casimirs, as we compute through the spectral sequence.

5.2 Vertically positive double complex

5.2.1 Definition

Let (K,d) be a complex, i.e. a differential space, made up by a graded vector space $K = \oplus_{n\in\mathbb{N}} K^n$ and a differential $d : K^n \to K^{n+1}$ that has weight 1 with respect to this grading. Assume that each term K^n is itself graded,

$$K^n = \oplus_{r,s\in\mathbb{N},r+s=n} K^{rs},$$

so that $K = \oplus_{r,s\in\mathbb{N}} K^{rs}$ is bigraded. We will refer to grading $K = \oplus_{n\in\mathbb{N}} K^n$ as the diagonal grading. Let $p,q \in \mathbb{N}, p+q = n$. Differential $d : K^{pq} \to \oplus_{r,s\in\mathbb{N},r+s=n+1} K^{rs}$

induces linear maps

$$d_{ab} : K^{pq} \to K^{p+a,q+b} \quad (a,b \in \mathbb{Z}, a+b=1),$$

such that

$$d = \sum_{a,b \in Z, a+b=1} d_{ab}.$$

If $d_{ab} = 0, \forall b < 0$ (resp. $d_{ab} = 0, \forall a < 0$), the preceding complex is a *vertically positive double complex* (VPDC) (resp. a *horizontally positive double complex* (HPDC)). Vertically positive and horizontally positive double complexes are *semi-positive double complexes*. A complex that is simultaneously a VPDC and a HPDC is a *double complex* (DC) in the usual sense.

We filter a VPDC (resp. a HPDC) using the *horizontal filtration* (resp. *vertical filtration*)

$${}^hK_p = \oplus_{r \in \mathbb{N}, s \geq p} K^{rs} \quad (\text{resp. } {}^vK_p = \oplus_{r \geq p, s \in \mathbb{N}} K^{rs}).$$

These filtrations are compatible (in the usual sense) with the diagonal grading and differential d. Moreover, they are regular, i.e. $K_p \cap K^n = 0, \forall p > n$ (as well for $K_p = {}^hK_p$ as for $K_p = {}^vK_p$), and verify $K_0 = K$ and $K_{+\infty} = 0$.

The (convergent) spectral sequence (SpecSeq) associated with this graded filtered differential space is extensively studied below. Let us stress that in the following we prove several general results on spectral sequences, which we could not find in literature. In order to increase the reader-friendliness of this chapter and to avoid scrolling, we chose to give these upshots in separate subsections that directly precede those where the results are needed.

5.2.2 Application to twisted r-matrix induced Poisson structures

We will now associate a VPDC to twisted r-matrix induced Poisson tensors. Let

$$\Lambda = \Lambda_I + \Lambda_{II} = aY_{23} + bY_{31} + cY_{12} + \Pi_\phi$$

be as in Equation (5.1).

Set $Y_i = \ell_{ij}\partial_j$, $\ell_{ij} \in \mathbb{R}^{3*}$ (we use the Einstein summation convention) and $D = \det \ell = \det(\ell_{ij}) \in \mathscr{S}^3\mathbb{R}^{3*}$. If $L \in \mathrm{gl}(3, \mathscr{S}^2\mathbb{R}^{3*})$ is the matrix of algebraic (2×2)-minors of ℓ, we have $\partial_i = \frac{L_{ji}}{D} Y_j$. The formal Poisson cochain space $\mathscr{P}$ is made up by the $0-$, $1-$, $2-$, and $3-$cochains

$$C^0 = \frac{\sigma}{D}, C^1 = \frac{\sigma_1}{D}Y_1 + \frac{\sigma_2}{D}Y_2 + \frac{\sigma_3}{D}Y_3, C^2 = \frac{\sigma_1}{D}Y_{23} + \frac{\sigma_2}{D}Y_{31} + \frac{\sigma_3}{D}Y_{12}, C^3 = \frac{\sigma}{D}Y_{123}, \tag{5.3}$$

where $\sigma,\sigma_1,\sigma_2,\sigma_3 \in \mathbb{R}[[x_1,x_2,x_3]]$ and where σ, $\ell_{ij}\sigma_i$, $L_{ij}\sigma_i$ are divisible by D (for any j; 3-cochains do not generate any divisibility condition). In order to understand these results, note first that, if $\mathscr{L} \in \mathrm{gl}(3,\mathscr{S}^4\mathbb{R}^{3*})$ denotes the matrix of algebraic (2×2)-minors of L, we have $\mathscr{L} = (\det L)\tilde{L}^{-1}$ and $L = (\det \ell)\tilde{\ell}^{-1}$. The last equation entails that $\det L = (\det \ell)^2$ and that $L^{-1} = \frac{1}{\det \ell}\tilde{\ell}$. Hence, it follows from the first equation that $\mathscr{L} = (\det \ell)\ell = D\ell$. Let now $C^2 = \sigma_1\partial_{23} + \sigma_2\partial_{31} + \sigma_3\partial_{12}$ be an arbitrary 2-cochain. Since its first term reads

$$\begin{aligned}\sigma_1\partial_{23} = \frac{\sigma_1}{D^2}L_{j2}L_{k3}Y_{jk} &= \frac{\sigma_1}{D^2}\left(\mathscr{L}_{11}Y_{23} + \mathscr{L}_{21}Y_{31} + \mathscr{L}_{31}Y_{12}\right)\\ &= \frac{\sigma_1}{D}\left(\ell_{11}Y_{23} + \ell_{21}Y_{31} + \ell_{31}Y_{12}\right),\end{aligned} \tag{5.4}$$

its is clear that any 2-cochain can be written as announced. Conversely, the first term of any 2-vector $C^2 = \frac{\sigma_1}{D}Y_{23} + \frac{\sigma_2}{D}Y_{31} + \frac{\sigma_3}{D}Y_{12}$ reads

$$\frac{\sigma_1}{D}Y_{23} = \frac{\sigma_1}{D}\ell_{2j}\ell_{3k}\partial_{jk} = \frac{\sigma_1}{D}\left(L_{11}\partial_{23} + L_{12}\partial_{31} + L_{13}\partial_{12}\right).$$

Thus, such a 2-vector C^2 is a formal Poisson 2-cochain if and only if $L_{ij}\sigma_i$ is divisible by D for any j. The proofs of the statements concerning 0-, 1-, and 3-cochains are similar.

Hence, if we substitute the Y_i for the standard basic vector fields ∂_i, the cochains assume—roughly speaking—the shape $\sum f\mathbf{Y}$, where f is a function and $\mathbf{Y}$ is a wedge product of basic fields Y_i. Then the Lichnerowicz-Poisson coboundary operator $\partial_{\Lambda_I} = [\Lambda_I,\cdot]$ is just

$$\partial_{\Lambda_I}(f\mathbf{Y}) = [\Lambda_I, f\mathbf{Y}] = [\Lambda_I, f]\wedge\mathbf{Y}. \tag{5.5}$$

More precisely, the coboundary operator associated with Λ_I is given by

$$[\Lambda_I,C^0] = \nabla C^0, [\Lambda_I,C^1] = \nabla\wedge C^1, [\Lambda_I,C^2] = \nabla . C^2, \text{ and } [\Lambda_I,C^3] = 0, \tag{5.6}$$

where $\nabla = \sum_i X_i(\cdot)Y_i$, $X_1 = cY_2 - bY_3, X_2 = aY_3 - cY_1, X_3 = bY_1 - aY_2$, and where the RHS have to be viewed as notations that give the coefficients of the coboundaries in the Y_i-basis. For instance, $[\Lambda_I,C^2] = (\sum_i X_i(\frac{\sigma_i}{D}))Y_{123}$.

Of course the formal power series $\sigma,\sigma_1,\sigma_2,\sigma_3$ in Equation (5.3) read

$$\sum_{J\in\mathbb{N}^3} c_J X^J = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} c_{j_1j_2j_3}x_1^{j_1}x_2^{j_2}x_3^{j_3} \quad (c_{j_1j_2j_3}\in\mathbb{R}).$$

The degrees $j_1,j_2,j_3 \in \mathbb{N}$ and the cochain degree $c\in\{0,1,2,3\}$ induce a 4-grading of the formal Poisson cochain space $\mathscr{P}$ of polyvector fields with coefficients in formal power series. Let us emphasize that the degrees j_i are read in the numerators σ

of the decomposition $C = \sum \frac{\sigma}{D}\mathbf{Y}$. They are tightly related with the r-matrix induced nature of Λ_I and were basic in the method developed in [MP06]. In the following we use the degrees $r = j_1 + j_2 + c$ and $s = j_3$ (depending on the considered Poisson tensor, other degrees could be used, but the preceding ones encompass the majority of twisted structures) that generate a bigrading of $\mathscr{P}$, $\mathscr{P} = \oplus_{r,s\in\mathbb{N}}\mathscr{P}^{rs}$. When defining the diagonal degree $n = r + s$, we get a graded space

$$\mathscr{P} = \oplus_{n\in\mathbb{N}}\mathscr{P}^n, \mathscr{P}^n = \oplus_{r,s\in\mathbb{N},r+s=n}\mathscr{P}^{rs}.$$

We now determine the weights of the coboundary operators ∂_{Λ_I} and $\partial_{\Lambda_{II}}$ with respect to r and s. Actually D is an eigenvector of the basic fields Y_i, hence of the fundamental fields X_i, $Y_iD = \lambda_i D$, $X_iD = \mu_i D$, $\lambda_i, \mu_i \in \mathbb{R}$. Indeed, since $\pi_{\lambda D} = \lambda(\partial_1 D\,\partial_{23} + \partial_2 D\,\partial_{31} + \partial_3 D\,\partial_{12})$, it follows from Equation (5.4) (take $\sigma_j = \lambda\,\partial_j D$) (and its cyclic permutations) that

$$\pi_{\lambda D} = \frac{\lambda}{D}\left(Y_1 D Y_{23} + Y_2 D Y_{31} + Y_3 D Y_{12}\right).$$

But $\pi_{\lambda D}$ is part of Λ_I and is—more precisely—of type (5.1), i.e. reads

$$\pi_{\lambda D} = \mathfrak{l}_1 Y_{23} + \mathfrak{l}_2 Y_{31} + \mathfrak{l}_3 Y_{12}$$

($\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3 \in \mathbb{R}$). Hence,

$$Y_i D = \frac{\mathfrak{l}_i}{\lambda} D =: \lambda_i D, \forall i \in \{1,2,3\}.$$

In view of Equations (5.3) and (5.6), the degrees j_1, j_2, j_3 of the Λ_I-coboundary $\partial_{\Lambda_I} C$ of any cochain C only depend on the values $X_i\left(\frac{\sigma}{D}\right)$ of the fundamental linear fields X_i for an arbitrary formal power series $\sigma = \sum_J c_J X^J$. Since

$$X_i\left(\frac{\sigma}{D}\right) = \sum_J c_J \frac{1}{D}\left(X_i - \mu_i\,\mathrm{id}\right) X^J,$$

it is clear that ∂_{Λ_I} preserves the total degree $\mathfrak{t} = j_1 + j_2 + j_3$.

In the following, we focus on the first twisted quadratic Poisson structures that appear in the DHC, i.e. on classes 4, 8, and 11, see [DH91]. Let us recall that

$$\begin{aligned}
\Lambda_4 &= ayz\partial_{23} + axz\partial_{31} + \left(bxy + z^2\right)\partial_{12} \\
&= aY_{23} + aY_{31} + bY_{12} + \frac{z^3}{D}Y_{12} = \Lambda_{4,I} + \Lambda_{4,II},
\end{aligned}$$

$$a \neq 0, b \neq 0, Y_1 = x\partial_1, Y_2 = y\partial_2, Y_3 = z\partial_3, D = xyz,$$

$$\begin{aligned}\Lambda_8 &= \left(\frac{a+b}{2}(x^2+y^2)\pm z^2\right)\partial_{12}+axz\partial_{23}+ayz\partial_{31}\\ &= aY_{23}+\frac{a+b}{2}Y_{12}\pm\frac{z^3}{D}Y_{12}=\Lambda_{8,I}+\Lambda_{8,II},\end{aligned}$$

$$a\neq 0, b\neq 0, Y_1=x_1\partial_1+x_2\partial_2, Y_2=x_1\partial_2-x_2\partial_1, Y_3=x_3\partial_3, D=(x^2+y^2)z,$$

$$\begin{aligned}\Lambda_{11} &= \left(ax^2+bz^2\right)\partial_{12}+(2a+1)xz\partial_{23}\\ &= Y_{23}+aY_{12}+b\frac{z^3}{D}((3a+1)Y_{12}+Y_{23})=\Lambda_{11,I}+\Lambda_{11,II},\end{aligned}$$

$$a\neq\frac{-1}{3}, b\neq 0, Y_1=\mathscr{E}, Y_2=x\partial_2, Y_3=(3a+1)z\partial_3, D=(3a+1)x^2z.$$

Owing to the above remarks, it is obvious that $\partial_{\Lambda_{i,I}}, i\in\{4,8,11\}$, preserves the partial degree $\mathfrak{p}=j_1+j_2$ (and, as aforementioned, the total degree $\mathfrak{t}$). Hence, its weight with respect to (r,s) is $(1,0)$:

$$d':=d_{10}:=\partial_{\Lambda_{i,I}}:\mathscr{P}^{rs}\to\mathscr{P}^{r+1,s}\quad(i\in\{4,8,11\})$$

(dependence on i omitted in d' and d_{10}).

As for the weight of $\partial_{\Lambda_{i,II}}, i\in\{4,8,11\}$, with respect to (r,s), let us first recall that, if f and g are some functions, and if $\mathbf{X}$ and $\mathbf{Y}$ denote wedge products of Y_1,Y_2,Y_3 with (non-shifted) degrees α and β respectively, we have

$$[f\mathbf{X},g\mathbf{Y}]=f[\mathbf{X},g]\wedge\mathbf{Y}+(-1)^{\alpha\beta-\alpha-\beta}g[\mathbf{Y},f]\wedge\mathbf{X}. \tag{5.7}$$

Of course, the RHS of the preceding equation is a linear combination of terms of the type $fY_i(g)\mathbf{Z}$ or $gY_i(f)\mathbf{Z}$, where $\mathbf{Z}$ is a wedge product of Y_1,Y_2,Y_3 of degree $\alpha+\beta-1$. It follows that $\partial_{\Lambda_{i,II}}C^c$, $i\in\{4,8,11\}$, $C^c\in\mathscr{P}$, is a formal series of terms of the type

$$[\frac{z^3}{D}\mathbf{X},\frac{X^J}{D}\mathbf{Y}].$$

Any such term is a linear combination of terms of the type

$$\frac{z^3}{D}Y_i\left(\frac{X^J}{D}\right)\mathbf{Z}\quad\text{and}\quad\frac{X^J}{D}Y_i\left(\frac{z^3}{D}\right)\mathbf{Z}.$$

As D is an eigenvector of Y_i, this entails that coboundary $\partial_{\Lambda_{i,II}}C^c$ has the form

$$\partial_{\Lambda_{i,II}}C^c=\sum\frac{\sum_K c_KX^K}{D^2}\mathbf{Z},$$

where in each term $k_1+k_2 = j_1+j_2$ and $k_3 = j_3+3$, and where the degree of wedge product $\mathbf{Z}$ is $\alpha+\beta-1 = c+1$. When dividing the preceding numerators by D (see above), we find that the weight of $\partial_{\Lambda_{i,II}}$ with respect to (r,s) is $(-1,2)$:

$$d'' := d_{-12} := \partial_{\Lambda_{i,II}} : \mathscr{P}^{rs} \to \mathscr{P}^{r-1,s+2} \quad (i \in \{4,8,11\})$$

(dependence on i omitted in d'' and d_{-12}).

Finally, $(\mathscr{P}, \partial_{\Lambda_i})$, $i \in \{4,8,11\}$, endowed with the previously mentioned gradings

$$\mathscr{P} = \oplus_{n\in\mathbb{N}} \mathscr{P}^n, \mathscr{P}^n = \oplus_{r,s\in\mathbb{N}, r+s=n} \mathscr{P}^{rs}$$

and the differential

$$d := \partial_{\Lambda_i} = \partial_{\Lambda_{i,I}} + \partial_{\Lambda_{i,II}} = d' + d'' = d_{10} + d_{-12},$$

is a VPDC. We will compute the cohomology $H(\Lambda_i) = H(\mathscr{P}, d)$ using the SpecSeq associated with this VPDC (see above).

5.3 Model of the spectral sequence associated with a VPDC

As mentioned above, a VPDC, a HPDC, and a DC can canonically be viewed as regular filtered graded differential spaces. Hence, a SpecSeq (two, for any DC) is associated with each one of these complexes.

In order to introduce notations, let us recall that, if (K, d, K_p, K^n) is any (regular, i.e. $K_p \cap K^n = 0, \forall p > n$) filtered (subscripts) graded (superscripts) differential space (in our work p and n can be regarded as positive integers), the associated SpecSeq (E_r, d_r) $(r \in \mathbb{N})$ is defined by

$$E_r^{pq} = Z_r^{pq}/(Z_{r-1}^{p+1,q-1} + B_{r-1}^{pq}),$$

where $Z_r^{pq} = K_p \cap d^{-1}K_{p+r} \cap K^{p+q}$ and $B_r^{pq} = K_p \cap dK_{p-r} \cap K^{p+q}$ are the spaces of "weak cocycles" and "strong coboundaries" of order r in $K_p \cap K^{p+q}$, and

$$d_r : E_r^{pq} \ni [\mathfrak{z}_r^{pq}]_{E_r^{pq}} \to [d\mathfrak{z}_r^{pq}]_{E_r^{p+r,q+1-r}} \in E_r^{p+r,q+1-r}.$$

In the following, we also use the vector space isomorphism

$$\sigma_r : E_{r+1}^{pq} \to H^{pq}(E_r, d_r),$$

which assigns to each $[\mathfrak{z}^{pq}_{r+1}]_{E^{pq}_{r+1}}$, $\mathfrak{z}^{pq}_{r+1} \in Z^{pq}_{r+1} \subset Z^{pq}_r$, the d_r-cohomology class $[[\mathfrak{z}^{pq}_{r+1}]_{E^{pq}_r}]_{d_r}$, $[\mathfrak{z}^{pq}_{r+1}]_{E^{pq}_r} \in E^{pq}_r \cap \ker d_r$. For more detailed results on spectral sequences, we refer the reader to [Cle85], [God52], [CE56], [Vai73], ... In these monographs, a model for the SpecSeq associated with a (HP)DC is partially depicted up to $r = 2$. It is well-known that spectral sequences are particularly easy to use, if many spaces E^{pq}_2 (or E^{pq}_r $(r > 2)$) vanish. Due to richness of Poisson cohomology, this lacunary phenomenon is less pronounced in our setting. Since we have thus to compute through the whole SpecSeq, we need the complete description of the entire model of the SpecSeq (E_r, d_r) $(r \in \mathbb{N})$ associated with a VPDC.

So consider an arbitrary VPDC and let $G^{pq}(K)$ $(p, q \in \mathbb{N})$ be the term of degree (p, q) of the bigraded space associated with the filtered graded space K. It is clear that the mapping

$$I_0 : E^{pq}_0 = K_p \cap K^{p+q}/K_{p+1} \cap K^{p+q} = G^{pq}(K) \ni [\mathfrak{z}^{pq}_0 = \sum_{i=0}^{q} z^{q-i,p+i}]_{E^{pq}_0} \to z^{qp} \in K^{qp},$$

where z^{rs} (as well as—in the following—all Latin characters with double superscript) is an element of K^{rs} (whereas German Fraktur characters with double superscript, such as $\mathfrak{z}^{pq}_0$, do not refer to the bigrading of K), is an isomorphism of bigraded vector spaces (i.e. a vector space isomorphism that respects the bigrading). It is easily seen that, when reading d_0 through this isomorphism, we get the compound map

$$\overline{d}_0 = I_0 d_0 I_0^{-1} = d_{10}.$$

Thus $I_0 : (E_0, d_0) \to (K, \overline{d}_0)$ is an isomorphism between bigraded differential spaces, and induces an isomorphism

$$I_{0\sharp} : H^{pq}(E_0, d_0) \ni [[\mathfrak{z}^{pq}_0 = \textstyle\sum_{i=0}^{q} z^{q-i,p+i}]_{E^{pq}_0}]_{d_0}$$
$$\longrightarrow [z^{qp}]_{\overline{d}_0} \in H^{pq}(K, \overline{d}_0) =: {}^0H^{pq}(K) = {}^0H^q(K^{*p})$$

of bigraded vector spaces, where the last space is the q-term of the cohomology space of $(K^{*p}, \overline{d}_0 = d_{10})$. Hence the bigraded vector space isomorphism

$$I_1 = I_{0\sharp}\sigma_0 : E^{pq}_1 \ni [\mathfrak{z}^{pq}_1 = \sum_{i=0}^{q} z^{q-i,p+i}]_{E^{pq}_1} \to [[\mathfrak{z}^{pq}_1]_{E^{pq}_0}]_{d_0} \to [z^{qp}]_{\overline{d}_0} \in {}^0H^q(K^{*p}).$$

We now again verify straightforwardly that differential d_1 read on model ${}^0H(K)$ is induced by d_{01}, i.e. that

$$\overline{d}_1 = I_1 d_1 I_1^{-1} = d_{01\sharp}.$$

Finally,

$$I_2 = I_{1\sharp}\sigma_1 : E_2^{pq} \ni [\mathfrak{z}_2^{pq} = \sum_{i=0}^{q} z^{q-i,p+i}]_{E_2^{pq}} \to [[\mathfrak{z}_2^{pq}]_{E_1^{pq}}]_{d_1} \to [[z^{qp}]_{\overline{d}_0}]_{\overline{d}_1} \in {}^1H^p({}^0H^q(K))$$

is an isomorphism of bigraded vector spaces. As for the sense of the last space, note that $({}^0H^q(K) = \oplus_p {}^0H^q(K^{*p}), \overline{d}_1)$ is a complex. Observe now that the inverse I_2^{-1} is less straightforward than I_0^{-1} and I_1^{-1}. Indeed, if $[[z^{qp}]_{\overline{d}_0}]_{\overline{d}_1} \in {}^1H^p({}^0H^q(K))$, representative z^{qp} is generally not a member of Z_2^{pq}. However, since the considered class makes sense,

$$\begin{aligned} &d_{10}z^{qp} = 0 \\ &d_{01}z^{qp} + d_{10}z^{q-1,p+1} = 0, \end{aligned}$$

where $z^{q-1,p+1} \in K^{q-1,p+1}$. Thus, $\mathfrak{z}_2^{pq} := z^{qp} + z^{q-1,p+1} \in Z_2^{pq}$ and

$$I_2^{-1}[[z^{qp}]_{\overline{d}_0}]_{\overline{d}_1} = [\mathfrak{z}_2^{pq}]_{E_2^{pq}}.$$

So

$$\overline{d}_2[[z^{qp}]_{\overline{d}_0}]_{\overline{d}_1} = I_2[d\mathfrak{z}_2^{pq}]_{E_2^{p+2,q-1}} = [[d_{-12}z^{qp} + d_{01}z^{q-1,p+1}]_{\overline{d}_0}]_{\overline{d}_1}.$$

The preceding results extend those given in [Vai73] (for a HPDC). They can easily be adapted to the most frequently encountered situations where only some terms d_{ab} of d do not vanish.

In the following, we complete the description of the SpecSeq associated with a VPDC, assuming that $d = d_{10} + d_{-12} := d' + d''$. This hypothesis entails that $d'^2 = d''^2 = d'd'' + d''d' = 0$, i.e. that d' and d'' are two anticommuting differentials. Hereafter, we denote by ${}^rH(.)$ $(r \in \mathbb{N})$ the cohomology of differential $\overline{d}_{2r}$ and by $[.]_r$ the corresponding classes. Moreover, we will deal with strongly triangular systems of type

$$\begin{aligned} &d'z^{qp} = 0 && (\mathfrak{E}_0) \\ &d''z^{qp} + d'z^{q-2,p+2} = 0 && (\mathfrak{E}_1) \\ &\ldots && \\ &d''z^{q-2(k-2),p+2(k-2)} + d'z^{q-2(k-1),p+2(k-1)} = 0. && (\mathfrak{E}_{k-1}) \end{aligned}$$

Note that when solving such a system, we prove at each stage that some d'-cocycle is actually a d'-coboundary. We refer to this kind of system using the notation $S(z^{qp};k)$ or $S(k;z^{q-2(k-1),p+2(k-1)})$ depending on the necessity to emphasize the first or the last unknown or entry of an ordered solution.

Proposition 35. *The spectral sequence associated to a VPDC with differential $d = d_{10} + d_{-12} = d' + d''$ admits the following model. The model of E_0, isomorphisms I_0 and I_0^{-1}, and differential $\overline{d}_0$ are the same as above. For any $r \in \{1,2,\ldots\}$,*

(i) The map

$$I_{2r-1}: \qquad E^{pq}_{2r-1} \ni [\mathfrak{z}^{pq}_{2r-1} = \sum_{i=0}^{q} z^{q-i,p+i}]_{E^{pq}_{2r-1}} \longrightarrow [[[z^{qp}]_0]_1 \ldots]_{r-1} \in {}^{r-1}H^{pq}({}^{r-2}H(\ldots({}^0H(K))))$$

is a bigraded vector space isomorphism. Its inverse I^{-1}_{2r-1} associates to any RHS-class the LHS-class with representative $\mathfrak{z}^{pq}_{2r-1} = \sum_{i=0}^{r-1} z^{q-2i,p+2i}$, where $(z^{qp},\ldots,z^{q-2(r-1),p+2(r-1)})$ is any solution of system $S(z^{qp};r)$. Furthermore, $\overline{d}_{2r-1} = 0$.

(ii) The model of E^{pq}_{2r} and the corresponding isomorphisms I_{2r} and I^{-1}_{2r} coincide with those pertaining to E^{pq}_{2r-1}. Moreover,

$$\overline{d}_{2r}[[[z^{qp}]_0]_1\ldots]_{r-1} = [[[d''z^{q-2(r-1),p+2(r-1)}]_0]_1\ldots]_{r-1}, \tag{5.8}$$

where $z^{q-2(r-1),p+2(r-1)}$ is the last entry of an arbitrary solution of $S(z^{qp};r)$.

Proof. It is easier to prove an extended version of Proposition 35. Indeed, let us complete assertions (i) and (ii) by item

(iii) Existence (resp. vanishing) of a class $[[[z^{qp}]_0]_1\ldots]_{r-1}$ is equivalent with existence of at least one solution of system $S(z^{qp};r)$ (resp. with existence of $z^{q-1,p}$ and of $z_i^{q+1,p-2}$, $i \in \{1,\ldots,r-1\}$, which induce systems $S(i;z_i^{q+1,p-2})$ with solution, such that

$$z^{qp} + d'z^{q-1,p} + d''\sum_{i=1}^{r-1} z_i^{q+1,p-2} = 0.)$$

The proof is by induction on r. Observe first that the assertions are valid for $r = 1$ (see above). Assume now that all items hold for $r \in \{1,\ldots,\ell-1\}$. Proceeding as above, we easily show that $I_{2\ell-1} := I_{2(\ell-1)\sharp}\sigma_{2(\ell-1)}$ is the appropriate bigraded vector space isomorphism. In order to determine $I^{-1}_{2\ell-1}$, take any RHS-class $[[[z^{qp}]_0]_1\ldots]_{\ell-1}$.

Let us first prove assertion (iii). Existence of class $[[[z^{qp}]_0]_1\ldots]_{\ell-1}$ is equivalent with existence of class $[[[z^{qp}]_0]_1\ldots]_{\ell-2}$ (itself equivalent to existence of at least one solution

$$z^{q-2j,p+2j}\quad (0\leq j\leq \ell-2)$$

for $S(z^{qp};\ell-1)$, by induction) and condition

$$\overline{d}_{2(\ell-1)}[[[z^{qp}]_0]_1\ldots]_{\ell-2}=0.$$

Using the induction assumptions, we see that the last condition is equivalent, first with

$$[[[d''z^{q-2(\ell-2),p+2(\ell-2)}]_0]_1\ldots]_{\ell-2}=0,$$

then with existence of

$$z^{q-2(\ell-1),p+2(\ell-1)}$$

and $z_i^{q-2(\ell-2),p+2(\ell-2)}$ $(1\leq i\leq \ell-2)$, which implement systems $S(i;z_i^{q-2(\ell-2),p+2(\ell-2)})$ with solution, say

$$z_i^{q-2j,p+2j}\quad (1\leq \ell-i-1\leq j\leq \ell-2),$$

such that

$$d''\left(z^{q-2(\ell-2),p+2(\ell-2)}+\sum_{i=1}^{\ell-2}z_i^{q-2(\ell-2),p+2(\ell-2)}\right)+d'z^{q-2(\ell-1),p+2(\ell-1)}=0. \quad (5.9)$$

Assume now that all this holds and define new $z^{q-2j,p+2j}$ $(0\leq j\leq \ell-1)$. For each j, take just the sum of the old $z^{q-2j,p+2j}$ and of all existing $z_i^{q-2j,p+2j}$. These new $z^{q-2j,p+2j}$ form a solution of $S(z^{qp};\ell)$. Note first that for $j\in\{0,\ell-1\}$, the old and new $z^{q-2j,p+2j}$ coincide. Hence, the last equation $(\mathfrak{E}_{\ell-1})$ of $S(z^{qp};\ell)$ is nothing but Equation (5.9). Moreover, it is easily checked that Equations $(\mathfrak{E}_{\ell-2}),\ldots,(\mathfrak{E}_0)$ are also verified. Conversely, if $S(z^{qp};\ell)$ has a solution, the successive classes $[z^{qp}]_0,[[z^{qp}]_0]_1,\ldots,[[[z^{qp}]_0]_1\ldots]_{\ell-1}$ are actually defined. It suffices to note that $\overline{d}_0z^{qp}=0$ and that, by induction,

$$\begin{aligned}\overline{d}_{2r}[[[z^{qp}]_0]_1\ldots]_{r-1} &= [[[d''z^{q-2(r-1),p+2(r-1)}]_0]_1\ldots]_{r-1}\\ &= -[[[d'z^{q-2r,p+2r}]_0]_1\ldots]_{r-1}=0,\end{aligned}$$

for any $r\in\{1,\ldots,\ell-1\}$.

As for the second part of (iii), note that a class $[[[z^{qp}]_0]_1\ldots]_{\ell-1}$ vanishes if and only if there is $z_{\ell-1}^{q+2(\ell-1)-1,p-2(\ell-1)}$ that generates a system $S(z_{\ell-1}^{q+2(\ell-1)-1,p-2(\ell-1)};\ell-1)$ with solution, say

$$z_{\ell-1}^{q+2(\ell-j-1)-1,p-2(\ell-j-1)} \quad (0\le j\le \ell-2),$$

such that

$$\begin{aligned}[[[z^{qp}]_0]_1\ldots]_{\ell-2} &= -\overline{d}_{2(\ell-1)}[[[z_{\ell-1}^{q+2(\ell-1)-1,p-2(\ell-1)}]_0]_1\ldots]_{\ell-2}\\ &= -[[[d''z_{\ell-1}^{q+1,p-2}]_0]_1\ldots]_{\ell-2}.\end{aligned}$$

But, by induction, $[[[z^{qp}+d''z_{\ell-1}^{q+1,p-2}]_0]_1\ldots]_{\ell-2}=0$ if and only if there are $z^{q-1,p}$ and $z_i^{q+1,p-2}$ $(1\le i\le \ell-2)$, which induce systems $S(i;z_i^{q+1,p-2})$ with solution, such that

$$z^{qp}+d''z_{\ell-1}^{q+1,p-2}+d''\sum_{i=1}^{\ell-2}z_i^{q+1,p-2}+d'z^{q-1,p}=0.$$

Hence the conclusion.

We now revert to items (i) and (ii). For any RHS-class $[[[z^{qp}]_0]_1\ldots]_{\ell-1}\in {}^{\ell-1}H^{pq}({}^{\ell-2}H(\ldots({}^0H(K))))$, the corresponding system $S(z^{qp};\ell)$ admits, as just explained, at least one solution $z^{q-2j,p+2j}$ $(0\le j\le \ell-1)$. Set

$$\mathfrak{z}_{2\ell-1}^{pq}:=\sum_{j=0}^{\ell-1}z^{q-2j,p+2j}.$$

As $d\mathfrak{z}_{2\ell-1}^{pq}=d''z^{q-2(\ell-1),p+2(\ell-1)}\in K^{q-2\ell+1,p+2\ell}$, we see that $\mathfrak{z}_{2\ell-1}^{pq}\in Z_{2\ell-1}^{pq}=K_p\cap d^{-1}K_{p+2\ell-1}\cap K^{p+q}$. Hence $I_{2\ell-1}^{-1}$. As $d_{2\ell-1}[\mathfrak{z}_{2\ell-1}^{pq}]_{E_{2\ell-1}^{pq}}\in E_{2\ell-1}^{p+2\ell-1,q-2\ell+2}$, it is clear that $\overline{d}_{2\ell-1}=0$. Thus, the statement concerning the model of $E_{2\ell}^{pq}$ and the isomorphisms $I_{2\ell}$ and $I_{2\ell}^{-1}$ is obvious. Finally, as $d_{2\ell}[\mathfrak{z}_{2\ell}^{pq}]_{E_{2\ell}^{pq}}\in E_{2\ell}^{p+2\ell,q-2\ell+1}$, we get

$$\overline{d}_{2\ell}[[[z^{qp}]_0]_1\ldots]_{\ell-1}=[[[d''z^{q-2(\ell-1),p+2(\ell-1)}]_0]_1\ldots]_{\ell-1}. \quad \blacksquare$$

Remark. Result (5.8) can be rephrased as $\overline{d}_{2r}=\left((-1)^{r-1}d''(d'^{-1}d'')^{r-1}\right)_\sharp$, for any $r\in\{1,2,\ldots\}$.

5.4 Formal cohomology of Poisson tensor Λ_4

As aforementioned, we use the just depicted SpecSeq associated with the above detailed VPDC implemented by the twisted r-matrix induced Poisson structure Λ_4.

5.4.1 Computation of the second term of the SpecSeq

In this section, we give the second term $E_2 \simeq {}^0H(\mathscr{P})$ of the SpecSeq. Note that ${}^0H(\mathscr{P})$ is the formal Poisson cohomology of $\overline{d}_0 = d' = d_{10} = \partial_{\Lambda_{4,I}}$. As already elucidated in the Introduction, we came up with decomposition (5.1), since the cohomology of ∂_{Λ_I} is always accessible by the technique proposed in [MP06]. Hence, cohomology space ${}^0H(\mathscr{P})$ can be obtained (quite straightforwardly) by this modus operandi. Let us emphasize that our results are in accordance, as well with similar upshots in [Mon02,2], as with our comments in [MP06], regarding the tight relation between Casimir functions and Koszul-exactness or "quasi-exactness", the appearance of "accidental Casimir-like" non bounding cocycles, and the increase of the "weight in cohomology" of the singularities, with closeness of the considered Poisson structure to Koszul-exactness.

If $\frac{b}{a} \in \mathbb{Q}^*$, we denote by $(\beta,\alpha) \sim (b,a)$, $\alpha \in \mathbb{N}^*$, the irreducible representative of $\frac{b}{a}$. Remember that, see [MP06], for $\frac{b}{a} \in \mathbb{Q}^*_+$, a quasi-exact structure

$$\Lambda = a\partial_1(pq)\partial_{23} + a\partial_2(pq)\partial_{31} + b\partial_3(pq)\partial_{12}, \tag{5.10}$$

$p = p(x,y)$, $q = q(z)$, exhibits the basic Casimir $p^\alpha q^\beta$. Furthermore, we set $D = xyz$, $D' = xy$, and write $\mathscr{A}_\alpha Y_3$, $\alpha \in \mathbb{N}^*$, instead of $D'^\alpha z^{-1} Y_3 = D'^\alpha \partial_3$, and $\oplus_{ij} \ldots Y_{ij}$ instead of $\ldots Y_{23} + \ldots Y_{31} + \ldots Y_{12}$. Remark also that the algebra of polynomials of the algebraic variety of singularities of $\Lambda_{4,I}$ is $\mathbb{R}[[x]] \oplus \mathbb{R}[[y]] \oplus \mathbb{R}[[z]]$, where it is understood that term $\mathbb{R}$ is considered only once.

The following proposition is now almost obvious.

Proposition 36.

1. *If $\frac{b}{a} \in \mathbb{Q}^*_+$, the algebra of $\Lambda_{4,I}$-Casimirs is* $\mathrm{Cas}(\Lambda_{4,I}) = \oplus_{i\in\mathbb{N}} \mathbb{R} D'^{\alpha i} z^{\beta i}$ *and the cohomology space ${}^0H(\mathscr{P}) \simeq E_2$ is given by*

$$\begin{aligned} {}^0H(\mathscr{P}) \quad = \quad & \mathrm{Cas}(\Lambda_{4,I}) \oplus \bigoplus_i \mathrm{Cas}(\Lambda_{4,I}) Y_i \oplus \bigoplus_{ij} \mathrm{Cas}(\Lambda_{4,I}) Y_{ij} \oplus \mathrm{Cas}(\Lambda_{4,I}) Y_{123} \\ & \oplus \begin{cases} \mathbb{R}[[x]]\partial_{23} \oplus \mathbb{R}[[y]]\partial_{31} \oplus (\mathbb{R}[[x]] \oplus \mathbb{R}[[y]])\partial_{123}, \textit{if } b = a \\ 0, \textit{otherwise} \end{cases} \\ & \oplus \mathbb{R}[[z]]\partial_{12} \oplus \mathbb{R}[[z]]\partial_{123} \end{aligned}$$

2. *If $\frac{b}{a} \in \mathbb{R}^* \setminus \mathbb{Q}^*_+$, we have* $\mathrm{Cas}(\Lambda_{4,I}) = \mathbb{R}$ *and the cohomology space* ${}^0H(\mathscr{P}) \simeq$

E_2 is given by

$$
\begin{aligned}
{}^{0}H(\mathscr{P}) &= \mathrm{Cas}(\Lambda_{4,I}) \oplus \bigoplus_i \mathrm{Cas}(\Lambda_{4,I})Y_i \oplus \bigoplus_{ij} \mathrm{Cas}(\Lambda_{4,I})Y_{ij} \oplus \mathrm{Cas}(\Lambda_{4,I})Y_{123} \\
&\oplus \begin{cases} \mathbb{R}\mathscr{A}_\alpha Y_3 \oplus \mathscr{A}_\alpha(\mathbb{R}Y_{23} + \mathbb{R}Y_{31}) \oplus \mathbb{R}\mathscr{A}_\alpha Y_{123}, \ \textit{if } (-1,\alpha) \sim (b,a) \\ 0, \textit{otherwise} \end{cases} \\
&\oplus \mathbb{R}[[z]]\partial_{12} \oplus \mathbb{R}[[z]]\partial_{123}
\end{aligned}
$$

Remark. Due to the properties—used below—of the preceding (non bounding) $\Lambda_{4,I}$-cocycles, we classify these representatives as follows:

1. Representatives of type 1: All cocycles with cochain degree 0, the $1-$ and $2-$cocycles that contain a Casimir (maybe the accidental Casimir $\mathscr{A}_\alpha$), except cocycles $\mathrm{Cas}(\Lambda_{4,I})Y_{12}$
2. Representatives of type 2: All $3-$cocycles, all cocycles with singularities, and cocycles $\mathrm{Cas}(\Lambda_{4,I})Y_{12}$

5.4.2 Prolongable systems $\mathbf{S}(\mathbf{z}^{\mathbf{qp}};\mathbf{r})$

Since computation through the whole SpecSeq will shape up as inescapable, we need the below corollary of Proposition (35). It allows to short-circuit the process of computing the successive terms of the sequence. Let us specify that in the following a system of representatives of a space of classes is made up by representatives that are in 1-to-1 correspondence with the considered classes.

Corollary 4. *If, for some fixed $r \in \mathbb{N}^*$, all the classes $[[[z^{qp}]_0]_1 \ldots]_{r-1}$ in model space ${}^{r-1}H({}^{r-2}H(\ldots {}^{0}H(K)))$, appendant on a SpecSeq associated to a VPDC with differential $d = d_{10} + d_{-12} = d' + d''$, give rise to an enlarged system $S(z^{qp};s)$ with solution, for some fixed $s \geq r$, the following upshots hold:*

1. *All the differentials $\overline{d}_{2r-1}, \overline{d}_{2r}, \ldots, \overline{d}_{2s-1}$ vanish*
2. *Differential $\overline{d}_{2s}$ is defined by*
$$\overline{d}_{2s}[[[z^{qp}]_0]_1 \ldots]_{r-1} = [[[d'' z^{q-2(s-1),p+2(s-1)}]_0]_1 \ldots]_{r-1}$$
3. *Any system (z^{qp}) of representatives of ${}^{r-1}H({}^{r-2}H(\ldots {}^{0}H(K)))$ is in 1-to-1 correspondence with the system $(\mathfrak{z}_{2s}^{pq} := \sum_{k=0}^{s-1} z^{q-2k,p+2k})$ of representatives of E_{2s}*

Proof. Induction on s. ∎

5.4.3 Forecast

In order to increase readability of this chapter, some intuitive advisements are necessary.

The basic idea of the theory of spectral sequences is that computation of the successive terms $E_r \simeq H(E_{r-1}, d_{r-1})$ $(r \in \mathbb{N}^*)$ allows to detect their inductive limit E_∞, which—for a convergent sequence—is isomorphic with the graded space $G(H)$ associated to the sought-after filtered cohomology space H. We then hope to be able to reconstruct this filtered space H from the corresponding graded space $G(H)$. Let us recall that space H is of course the cohomology of the filtered graded differential space associated with the SpecSeq. Hence, in our case, $H = H(\Lambda_4)$. It is clear that the successive cohomology computations take place on the concrete model side. To determine H, we have to pull our results back to the theoretical side, and more precisely to read them through the numerous isomorphisms involved.

Actually the application of spectral sequences presented in this work, provides a beautiful insight into the operating mode of a SpecSeq. Since—roughly spoken—the "weak cocycle condition" in the definition of Z_r^{pq} (resp. the "strong coboundary condition" in the definition of B_r^{pq}) converges to the usual cocycle condition (resp. the usual coboundary condition), we understand that, when passing from one estimate E_{r-1} of H to the next approximation E_r, we obtain an increasing number of conditions on our initial weak non bounding cocycles of E_2 and an increasing number of bounding cocycles. Moreover, when we compute through the SpecSeq, the aforementioned pullbacks, see Proposition (35), add up solutions of crescive systems,

$$\mathfrak{z}_{2r}^{pq} = z^{qp} + \sum_{k=1}^{r-1} z^{q-2k,p+2k}.$$

The next remarks aim at anticipation of these systems. The reader is already familiar with Casimirs of exact and quasi-exact structures. When taking an interest in slightly more general quasi-exact tensors,

$$\Lambda = a\partial_1((p+r)q)\partial_{23} + a\partial_2((p+r)q)\partial_{31} + b\partial_3((p+r)q)\partial_{12}, \qquad (5.11)$$

$a, b \in \mathbb{R}^*$, $p = p(x,y)$, $q = q(z)$, $r = r(z)$, it is natural to ask which polynomials of the type $(p + c\,r)^n q^m$, $c \in \mathbb{R}$, $n, m \in \mathbb{N}$, $(n,m) \neq (0,0)$, are Casimir functions. It is easily checked that structure Λ_4 has this form and that the Casimir conditions read $am = bn$ and $3bn = ca(2n+m)$. So, for $\frac{b}{a} \in \mathbb{Q}_+^*$, the basic Casimir C of Λ_4 and its powers C^i, $i \in \mathbb{N}$, are given by

$$\begin{aligned} C^i &= (p + \frac{3b}{2a+b} r)^{\alpha i} q^{\beta i} = (D' + \frac{z^2}{2a+b})^{\alpha i} z^{\beta i} \\ &= D'^{\alpha i} z^{\beta i} + \sum_{k=1}^{\alpha i} \frac{\mathtt{C}^k_{\alpha i}}{(2a+b)^k} D'^{\alpha i - k} z^{\beta i + 2k}. \end{aligned}$$

These powers C^i (non bounding cocycles of $H = H(\Lambda_4)$) will be obtained—while we compute through the SpecSeq—from those, $D'^{\alpha i} z^{\beta i}$, of the Casimir of $\Lambda_{4,I}$ (non bounding cocycles of $E_2 \simeq {}^0H(\mathscr{P})$). Hence, the above-quoted solutions and corresponding systems $S(D'^{\alpha i} z^{\beta i}, \alpha i + 1)$.

5.4.4 Computation through the SpecSeq

In view of the preceding awareness, it is natural to set

$$Z^{q_{ic}-2k, p_i+2k} = \frac{\mathtt{C}^k_{\alpha i}}{(2a+b)^k} D'^{\alpha i - k} z^{\beta i + 2k} \begin{cases} A_{ik} \\ B_{ik} Y_1 + C_{ik} Y_2 + D_{ik} Y_3 \\ E_{ik} Y_{23} + F_{ik} Y_{31} \end{cases},$$

where $k \in \{0, 1, \ldots, \alpha i\}$ and $A_{ik}, B_{ik}, C_{ik}, D_{ik}, E_{ik}, F_{ik} \in \mathbb{R}$. More precisely, if $\frac{b}{a} \in \mathbb{Q}^*_+$, we have $(b,a) \sim (\beta, \alpha)$, $\alpha, \beta \in \mathbb{N}^*$, and we ask that $i \in \mathbb{N}$, if $\frac{b}{a} \in \mathbb{R}^* \setminus \mathbb{Q}^*_+$, we choose $i = 0$, and if moreover $(b,a) \sim (\beta, \alpha) = (-1, \alpha)$, $\alpha \in \mathbb{N}^*$, we also accept the value $i = 1$, but add the conditions $A_{10} = B_{10} = C_{10} = 0$. We define $(q_{ic}, p_i) := (2\alpha i + 2 + c, \beta i + 1)$, where $c \in \{0,1,2\}$ denotes the cochain degree, so that the double superscript in the LHS is the bidegree $(r,s) = (j_1 + j_2 + c, j_3)$ of the RHS.

Observe that the Z^{q_{ic}, p_i} are exactly the representatives of type 1 of the classes of $E_2 \simeq {}^0H(\mathscr{P})$.

Lemma 7. *For any admissible exponent i and any cochain degree $c \in \{0,1,2\}$, the cochains $Z^{q_{ic}-2k, p_i+2k}$, $k \in \{0, 1, \ldots, \alpha i\}$, constitute a solution of system $S(Z^{q_{ic}, p_i}; \alpha i + 1)$, if and only if, for any $k \in \{0, 1, \ldots, \alpha i - 1\}$,*

$$A_{i,k+1} = A_{ik}, \text{ if } c = 0, \quad (C_0)$$

$$B_{i,k+1} + C_{i,k+1} = \frac{(\alpha i - k + 1)(B_{ik} + C_{ik}) - 2D_{ik}}{\alpha i - k} \text{ and } D_{i,k+1} = D_{ik}, \text{ if } c = 1, \quad (C_1)$$

$$E_{i,k+1} - F_{i,k+1} = \frac{\alpha i - k + 1}{\alpha i - k} (E_{ik} - F_{ik}), \text{ if } c = 2. \quad (C_2)$$

Furthermore,

$$
\begin{aligned}
& d'' Z^{q_{ic}-2\alpha i, p_i+2\alpha i} = d'' Z^{2+c, p_i+2\alpha i} \\
& = \begin{cases} 0, \textit{ for } c = 0, \\ (2a+b)^{-\alpha i}(B_{i,\alpha i} + C_{i,\alpha i} - 2D_{i,\alpha i})\, z^{2+i(2\alpha+\beta)}\partial_{12}, \textit{ for } c = 1, \\ (2a+b)^{-\alpha i}(E_{i,\alpha i} - F_{i,\alpha i})\, z^{3+i(2\alpha+\beta)}\partial_{123}, \textit{ for } c = 2, \end{cases}
\end{aligned}
$$

is a d'-coboundary if and only if the coefficient vanishes.

Proof. We must compute the differentials $d' = \partial_{\Lambda_{4,I}} = [\Lambda_{4,I}, .]$ and $d'' = \partial_{\Lambda_{4,II}} = [D^{-1}z^3 Y_{12}, .] =: [f\mathbf{X}, .]$ on the $Z^{q_{ic}-2k, p_i+2k}$. These cochains have the form $g\mathscr{Y} := D^{-1}X^J\mathscr{Y} := D^{-1}D'^n z^m \sum_j r_j \mathbf{Y}_j,\ n, m \in \mathbb{N}, r_j \in \mathbb{R}$, where the degree c of wedge product $\mathbf{Y}_j$ is independent of j. Hence, Equation (5.7) gives

$$d''(g\mathscr{Y}) = [f\mathbf{X}, g\mathscr{Y}] = f[\mathbf{X}, g] \wedge \mathscr{Y} + (-1)^c g[\mathscr{Y}, f] \wedge \mathbf{X}.$$

On the other hand, Equations (5.5) and (5.6) entail $d'(g\mathscr{Y}) = [\Lambda_{4,I}, g\mathscr{Y}] = [\Lambda_{4,I}, g] \wedge \mathscr{Y} = \sum_\ell X_\ell(g)\ Y_\ell \wedge \mathscr{Y}$, where $X_1 = bY_2 - aY_3, X_2 = aY_3 - bY_1, X_3 = a(Y_1 - Y_2)$. Since

$$Y_\ell\left(\frac{X^J}{D}\right) = (j_\ell - 1)\frac{X^J}{D}$$

(same notations as above), we get

$$d'(g\mathscr{Y}) = g\,(b(n-1) - a(m-1))\,(Y_1 - Y_2) \wedge \mathscr{Y}.$$

In particular, we recover the result $d' Z^{q_{ic}, p_i} = ig(b\alpha - a\beta)(Y_1 - Y_2) \wedge \mathscr{Y} = 0$, and, when setting $a = 0, b = 1, \mathscr{Y} = 1$, we find

$$[\mathbf{X}, g] = g(n-1)(Y_1 - Y_2).$$

We now compute $d'' Z^{q_{ic}-2k, p_i+2k}$, $k \in \{0, 1, \ldots, \alpha i\}$, and $d' Z^{q_{ic}-2(k+1), p_i+2(k+1)}$, $k \in \{0, 1, \ldots, \alpha i - 1\}$.

1. $c = 0$

It follows from the preceding equations that

$$d'' Z^{q_{i0}-2k, p_i+2k} = \complement^k_{\alpha i}(\alpha i - k) A_{ik}(2a+b)^{-k} D^{-1} D'^{\alpha i - k} z^{3+\beta i+2k}(Y_1 - Y_2)$$

and that

$$d'Z^{q_{i0}-2(k+1),p_i+2(k+1)} = -\complement_{\alpha i}^{k+1}(k+1)A_{i,k+1}(2a+b)^{-k}D^{-1}D'^{\alpha i-k}z^{3+\beta i+2k}(Y_1-Y_2).$$

Since for any $p,n\in\mathbb{N}, p<n$, we have $\complement_n^p(n-p)=\complement_n^{p+1}(p+1)$, the sum of these coboundaries vanishes if and only if $A_{i,k+1}=A_{ik}$, for any $k\in\{0,1,\dots,\alpha i-1\}$. For $k=\alpha i$, we get

$$d''Z^{q_{i0}-2\alpha i,p_i+2\alpha i}=d''Z^{2,p_i+2\alpha i}=0.$$

2. $c=1$
A short computation shows that

$$\begin{aligned}
&d''Z^{q_{i1}-2k,p_i+2k}\\
&\quad=\complement_{\alpha i}^{k}(2a+b)^{-k}D^{-1}D'^{\alpha i-k}z^{3+\beta i+2k}[-D_{ik}(\alpha i-k)Y_{23}-D_{ik}(\alpha i-k)Y_{31}\\
&\quad+((B_{ik}+C_{ik})(\alpha i-k+1)-2D_{ik})Y_{12}]
\end{aligned}$$

and that

$$\begin{aligned}
&d'Z^{q_{i1}-2(k+1),p_i+2(k+1)}\\
&\qquad=-\complement_{\alpha i}^{k+1}(k+1)(2a+b)^{-k}D^{-1}D'^{\alpha i-k}z^{3+\beta i+2k}[-D_{i,k+1}Y_{23}\\
&\qquad-D_{i,k+1}Y_{31}+(B_{i,k+1}+C_{i,k+1})Y_{12}].
\end{aligned}$$

If $k\in\{0,1,\dots,\alpha i-1\}$, the sum of these coboundaries vanishes if and only if

$$B_{i,k+1}+C_{i,k+1}=\frac{(\alpha i-k+1)(B_{ik}+C_{ik})-2D_{ik}}{\alpha i-k}\quad\text{and}\quad D_{i,k+1}=D_{ik}.$$

Furthermore, for $k=\alpha i$, the first of the preceding "coboundary equations" provides the announced result for $d''Z^{q_{i1}-2\alpha i,p_i+2\alpha i}$. As $\mathbb{R}[[z]]\partial_{12}$ is part of the cohomology of $\overline{d}_0=d'$, this d''-coboundary is a d'-coboundary if and only if its coefficient vanishes.

3. $c=2$
We immediately obtain

$$d''Z^{q_{i2}-2k,p_i+2k}=\complement_{\alpha i}^{k}(\alpha i-k+1)(2a+b)^{-k}D^{-1}D'^{\alpha i-k}z^{3+\beta i+2k}(E_{ik}-F_{ik})Y_{123}$$

and

$$\begin{aligned}
&d'Z^{q_{i2}-2(k+1),p_i+2(k+1)}\\
&\qquad=-\complement_{\alpha i}^{k+1}(k+1)(2a+b)^{-k}D^{-1}D'^{\alpha i-k}z^{3+\beta i+2k}(E_{i,k+1}-F_{i,k+1})Y_{123}.
\end{aligned}$$

Hence the announced upshots. ∎

Let us recall that the admissible values of i (and the potential conventions on coefficients A_{10}, B_{10}, C_{10}) depend on quotient b/a. Moreover, for $b/a \in \mathbb{R}^* \setminus \mathbb{Q}^*_+, (b,a) \not\approx (-1,\alpha), \alpha \in \mathbb{N}^*$, we set $\alpha = 1 \in \mathbb{N}^*$. Actually, in this case, α needed not be defined before, as it was systematically multiplied by $i = 0$.

The following theorem provides the complete description of the considered SpecSeq.

Theorem 18. *The even terms $E_{2(n-1)\alpha+4} = E_{2(n-1)\alpha+6} = \ldots = E_{2n\alpha+2}$ $(n \in \mathbb{N}$; for $n = 0$, this package contains only term $E_2)$ of the above defined SpecSeq are canonically isomorphic (i.e. $d_{2(n-1)\alpha+4} = d_{2(n-1)\alpha+6} = \ldots = d_{2n\alpha} = 0$) and admit the below system of representatives:*
1. All representatives of type 2 of $E_2 \sim {}^0H(\mathscr{P})$, except

$$\mathbb{R}z^{i(2\alpha+\beta)+2}\partial_{12} \quad \textit{and} \quad \mathbb{R}z^{i(2\alpha+\beta)+3}\partial_{123},$$

for all admissible $i \in \{0,1,\ldots,n-1\}$.
2. All representatives of type 1 of $E_2 \sim {}^0H(\mathscr{P})$, altered as follows:

- *For all admissible $i \in \{n, n+1, \ldots\}$,*

$$Z^{q_{ic},p_i} \rightsquigarrow \sum_{k=0}^{\alpha n} Z^{q_{ic}-2k,p_i+2k},$$

where the coefficients $A_{ik}, B_{ik}, C_{ik}, D_{ik}, E_{ik}, F_{ik}$ incorporated into the terms of the RHS verify conditions $(C_0)-(C_2)$ of Lemma 7 up to $k = \alpha n - 1$.

- *For all admissible $i \in \{0,1,\ldots,n-1\}$,*

$$Z^{q_{ic},p_i} \rightsquigarrow \begin{cases} \left(D' + \frac{z^2}{2a+b}\right)^{\alpha i} z^{\beta i} A_{i0}, & \textit{if} \quad c = 0, \\ \left(D' + \frac{z^2}{2a+b}\right)^{\alpha i} z^{\beta i} \left(B_{i0}(Y_1 + \frac{1}{2}Y_3) + C_{i0}(Y_2 + \frac{1}{2}Y_3)\right), & \textit{if}\ c = 1, \\ D'^{\alpha i} z^{\beta i} E_{i0}(Y_{23} + Y_{31}), & \textit{if} \quad c = 2. \end{cases}$$

Proof. The proof is by induction on n. For $n = 0$, Theorem 18 is obviously valid. Assume now that it holds true for $0,1,\ldots,n-1$ $(n \in \mathbb{N}^*)$. We first transfer the description of $E_{2(n-2)\alpha+4} = \ldots = E_{2(n-1)\alpha+2}$ to the concrete model side, in order to compute $\overline{d}_{2(n-1)\alpha+2}$. When having a look at the packages of terms that are known to be isomorphic, we see that the only differentials (under $\overline{d}_{2(n-1)\alpha+2}$) that do not vanish are $\overline{d}_{2m\alpha+2}$ ($m \in \{0,1,\ldots,n-2\}$). Hence the target of vector space isomorphism

$$I_{2(n-1)\alpha+2} : E_{2(n-1)\alpha+2} \to {}^{(n-2)\alpha+1}H\big({}^{(n-3)\alpha+1}H(\ldots{}^{1}H({}^{0}H(\mathscr{P})))\big),$$

which—as it appears from its general description—maps the system of $E_{2(n-1)\alpha+2}$-representatives onto the system evidently made up by:

1. All representatives of type 2 of E_2, except $\mathbb{R}z^{i(2\alpha+\beta)+2}\partial_{12}$ and $\mathbb{R}z^{i(2\alpha+\beta)+3}\partial_{123}$, for all admissible $i \in \{0,1,\ldots,n-2\}$.

2. All representatives of type 1 of E_2, Z^{q_{ic},p_i}, i admissible, $c \in \{0,1,2\}$, with, for all admissible $i \in \{0,1,\ldots,n-2\}$, $B_{i0}+C_{i0} = 2D_{i0}$, if $c=1$, and $E_{i0} = F_{i0}$, if $c=2$.

We now compute the cohomology of space

$$\big({}^{(n-2)\alpha+1}H(\ldots{}^{0}H(\mathscr{P})),\overline{d}_{2(n-1)\alpha+2}\big).$$

If z^{qp} is one of the representatives of the preceding system,

$$\overline{d}_{2(n-1)\alpha+2}[[z^{qp}]_0\ldots]_{(n-2)\alpha+1} = [[d'' z^{q-2\alpha(n-1),p+2\alpha(n-1)}]_0\ldots]_{(n-2)\alpha+1}, \quad (5.12)$$

where $z^{q-2\alpha(n-1),p+2\alpha(n-1)}$ is the last entry of an arbitrary solution of $S(z^{qp};\alpha(n-1)+1)$.

The d''-coboundary of any z^{qp} of type 2 vanishes. This is obvious if z^{qp} is a 3-cochain or has the form $\mathrm{Cas}(\Lambda_{4,I})Y_{12}$ (as $d'' = [\Lambda_{4,II},.] = [D^{-1}z^3Y_{12},.]$). If z^{qp} is a 2-cochain with singularities, e.g. $D^{-1}p(x)Y_{23}$, where $p(x)$ is a polynomial in x, we get $d''z^{qp} = [D^{-1}z^3Y_{12}, D^{-1}p(x)Y_{23}] = -z^3p(x)D^{-2}Y_{123} + z^3p(x)D^{-2}Y_{123} = 0$. Hence, for any type 2 representative z^{qp}, system $S(z^{qp};s)$ admits solution $(z^{qp},0,\ldots,0)$, for any $s \in \mathbb{N}^*$ ($\mathbf{S_1}$, representative extended by 0 [reference needed in the following]), and Coboundary (5.12) vanishes.

Let now z^{qp} be a representative Z^{q_{ic},p_i} of the first type. We know from Lemma 7 that $Z^{q_{ic}-2k,p_i+2k}$, $k \in \{0,1,\ldots,\alpha i\}$, with coefficients that verify (C_0)-(C_2), is a solution of $S(Z^{q_{ic},p_i};\alpha i+1)$.

1. For any admissible $i \in \{n,n+1,\ldots\}$, this solution can be truncated to a solution of $S(Z^{q_{ic},p_i};\alpha n+1)$ ($\mathbf{S_2}$, truncated standard solution). Hence, Coboundary (5.12) vanishes.

2. If i is admissible in $\{0,1,\ldots,n-2\}$, we have

$$B_{i0}+C_{i0}=2D_{i0} \quad \text{and} \quad E_{i0}=F_{i0}.$$

It then follows from (C_1) and (C_2) that the same relation holds for $k=\alpha i$, i.e. that $B_{i,\alpha i}+C_{i,\alpha i}=2D_{i,\alpha i}$ and $E_{i,\alpha i}=F_{i,\alpha i}$. This however implies that $d''Z^{q_{ic}-2\alpha i,p_i+2\alpha i}=0$, so that system $S(Z^{q_{ic},p_i};\alpha n+1)$ admits an obvious solution ($\mathbf{S_3}$, standard solution extended by 0) and that Coboundary (5.12) vanishes again.

3. If $i=n-1$ is admissible,

$$\begin{aligned}&\overline{d}_{2(n-1)\alpha+2}[[Z^{q_{n-1,c},p_{n-1}}]_0\ldots]_{(n-2)\alpha+1}\\ &\qquad\qquad\qquad=[[d''Z^{q_{n-1,c}-2\alpha(n-1),p_{n-1}+2\alpha(n-1)}]_0\ldots]_{(n-2)\alpha+1}.\end{aligned}$$

In view of Lemma 7, this class vanishes for $c=0$, and coincides, if $c=1$ (resp. $c=2$), up to a coefficient, with class $[[z^{(n-1)(2\alpha+\beta)+2}\partial_{12}]_0\ldots]_{(n-2)\alpha+1}$ (resp. $[[z^{(n-1)(2\alpha+\beta)+3}\partial_{123}]_0\ldots]_{(n-2)\alpha+1}$). The above depicted system of representatives of ${}^{(n-2)\alpha+1}H(\ldots{}^0H(\mathscr{P}))$ shows that the preceding two classes do not vanish. Hence, the cocycle-condition is equivalent with the annihilation of the mentioned coefficient, i.e. with

$$B_{n-1,\alpha(n-1)}+C_{n-1,\alpha(n-1)}=2D_{n-1,\alpha(n-1)} \quad (\text{resp.} \quad E_{n-1,\alpha(n-1)}=F_{n-1,\alpha(n-1)}),$$

or, as already explained,

$$B_{n-1,0}+C_{n-1,0}=2D_{n-1,0} \quad (\text{resp.} \quad E_{n-1,0}=F_{n-1,0}). \tag{5.13}$$

Since it clearly follows from our computations that the space of $\overline{d}_{2(n-1)\alpha+2}$-coboundaries is generated by the two just encountered non-vanishing classes, cohomology space ${}^{(n-1)\alpha+1}H({}^{(n-2)\alpha+1}H(\ldots{}^0H(\mathscr{P})))$ has the same system of representatives than its predecessor ${}^{(n-2)\alpha+1}H(\ldots{}^0H(\mathscr{P}))$, but with exclusions carried out and conditions on $B_{i0},C_{i0},D_{i0},E_{i0},F_{i0}$ valid for all admissible $i\in\{0,1,\ldots,n-1\}$.

It now suffices to apply Corollary 4 to cohomology space ${}^{(n-1)\alpha+1}H({}^{(n-2)\alpha+1}H(\ldots{}^0H(\mathscr{P})))$. Observe first that ($\mathbf{S_1}$)-($\mathbf{S_3}$) entail existence of a solution of $S(z^{qp};\alpha n+1)$, for all representatives z^{qp} dissimilar from $Z^{q_{n-1,c},p_{n-1}}$. But, as the coefficients of these last representatives—viewed as representatives of the preceding $\overline{d}_{2(n-1)\alpha+2}$-cohomology space—satisfy Conditions

(5.13), the coboundaries $d''Z^{q_{n-1,c}-2\alpha(n-1),p_{n-1}+2\alpha(n-1)}$ vanish. So the previously met solution of $S(Z^{q_{n-1,c},p_{n-1}};\alpha\,(n-1)+1)$ can be indefinitely extended by 0 (**S$_4$**, standard solution extended by 0). Finally, Corollary 4 is applicable for $s=\alpha\,n+1$.

Hence, spaces $E_{2(n-1)\alpha+4}=\ldots=E_{2n\alpha+2}$ coincide and we build, from the known system z^{qp} of representatives of ${}^{(n-1)\alpha+1}H(\ldots{}^{0}H(\mathscr{P}))$, a system of $E_{2n\alpha+2}$ by just summing-up the entries of any solutions of the systems $S(z^{qp};\alpha\,n+1)$. For any Z^{q_{ic},p_i}, the coefficients of which verify

$$B_{i0}+C_{i0}=2D_{i0}\quad(c=1)\quad\text{and}\quad E_{i0}=F_{i0}\quad(c=2),$$

the standard $Z^{q_{ic}-2k,p_i+2k}$, $k\in\{0,1,\ldots,\alpha\,i\}$, are solution, see Lemma 7, of $S(Z^{q_{ic},p_i};\alpha\,i+1)$, e.g. if we choose

$$\begin{aligned}&A_{ik}=A_{i0}\ \ (c=0),\ \ B_{ik}=B_{i0},\ \ C_{ik}=C_{i0},\\&D_{ik}=D_{i0}\ \ (c=1),\quad\text{and}\quad E_{ik}=F_{ik}=0\ \ (c=2,k\neq 0).\end{aligned}\tag{5.14}$$

If we pull the concrete side representatives back to theoretical side representatives using these solutions, we exactly get, see S_1-S_4, the sought-after system. ∎

Remark.

1. We already observed previously the obvious fact that when pulling RHS-representatives back, using different solutions of the standard system, we obtain equivalent LHS-representatives. These equivalent LHS-representatives would implement cohomologous cocycles in cohomology space $H(\Lambda_4)$. Choice (5.14) will induce in cohomology the most basic possible cocycles.

2. Note also that in view of Theorem 18 and our conventions on coefficients B_{10},C_{10}, cocycle $\mathbb{R}\mathscr{A}_\alpha Y_3$ disappears from all spaces E_{2r}, $r\geq 2\alpha+4$.

5.4.5 Limit of the SpecSeq and reconstruction of the cohomology

The limit of the SpecSeq can be guessed from Theorem 18. However, we already stressed the importance of a careful reading of all results through the isomorphisms involved in the theory of spectral sequences. The proof of Theorem 18 shows for instance that the appropriate Casimir functions appear, when we pull the RHS-representatives back to the LHS, i.e. read them through isomorphism $I^{-1}_{2(n-1)\alpha+3}$. Hence, a precise description of the isomorphisms that lead now to the cohomology of Λ_4 is essential.

General results

Let us consider the SpecSeq associated with a (regular) filtered graded differential space (K, d, K_p, K^n) and recall that the limit spaces $E_\infty^{pq}, Z_\infty^{pq}, B_\infty^{pq}$ are defined exactly as spaces $E_r^{pq}, Z_r^{pq}, B_r^{pq}$, see Section 5.3, so that Z_∞^{pq} and B_∞^{pq} are the spaces of cocycles and coboundaries in $K_p \cap K^{p+q}$ respectively. For any fixed p and q, regularity implies that the target space of the restriction of d_r to E_r^{pq} vanishes, if $r > q+1$. Thus, there is a canonical linear surjective map

$$\vartheta_r^{pq} : E_r^{pq} \to H^{pq}(E_r, d_r) \to E_{r+1}^{pq}.$$

For $s \geq r > q+1$, we define $\theta_{rs}^{pq} := \vartheta_{s-1}^{pq} \circ \ldots \circ \vartheta_r^{pq} : E_r^{pq} \to E_s^{pq}$, and for $r > q+1$, we set

$$\theta_r^{pq} : E_r^{pq} \ni [\mathfrak{z}_r^{pq}]_{E_r^{pq}} \to [\mathfrak{z}_r^{pq}]_{E_\infty^{pq}} \in E_\infty^{pq}. \tag{5.15}$$

Due to regularity, the first two of the well-known inclusions $Z_\infty^{pq} \subset Z_r^{pq}, Z_\infty^{p+1,q-1} \subset Z_{r-1}^{p+1,q-1}$, and $B_{r-1}^{pq} \subset B_\infty^{pq}$ are actually double inclusions, and $Z_\infty^{p+1,q-1} + B_{r-1}^{pq} \subset Z_\infty^{p+1,q-1} + B_\infty^{pq} \subset Z_\infty^{pq}$. Hence, map θ_r^{pq} is canonical, linear and surjective. It is known that space E_∞^{pq} together with the preceding linear surjections θ_r^{pq} is a model of the inductive limit of the inductive system $(E_r^{pq}, \theta_{rs}^{pq})$. Consider now a first quadrant SpecSeq (i.e. $p, q \in \mathbb{N}$) and assume that $K_0 = K$. For any p, q, the SpecSeq collapses at

$$r > \sup(p, q+1),$$

more precisely, $E_r^{pq} = E_\infty^{pq}$ and $\theta_r^{pq} = \mathrm{id}$. Indeed, in this case, in addition to the aforementioned double inclusions $(r > q+1)$, we now have also $B_{r-1}^{pq} = K_p \cap dK_{p+1-r} \cap K^{p+q} = K_p \cap dK_0 \cap K^{p+q} = B_\infty^{pq}$ $(r > p)$. Hence the announced results.

The SpecSeq associated with any filtered graded differential space is convergent in the sense that limit E_∞^{pq} is known to be isomorphic as a vector space with term G^{pq} of the bigraded space $G(H(K))$, G for short, associated with the filtered graded space $H(K)$. Let us recall that the filtration of $H(K)$ is induced by that of K. More precisely, injection $i : (K_p, d) \to (K, d)$ is a morphism of differential spaces and $H_p := i_\sharp H(K_p) \subset H(K)$ is the mentioned filtration of $H(K)$. In order to reduce notations, we denote the terms of the grading of $H(K)$ simply by H^n. It is a fact that the filtration and the grading of $H(K)$ are compatible and that filtration H_p is regular if its generatrix K_p is. Hence, $H_p = \oplus_{q \in \mathbb{N}} H_p \cap H^{p+q} =: \oplus_{q \in \mathbb{N}} H_p^{p+q}$. Finally, it is a matter of knowledge that the isomorphism, say ι, between $G^{pq} := H_p^{p+q}/H_{p+1}^{p+q}$ and E_∞^{pq} is canonical,

$$\iota : E_\infty^{pq} \ni [\mathfrak{z}_\infty^{pq}]_{E_\infty^{pq}} \to [[\mathfrak{z}_\infty^{pq}]_{H_p^{p+q}}]_{G^{pq}} \in G^{pq}. \tag{5.16}$$

We now reconstruct $H(K)$ from G. Let us again focus on a first quadrant SpecSeq associated with a (regular) filtered complex (K,d,K_p,K^n) (such that $K_0=K$). For any $n\in\mathbb{N}$, we denote by $G^{n-j_1,j_1},G^{n-j_2,j_2},\ldots,G^{n-j_{k_n},j_{k_n}}$, $n\geq j_1>j_2>\ldots>j_{k_n}\geq 0$, the non vanishing $G^{pq}=H_p^{p+q}/H_{p+1}^{p+q}$, $p+q=n$. Since $H_0=H(K)$ and $H_p^n=H_p\cap H^n=0,\forall p>n$, it follows that

$$\begin{aligned}H^n=H_0^n=\ldots=H^n_{n-j_1}\supset H^n_{n-j_1+1}=\ldots=H^n_{n-j_2}\\ \supset H^n_{n-j_2+1}\ldots H^n_{n-j_{k_n}}\supset H^n_{n-j_{k_n}+1}=\ldots=H^n_n=0.\end{aligned}$$

Hence,

$$H^n/H^n_{n-j_2}=G^{n-j_1,j_1},\ldots,H^n_{n-j_{k_n-1}}/H^n_{n-j_{k_n}}=G^{n-j_{k_n-1},j_{k_n-1}},H^n_{n-j_{k_n}}=G^{n-j_{k_n},j_{k_n}}.$$

However, if $B/A=C$, A a vector subspace of B, the sequence $0\to A\xrightarrow{i}B\xrightarrow{p}C\to 0$, is a short exact sequence of vector spaces. A short exact sequence in a category is split if and only if kernel A admits in vector space B a complementary subspace that is a subobject, or—alternatively—if and only if there is a right inverse morphism $\chi:C\to B$ of projection p. Of course, in the category of vector spaces such a sequence is always split. If χ is a linear right inverse of p, we have $B=A\oplus\chi(C)$.

Let us now come back to our circumstances. If $\chi_1,\ldots,\chi_{k_n-1}$ denote splittings of the involved sequences, central extension H^n is given by

$$H^n=\chi_1(G^{n-j_1,j_1})\oplus\ldots\oplus\chi_{k_n-1}(G^{n-j_{k_n-1},j_{k_n-1}})\oplus G^{n-j_{k_n},j_{k_n}}. \qquad (5.17)$$

It follows of course from Equation (5.17) that $H(K)$ is—in this vector space setting—isomorphic with $G=G(H(K))$. It is known that in the case of ring coefficients, extension problems may prevent the reconstruction of $H(K)$ from $G(H(K))$.

Application to Poisson tensor Λ_4

The next proposition provides a system of representatives of the cohomology space of

$$\Lambda_4=ayz\partial_{23}+axz\partial_{31}+(bxy+z^2)\partial_{12}\quad(a\neq 0,b\neq 0).$$

Remember that $D'=xy$ and $Y_1=x\partial_1,Y_2=y\partial_2,Y_3=z\partial_3$. If $\frac{b}{a}\sim\frac{\beta}{\alpha}\in\mathbb{Q}_+^*$, we define

$$\mathrm{Cas}(\Lambda_4):=\oplus_{i\in\mathbb{N}}\mathbb{R}\left(D'+\frac{z^2}{2a+b}\right)^{\alpha i}z^{\beta i}$$

and use the above introduced notation $\mathrm{Cas}(\Lambda_{4,I})=\oplus_{i\in\mathbb{N}}\mathbb{R}D'^{\alpha i}z^{\beta i}$. If $\frac{b}{a}\in\mathbb{R}^*\setminus\mathbb{Q}_+^*$, we set $\mathrm{Cas}(\Lambda_4):=\mathbb{R}$ and, as aforementioned, $\mathscr{A}_\alpha=D'^\alpha z^{-1}$.

Theorem 19. *1. If $\frac{b}{a} \in \mathbb{Q}_+^*$, the cohomology of Λ_4 is given by*

$$
\begin{aligned}
&E_\infty \sim G \sim H(\Lambda_4) = \\
&\mathrm{Cas}(\Lambda_4) \oplus \mathrm{Cas}(\Lambda_4)(Y_1 + \frac{1}{2}Y_3) \oplus \mathrm{Cas}(\Lambda_4)(Y_2 + \frac{1}{2}Y_3) \\
&\oplus \mathrm{Cas}(\Lambda_{4,I})(Y_{23} + Y_{31}) \oplus \mathrm{Cas}(\Lambda_{4,I})Y_{12} \oplus \mathrm{Cas}(\Lambda_{4,I})Y_{123} \\
&\oplus \bigoplus_{k \in \mathbb{N} \setminus \mathbb{N}(2\alpha+\beta)+2} \mathbb{R}z^k \partial_{12} \oplus \bigoplus_{k \in \mathbb{N} \setminus \mathbb{N}(2\alpha+\beta)+3} \mathbb{R}z^k \partial_{123} \\
&\oplus \begin{cases} \mathbb{R}[[x]]\partial_{23} \oplus \mathbb{R}[[y]]\partial_{31} \oplus (\mathbb{R}[[x]] \oplus \mathbb{R}[[y]])\partial_{123}, \textit{if } b = a \\ 0, \textit{otherwise} \end{cases}
\end{aligned}
$$

2. If $\frac{b}{a} \in \mathbb{R}^* \setminus \mathbb{Q}_+^*$, we have :

$$
\begin{aligned}
&E_\infty \sim G \sim H(\Lambda_4) = \\
&\mathrm{Cas}(\Lambda_4) \oplus \mathrm{Cas}(\Lambda_4)(Y_1 + \frac{1}{2}Y_3) \oplus \mathrm{Cas}(\Lambda_4)(Y_2 + \frac{1}{2}Y_3) \\
&\oplus \mathrm{Cas}(\Lambda_4)(Y_{23} + Y_{31}) \oplus \mathrm{Cas}(\Lambda_4)Y_{12} \oplus \mathrm{Cas}(\Lambda_4)Y_{123} \\
&\oplus \begin{cases} \oplus \mathbb{R}\mathscr{A}_\alpha(Y_{23} + Y_{31}) \oplus \mathbb{R}\mathscr{A}_\alpha Y_{123}, \ \textit{if } (b,a) \sim (-1,\alpha) \\ 0, \textit{otherwise} \end{cases} \\
&\oplus \begin{cases} \bigoplus_{k \in \mathbb{N} \setminus \{2, 2\alpha+1\}} \mathbb{R}z^k \partial_{12} \oplus \bigoplus_{k \in \mathbb{N} \setminus \{3, 2\alpha+2\}} \mathbb{R}z^k \partial_{123}, \ \textit{if } (b,a) \sim (-1,\alpha) \\ \bigoplus_{k \in \mathbb{N} \setminus \{2\}} \mathbb{R}z^k \partial_{12} \oplus \bigoplus_{k \in \mathbb{N} \setminus \{3\}} \mathbb{R}z^k \partial_{123}, \textit{otherwise} \end{cases}
\end{aligned}
$$

Proof. Fix $a, b \in \mathbb{R}^*$ and take any representative of E_2. Remember that the representatives of type 1 are exactly the cochains Z^{q_{ic}, p_i} (i admissible, $c \in \{0, 1, 2\}$). Moreover, we say that a representative of type 2 is critical if it has the form $\mathbb{R}z^{i(2\alpha+\beta)+2}\partial_{12}$ or $\mathbb{R}z^{i(2\alpha+\beta)+3}\partial_{123}$ (i admissible). If the considered representative z^{qp} is of type 2 and not critical (resp. of type 2 and critical, of type 1), we choose $n \in \mathbb{N}$ such that $2n\alpha + 2 > \sup(p, q+1)$ (resp. $2n\alpha + 2 > \sup(p, q+1, 2i\alpha + 2)$ [hence, we have $n - 1 \geq i$], $2n\alpha + 2 > \sup(p_i, q_{ic} + 1, 2i\alpha + 2)$). The system of representatives of $E_\infty \sim G$ specified in Theorem 19 arises now from Theorem 18 and from the canonical isomorphisms (5.15) (condition: $r > \sup(p, q+1)$) and (5.16). These representatives are representatives of bases of the non vanishing G^{pq}. In order to compute $H(K)$, it suffices to build arbitrary splittings in keeping with Equation (5.17). Hence, it suffices to choose, for any class of any basis of the concerned G^{pq}, an arbitrary representative, e.g. the aforementioned one. It follows that $H(K)$ admits exactly the same representatives as $E_\infty \sim G$. ∎

Remarks. Hence, the twist makes a threefold impact on cohomology. When applying our computing device, see Theorem 18, we get, at *each* turn of the handle, on the model level, roughly speaking, cocycle-conditions on the coefficients related with an *additional* power of the basic Casimir $C_{\Lambda_{4,I}}$ of $\Lambda_{4,I}$, and we exclude a *supplementary* pair of singularity-induced classes. These conditions appear in cohomology as terms Y_i or Y_{ij} with the same "Casimir-coefficient". Eventually, the cocycle-conditions allow to lift the mentioned *accessory* power of $C_{\Lambda_{4,I}}$ to the real level as power of Casimir C_{Λ_4} of Λ_4 or—depending on cochain degree—as power of Casimir $C_{\Lambda_{4,I}}$. We know that such a lift is not unique and that two different ones are cohomologous. It follows from Theorem 19 (resp. from the proof of Theorem 18) that any term of $\mathrm{Cas}(\Lambda_4)(Y_{23}+Y_{31})$ is a Λ_4-cocycle (resp. can be chosen as lift of the corresponding term in $\mathrm{Cas}(\Lambda_{4,I})(Y_{23}+Y_{31})$, as well as this term itself). So any term of $\mathrm{Cas}(\Lambda_4)(Y_{23}+Y_{31})$ is cohomologous to the analogous term in $\mathrm{Cas}(\Lambda_{4,I})(Y_{23}+Y_{31})$. Finally, the aforementioned proof allows to see that $\Lambda_{4,I}$-cocycle $\mathbb{R}A_\alpha Y_3 = \mathbb{R}D'^{\alpha} z^{-1} Y_3$, which is not a product of two Λ_4-cocycles, is a Λ_4-cocycle if and only if its coefficient vanishes.

Let us in the end have a look at singularities. The singular locus of $\Lambda_{4,I}$ (resp. Λ_4) is made up by the three coordinate axes (resp. the axis of abscissæ and the axis of ordinates). Comparing the results of Proposition 36 and of Theorem 19, we see that the twist $\Lambda_{4,II}$, which removes the z-axis from the singular locus, cancels only part of the corresponding polynomials in cohomology. We already observed in [MP06] that, for r-matrix induced tensors, some coefficients of non bounding 2- or 3-cocycles can just be interpreted as polynomials on singularities via an extension of the polynomial ring of the singular locus. In the case of twisted r-matrix induced structures, some of these polynomial coefficients are simply not polynomials on singularities.

5.5 Formal cohomology of Poisson tensor Λ_8

We now describe the cohomology space of the twisted quadratic Poisson structure

$$\Lambda_8 = \left(\frac{\mathfrak{a}+\mathfrak{b}}{2}(x^2+y^2) \pm z^2\right)\partial_{12} + \mathfrak{a}xz\partial_{23} + \mathfrak{a}yz\partial_{31} \quad (\mathfrak{a} \neq 0, \mathfrak{b} \neq 0).$$

If we substitute c (resp. b) for $-\mathfrak{b}$ (resp. $(\mathfrak{a}+\mathfrak{b})/2$), tensor Λ_8 reads

$$\Lambda_8 = b(x^2+y^2)\partial_{12} + (2b+c)xz\partial_{23} + (2b+c)yz\partial_{31} \pm z^2\partial_{12}. \qquad (5.18)$$

Henceforth we use parameters b and c. Assumptions $\mathfrak{a} \neq 0, \mathfrak{b} \neq 0$ are equivalent with $2b+c \neq 0, c \neq 0$. Moreover, the r-matrix induced part $\Lambda_{8,I} = b(x^2+y^2)\partial_{12} + (2b+c)xz\partial_{23} + (2b+c)yz\partial_{31}$ of Λ_8 is nothing but structure Λ_7 with parameter $a=0$, see [MP06, Section 9], so that term $E_2 \simeq H(\Lambda_{8,I})$ of the spectral sequence follows from [MP06, Theorems 6,8,9].

Let us recall that the Y_i stem from $\Lambda_{8,I}$, i.e. from Λ_7. Hence, $Y_1 = x\partial_1 + y\partial_2, Y_2 = x\partial_2 - y\partial_1, Y_3 = z\partial_3$. We set $D' = x^2+y^2$. Moreover, if $\frac{b}{c} \in \mathbb{Q}, b(2b+c) < 0$, we denote by $(\beta,\gamma) \sim (b,c)$ the irreducible representative of the rational number $\frac{b}{c}$, with positive denominator, $\beta \in \mathbb{Z}, \gamma \in \mathbb{N}^*$, and if $\frac{b}{c} \in \mathbb{Q}, b(2b+c) > 0$, $(\beta,\gamma) \sim (b,c)$ denotes the irreducible representative with positive numerator, $\beta \in \mathbb{N}^*, \gamma \in \mathbb{Z}^*$.

Theorem 20. *The terms of the cohomology space of Λ_8 (see (5.18)) are given by the following equations:*

1. *If $\frac{b}{c} \in \mathbb{Q}, b(2b+c) > 0$,*

$$
\begin{aligned}
H^0(\Lambda_8) &= \mathrm{Cas}(\Lambda_8) = \oplus_{i\in\mathbb{N},\gamma i\in 2\mathbb{Z}}\mathbb{R}\left(D' \pm \frac{z^2}{3b+c}\right)^{(\beta+\frac{\gamma}{2})i} z^{\beta i},\\
H^1(\Lambda_8) &= \mathrm{Cas}(\Lambda_{8,I})Y_2 \oplus \mathrm{Cas}(\Lambda_8)(Y_1+Y_3),\\
H^2(\Lambda_8) &= \mathrm{Cas}(\Lambda_{8,I})Y_{12} \oplus \mathrm{Cas}(\Lambda_{8,I})Y_{23} \oplus \bigoplus_{k\in\mathbb{N}\setminus\mathbb{N}(3\beta+\gamma)+2} \mathbb{R}z^k\partial_{12},\\
H^3(\Lambda_8) &= \mathrm{Cas}(\Lambda_{8,I})Y_{123} \oplus \bigoplus_{k\in\mathbb{N}\setminus\mathbb{N}(3\beta+\gamma)+3} \mathbb{R}z^k\partial_{123},
\end{aligned}
$$

where $\mathrm{Cas}(\Lambda_{8,I}) = \oplus_{i\in\mathbb{N},\gamma i\in 2\mathbb{Z}}\mathbb{R}D'^{(\beta+\frac{\gamma}{2})i}z^{\beta i}$.

2. *If* $\frac{b}{c} \notin \mathbb{Q}$ *or* $\frac{b}{c} \in \mathbb{Q}, b(2b+c) < 0$,

$$\begin{array}{rcl} H^0(\Lambda_8) & = & \mathrm{Cas}(\Lambda_8) = \mathbb{R}\,, \\ H^1(\Lambda_8) & = & \mathrm{Cas}(\Lambda_8)Y_2 \oplus \mathrm{Cas}(\Lambda_8)(Y_1+Y_3)\,, \\ H^2(\Lambda_8) & = & \mathrm{Cas}(\Lambda_8)Y_{12} \oplus \mathrm{Cas}(\Lambda_8)Y_{23} \\ & & \oplus \begin{cases} \bigoplus_{k\in\mathbb{N}\setminus\{2,\gamma-1\}} \mathbb{R}z^k\partial_{12}, \textit{if } (b,c) \sim (-1,\gamma),\ \gamma \in \{4,6,8,...\} \\ \bigoplus_{k\in\mathbb{N}\setminus\{2\}} \mathbb{R}z^k\partial_{12}, \textit{otherwise} \end{cases} \\ & & \oplus \begin{cases} \mathbb{R}\mathscr{A}_\gamma Y_{23}, \textit{if } (b,c) \sim (-1,\gamma), \gamma \in \{4,6,8,...\} \\ 0, \textit{otherwise}\,, \end{cases} \\ H^3(\Lambda_8) & = & \mathrm{Cas}(\Lambda_8)Y_{123} \\ & & \oplus \begin{cases} \bigoplus_{k\in\mathbb{N}\setminus\{3,\gamma\}} \mathbb{R}z^k\partial_{123}, \textit{if } (b,c) \sim (-1,\gamma), \gamma \in \{4,6,8,...\} \\ \bigoplus_{k\in\mathbb{N}\setminus\{3\}} \mathbb{R}z^k\partial_{123}, \textit{otherwise} \end{cases} \\ & & \oplus \begin{cases} \mathbb{R}\mathscr{A}_\gamma Y_{123}, \textit{if } (b,c) \sim (-1,\gamma), \gamma \in \{4,6,8,....\} \\ 0, \textit{otherwise}\,, \end{cases} \end{array}$$

where $\mathscr{A}_\gamma = D'^{\frac{\gamma}{2}-1}z^{-1}$.

3. *If* $b = 0$,

$$\begin{array}{rcl} H^0(\Lambda_8) & = & \mathrm{Cas}(\Lambda_8) = \oplus_{i\in\mathbb{N}}\mathbb{R}\left(D' \pm \dfrac{z^2}{c}\right)^i\,, \\ H^1(\Lambda_8) & = & \mathrm{Cas}(\Lambda_{8,I})Y_2 \oplus \mathrm{Cas}(\Lambda_8)(Y_1+Y_3)\,, \\ H^2(\Lambda_8) & = & \mathrm{Cas}(\Lambda_{8,I})Y_{12} \oplus \mathrm{Cas}(\Lambda_{8,I})Y_{23} \oplus \displaystyle\bigoplus_{k\in\mathbb{N}\setminus\{2\mathbb{N}+2\}} \mathbb{R}z^k\partial_{12}\,, \\ H^3(\Lambda_8) & = & \mathrm{Cas}(\Lambda_{8,I})Y_{123} \oplus \displaystyle\bigoplus_{k\in\mathbb{N}\setminus\{2\mathbb{N}+3\}} \mathbb{R}z^k\partial_{123}\,, \end{array}$$

where $\mathrm{Cas}(\Lambda_{8,I}) = \oplus_{i\in\mathbb{N}}\mathbb{R}D'^i$.

Chapter 6

Strongly *r*-matrix Induced Tensors, Koszul Cohomology, and Arbitrary-Dimensional Quadratic Poisson Cohomology

6.1 Introduction

In this chapter, we focus on the formal Poisson cohomology associated with the quadratic Poisson tensors (QPT) Λ of $\mathbb{R}^n$ that read as real linear combination

$$\Lambda = \sum_{i<j} \alpha^{ij} Y_i \wedge Y_j =: \sum_{i<j} \alpha^{ij} Y_{ij}, \quad \alpha^{ij} \in \mathbb{R} \tag{6.1}$$

of the wedge products of n commuting linear vector fields $Y_1, \ldots, Y_n$, such that $Y_1 \wedge \ldots \wedge Y_n =: Y_{1\ldots n} \neq 0$. Let us recall that "formal" means that we substitute the space $\mathbb{R}[[x_1, \ldots, x_n]] \otimes \wedge \mathbb{R}^n$ of multivectors with coefficients in the formal series for the usual Poisson cochain space $\mathscr{X}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \otimes \wedge \mathbb{R}^n$. Furthermore, the reader may think about QPT of type (6.1) as QPT implemented by a classical r-matrix in their stabilizer for the canonical matrix action.

Hence, in Section 2, we are interested in the characterization of the QPT that are images of a classical r-matrix. We comment on the tight relation between the fact that a QPT is induced by an r-matrix and the dimension of its stabilizer. We prove that if the stabilizer of a given QPT Λ of $\mathbb{R}^n$ contains n commuting linear vector fields Y_i, such that $Y_{1\ldots n} \neq 0$, then Λ is implemented by an r-matrix in its stabilizer, see Theorem 21. In the following, we refer to such tensors as strongly r-matrix induced (SRMI) structures and show that any structure of the DHC decomposes into the sum of a major SRMI structure and a small compatible (mostly exact) Poisson tensor, see Theorem 22. This decomposition constitutes the foundation of our cohomological techniques proposed in [MP06] and Chapter 5. The preceding description and the philosophy of the mentioned cohomological modus operandi allow understanding that our splitting is in some sense in opposition to the one proven in [LX92] that incorporates the largest possible part of the Poisson tensor into the exact term.

In [MP06], the authors developed a cohomological method in the Euclidean Three-Space that led to a significant simplification of Poisson cohomology computations for the SRMI structures of the DHC. Section 3 of the present note aims for extension of this procedure to arbitrary dimensional vector spaces. Nontrivial lemmata allow injecting the space $\mathscr{R}$ of "real" Poisson cochains (formal multivector fields) into a larger space $\mathscr{P}$ of "potential" cochains, see Theorem 23, and identifying the natural extension to $\mathscr{P}$ of the Poisson differential as the Koszul differential associated with n commuting endomorphisms $X_i - (\operatorname{div} X_i)\,\mathrm{id}$, $X_i = \sum_j \alpha^{ij} Y_j$, $\alpha^{ji} = -\alpha^{ij}$, of the space made up by the polynomials on $\mathbb{R}^n$ with some fixed homogeneous degree, Theorems 24 and 25. We then choose a space $\mathscr{S}$ supplementary to $\mathscr{R}$ in $\mathscr{P}$ and show that the Poisson differential induces a differential on $\mathscr{S}$. Eventually, we end up with a short exact sequence of differential spaces and an exact triangle in cohomology. It could be proven that the Poisson cohomology ($\mathscr{R}$-cohomology) reduces, essentially, to the above-depicted Koszul cohomology ($\mathscr{P}$-cohomology) and a relative cohomology ($\mathscr{S}$-cohomology), see Theorem 26.

In order to take advantage of these upshots, we investigate in Section 4 the Koszul cohomology associated to n commuting linear operators on a finite-dimensional complex vector space. We prove a homotopy-type formula, see Proposition 44, and—using spectral properties—we show that the Koszul cohomology is, roughly spoken, located inside (a direct sum of intersections of) the kernels of some transformations that can be constructed recursively from the initially considered operators, Proposition 45 and Corollary 7.

In Section 5, we apply this result, gain valuable insight into the structure of the Koszul cohomology implemented by SRMI tensors, and show that in order to compute this central part of Poisson cohomology it basically suffices to solve triangular systems of linear equations.

Section 6 contains a full description of the Poisson cohomology spaces of structures Λ_3 and Λ_9 of the DHC.

Eventually, the aforementioned general upshots and our growing list of explicit data allow describing the main Poisson cohomological phenomena, see Section 7.

6.2 Characterization of strongly r-matrix induced Poisson structures

6.2.1 Stabilizer dimension and r-matrix generation

In the following, we report on an idea regarding generation of quadratic Poisson tensors by classical r-matrices.

Set $G = \mathrm{GL}(n,\mathbb{R})$ and $\mathfrak{g} = \mathrm{gl}(n,\mathbb{R})$. The Lie algebra isomorphism between $\mathfrak{g}$ and the algebra $\mathscr{X}_0^1(\mathbb{R}^n)$ of linear vector fields, extends to a Grassmann algebra and a graded Poisson-Lie algebra homomorphism $J : \wedge\mathfrak{g} \to \oplus_k \left(\mathscr{S}^k\mathbb{R}^{n*} \otimes \wedge^k\mathbb{R}^n\right)$. It is known that its restriction

$$J^k : \wedge^k\mathfrak{g} \to \mathscr{S}^k\mathbb{R}^{n*} \otimes \wedge^k\mathbb{R}^n$$

is onto, but has a non-trivial kernel if $k,n \geq 2$. In particular,

$$J^3[r,r] = [J^2r, J^2r],\ r \in \mathfrak{g}\wedge\mathfrak{g},$$

where $[.,.]$ is the Schouten-Nijenhuis bracket. These observations allow to understand that the characterization of the quadratic Poisson structures that are implemented by a classical r-matrix, i.e. a bimatrix $r \in \mathfrak{g}\wedge\mathfrak{g}$ that verifies the Classical Yang-Baxter Equation $[r,r] = 0$, is an open problem.

Quadratic Poisson tensors Λ_1 and Λ_2 are equivalent if and only if there is $A \in G$ such that $A_*\Lambda_1 = \Lambda_2$, where $*$ denotes the standard action of G on tensors of $\mathbb{R}^n$. As J^2 is a G-module homomorphism, i.e.

$$A_*(J^2r) = J^2(\mathrm{Ad}(A)r),\ A \in G, r \in \mathfrak{g}\wedge\mathfrak{g},$$

the G-orbit of a quadratic Poisson structure $\Lambda = J^2r$ is the pointwise J^2-image of the G-orbit of r. Furthermore, representation Ad acts by graded Lie algebra homomorphisms, i.e.

$$\mathrm{Ad}(A)[r,r] = [\mathrm{Ad}(A)r, \mathrm{Ad}(A)r].$$

Hence, if $\Lambda = J^2r$, where r is a classical r-matrix, the whole orbit of this quadratic Poisson tensor is made up by r-matrix induced structures.

Of course, any quadratic Poisson tensor Λ is implemented by bimatrices $r \in \mathfrak{g}\wedge\mathfrak{g}$. In order to determine wether the G-orbit O_Λ of this tensor is generated by r-matrices, we have to take an interest in the preimage

$$(J^2)^{-1}(O_\Lambda) = \cup_{r\in(J^2)^{-1}\Lambda} O_r,$$

composed of the G-orbits O_r of all the bimatrices r that are mapped on Λ by J^2. We claim that the chances that a fiber of this bundle is located inside r-matrices are the bigger, the smaller is O_Λ. In other words, the dimension of the isotropy Lie group G_Λ of Λ, or of its Lie algebra, the stabilizer

$$\mathfrak{g}_\Lambda = \{a \in \mathfrak{g} : [\Lambda, Ja] = 0\}$$

of Λ for the corresponding infinitesimal action, should be big enough. In addition to the ostensible intuitive clearness of this conjecture, positive evidence comes from the fact that, in $\mathbb{R}^3$, the Poisson tensor $\Lambda = (x_1^2 + x_2x_3)\partial_{23}$, $\partial_{23} := \partial_2 \wedge \partial_3$, $\partial_i := \partial/\partial_{x_i}$, is not r-matrix induced, see [MMR02], and the dimension of its stabilizer is $\dim \mathfrak{g}_\Lambda = 2$, as well as from the following theorem (we implicitly identify stabilizer $\mathfrak{g}_\Lambda \subset \mathfrak{g}$ and the (isomorphic) Lie subalgebra $J^1\mathfrak{g}_\Lambda = \{Y \in \mathscr{X}_0^1(\mathbb{R}^n) : [\Lambda, Y] = 0\} \subset \mathscr{X}_0^1(\mathbb{R}^n)$ of linear vector fields of $\mathbb{R}^n$).

Theorem 21. *Let Λ be a quadratic Poisson tensor of $\mathbb{R}^n$. If its stabilizer $\mathfrak{g}_\Lambda$ contains n commuting linear vector fields Y_i, $i \in \{1,\ldots,n\}$, such that $Y_1 \wedge \ldots \wedge Y_n \neq 0$, then Λ is implemented by a classical r-matrix that belongs to the stabilizer, i.e. $\Lambda = J^2a$, $[a,a] = 0$, $a \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$.*

Proof. Let $(x_1,\ldots,x_n)$ be the canonical coordinates of $\mathbb{R}^n$. Set $\partial_r = \partial_{x_r}$ and $Y_i = \sum_{r=1}^n \ell_{ir}\partial_r$, with $\ell \in \mathrm{gl}(\mathrm{n}, \mathbb{R}^{\mathrm{n}*})$. The determinant $D = \det \ell$ does not vanish everywhere, since $Y_{1\ldots n} = D\,\partial_{1\ldots n}$ and $Y_{1\ldots n} \neq 0$. At any point of the nonempty open subset $Z = \{x \in \mathbb{R}^n, D(x) \neq 0\}$ of $\mathbb{R}^n$, the Y_i form a basis of the corresponding tangent space of $\mathbb{R}^n$. Moreover, in Z, we get

$$\partial_{ij} = D^{-2} \sum_{k<l} \left(\mathbf{L}^k_i \mathbf{L}^l_j - \mathbf{L}^l_i \mathbf{L}^k_j\right) Y_{kl} =: D^{-2} \sum_{k<l} Q^{kl}_{ij} Y_{kl},$$

where $\mathbf{L}$ denotes the matrix of maximal algebraic minors of ℓ, and where $Q^{kl}_{ij} \in \mathscr{S}^{2n-2}\mathbb{R}^{n*}$. Hence, if the quadratic Poisson tensor Λ reads $\Lambda = \sum_{i<j} \Lambda^{ij}\partial_{ij}$, $\Lambda^{ij} \in \mathscr{S}^2\mathbb{R}^{n*}$, we have in Z,

$$\Lambda = D^{-2} \sum_{k<l} \sum_{i<j} \Lambda^{ij} Q^{kl}_{ij} Y_{kl} =: D^{-2} \sum_{k<l} P^{kl} Y_{kl},$$

where $P^{kl} \in \mathscr{S}^{2n}\mathbb{R}^{n*}$. We now prove that the rational functions $D^{-2}P^{kl}$ are actually constants. Since the Y_i are commuting vector fields in $\mathfrak{g}_\Lambda$, the commutation relations $[Y_i, Y_j] = 0$ and $[\Lambda, Y_i] = 0$, $i, j \in \{1,\ldots,n\}$, hold true. It follows that

$$Y_i\left(D^{-2}P^{kl}\right) = \sum_{r=1}^n \ell_{ir}\partial_r \left(D^{-2}P^{kl}\right) = 0, \quad i \in \{1,\ldots,n\},$$

everywhere in Z, and, as ℓ is invertible in Z, that $\partial_r \left(D^{-2}P^{kl}\right) = 0$, $r \in \{1,\ldots,n\}$. Hence, $P^{kl} = \alpha^{kl} D^2$, $\alpha^{kl} \in \mathbb{R}$, in each connected component of Z. As these components are open subsets of $\mathbb{R}^n$, the last result holds in $\mathbb{R}^n$ (in particular the constants α^{kl} associated with different connected components coincide). Eventually,

$$\Lambda = \sum_{k<l} \alpha^{kl} Y_{kl} = J^2 \left(\sum_{k<l} \alpha^{kl} a_{kl} \right),$$

where $a_i = (J^1)^{-1} Y_i \in \mathfrak{g}_\Lambda$. Since the a_i are (just as the Y_i) mutually commuting, it is clear that the bimatrix $r = \sum_{k<l} \alpha^{kl} a_{kl} \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$ verifies the classical Yang-Baxter equation. ∎

Definition 39. *We refer to a quadratic Poisson structure Λ that is implemented by a classical r-matrix $r \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$, where $\mathfrak{g}_\Lambda$ denotes the stabilizer of Λ for the canonical matrix action, as a strongly r-matrix induced (SRMI) tensor.*

6.2.2 Classification theorem in Euclidean Three-Space

Two concepts of exact Poisson structure—tightly related with two special cohomology classes—are used below. Let Λ be a Poisson tensor on a smooth manifold M oriented by a volume element Ω. We say that Λ, which is of course a Poisson 2-cocycle, is Poisson-exact, if

$$\Lambda = [\Lambda, X],\ X \in \mathscr{X}^1(M),$$

and we term Λ K-exact (Koszul), if

$$\Lambda = \delta(T),\ T \in \mathscr{X}^3(M).$$

Operator $\delta := \phi^{-1} \circ d \circ \phi$ is the pullback of the de Rham differential d by the canonical vector space isomorphism $\phi := i.\Omega$. Although introduced earlier, the generalized divergence δ ($\delta(X) = \mathrm{div}_\Omega X,\ X \in \mathscr{X}^1(M)$) is prevalently attributed to J.-L. Koszul. The curl vector field $K(\Lambda) := \delta(\Lambda)$ of Λ (if Ω is the standard volume of $\mathbb{R}^3$ and Λ is identified with a vector field $\vec{\Lambda}$ of $\mathbb{R}^3$, $K(\Lambda)$ coincides with the standard curl $\vec{\nabla} \wedge \vec{\Lambda}$) is a Poisson 1-cocycle. In $\mathbb{R}^n$, $n \geq 3$, a Poisson tensor Λ is K-exact, if and only if it is "irrotational", i.e. $K(\Lambda) = 0$, and in $\mathbb{R}^3$, K-exact means "function-induced", i.e.

$$\Lambda = \Pi_f := \partial_1 f\, \partial_{23} + \partial_2 f\, \partial_{31} + \partial_3 f\, \partial_{12},\ f \in C^\infty(\mathbb{R}^3).$$

The K-exact quadratic Poisson tensors Π_p of $\mathbb{R}^3$, i.e. the K-exact Poisson structures that are induced by a homogeneous polynomial $p \in \mathscr{S}^3\mathbb{R}^{3*}$, represent class

14 of the DHC. The cohomology of this class has been studied in [Pic05] (actually the author deals with structures Π_p implemented by a weight homogeneous polynomial p with an isolated singularity). Hence, class 14 of the DHC will not be examined in the current work.

Let us also recall that two Poisson tensors Λ_1 and Λ_2 are compatible, if their sum is again a Poisson structure, i.e. if $[\Lambda_1, \Lambda_2] = 0$.

The following theorem classifies the quadratic Poisson classes according to their membership of the family of strongly r-matrix induced structures. Furthermore, we show that any structure reads as the sum of a *major* strongly r-matrix induced tensor and a *small* compatible Poisson structure. On one hand, this membership entails accessibility to the cohomological technique exemplified in [MP06], on the other, this splitting—which, by the way, differs from the decomposition suggested in [LX92] in the sense that we incorporate the biggest possible part of the structure into the strongly induced term—is of particular importance with regard to the cohomological approach detailed in Chapter 5.

Theorem 22. *Let $a,b,c \in \mathbb{R}$ and let Λ_i $(i \in \{1,\dots,13\})$ be the quadratic Poisson tensors of the DHC, see [DH91]. Denote the canonical coordinates of $\mathbb{R}^3$ by x,y,z (or x_1,x_2,x_3) and the partial derivatives with respect to these coordinates by $\partial_1, \partial_2, \partial_3$ $(\partial_{ij} = \partial_i \wedge \partial_j)$.*

If $\dim \mathfrak{g}_\Lambda > 3$ (subscript i omitted), there are mutually commuting linear vector fields Y_1, Y_2, Y_3, such that

$$\Lambda = \alpha Y_{23} + \beta Y_{31} + \gamma Y_{12} \ \ (\alpha, \beta, \gamma \in \mathbb{R}),$$

so that Λ is strongly r-matrix induced (SRMI), i.e. implemented by a classical r-matrix in $\mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$. In the following classification of the quadratic Poisson tensors with regard to property SRMI, we decompose each not SRMI tensor into the sum of a major SRMI structure and a smaller compatible quadratic Poisson tensor.

- *Set $Y_1 = x\partial_1, Y_2 = y\partial_2, Y_3 = z\partial_3$*

  1. $\Lambda_1 = ayz\partial_{23} + bxz\partial_{31} + cxy\partial_{12}$ is SRMI for all values of the parameters a,b,c. More precisely,

  $$\Lambda_1 = aY_{23} + bY_{31} + cY_{12}$$

2. $\Lambda_4 = ayz\partial_{23} + axz\partial_{31} + (bxy + z^2)\partial_{12}$ *is not SRMI if and only if* $(a,b) \neq (0,0)$*. We have,*

$$\Lambda_4 = a(Y_{23} + Y_{31}) + bY_{12} + \frac{1}{3}\Pi_{z^3}$$

- *Set* $Y_1 = x\partial_1 + y\partial_2, Y_2 = x\partial_2 - y\partial_1, Y_3 = z\partial_3$

1. $\Lambda_2 = (2ax - by)z\partial_{23} + (bx + 2ay)z\partial_{31} + a(x^2 + y^2)\partial_{12}$ *is SRMI for any* a, b*. More precisely,*

$$\Lambda_2 = 2aY_{23} + bY_{31} + aY_{12}$$

2. $\Lambda_7 = ((2a + c)x - by)z\partial_{23} + (bx + (2a + c)y)z\partial_{31} + a(x^2 + y^2)\partial_{12}$ *is SRMI for all* a, b, c*. More precisely,*

$$\Lambda_7 = (2a + c)Y_{23} + bY_{31} + aY_{12}$$

3. $\Lambda_8 = axz\partial_{23} + ayz\partial_{31} + \left(\frac{a+b}{2}(x^2 + y^2) \pm z^2\right)\partial_{12}$ *is not SRMI if and only if* $(a,b) \neq (0,0)$*. We have,*

$$\Lambda_8 = aY_{23} + \frac{a+b}{2}Y_{12} \pm \frac{1}{3}\Pi_{z^3}$$

- *Set* $Y_1 = x\partial_1 + y\partial_2, Y_2 = x\partial_2, Y_3 = z\partial_3$

1. $\Lambda_3 = (2x - ay)z\partial_{23} + axz\partial_{31} + x^2\partial_{12}$ *is SRMI for any a. More precisely,*

$$\Lambda_3 = 2Y_{23} + aY_{31} + Y_{12}$$

2. $\Lambda_5 = ((2a + 1)x + y)z\partial_{23} - xz\partial_{31} + ax^2\partial_{12}$ $(a \neq -\frac{1}{2})$ *is SRMI for any a. More precisely,*

$$\Lambda_5 = (2a + 1)Y_{23} - Y_{31} + aY_{12}$$

3. $\Lambda_6 = ayz\partial_{23} - axz\partial_{31} - \frac{1}{2}x^2\partial_{12}$ *is SRMI for any a. More precisely,*

$$\Lambda_6 = -aY_{31} - \frac{1}{2}Y_{12}$$

- *Set* $Y_1 = \mathscr{E} := x\partial_1 + y\partial_2 + z\partial_3, Y_2 = x\partial_2 + y\partial_3, Y_3 = x\partial_3$

1. $\Lambda_9 = (ax^2 - \frac{1}{3}y^2 + \frac{1}{3}xz)\partial_{23} + \frac{1}{3}xy\partial_{31} - \frac{1}{3}x^2\partial_{12}$ *is SRMI for any a. More precisely,*

$$\Lambda_9 = aY_{23} - \frac{1}{3}Y_{12}$$

2. $\Lambda_{10} = \left(ay^2 - (4a+1)xz\right)\partial_{23} + (2a+1)xy\partial_{31} - (2a+1)x^2\partial_{12}$ *is not SRMI if and only if* $a \neq -\frac{1}{3}$*. We have,*

$$\Lambda_{10} = -(2a+1)Y_{12} + (3a+1)(y^2 - 2xz)\partial_{23}$$

- *Set* $Y_1 = \mathscr{E}, Y_2 = x\partial_2, Y_3 = (ax + (3b+1)z)\partial_3$

1. $\Lambda_{11} = \left(ax^2 + (2b+1)xz\right)\partial_{23} + (bx^2 + cz^2)\partial_{12}$ $(a = 0)$ *is not SRMI if and only if* $c \neq 0$*. We have,*

$$\Lambda_{11} = Y_{23} + bY_{12} + \frac{c}{3}\Pi_{z^3}$$

2. $\Lambda_{12} = \left(ax^2 + (2b+1)xz\right)\partial_{23} + (bx^2 + cz^2)\partial_{12}$ $(a = 1)$ *is not SRMI if and only if* $c \neq 0$*. We have,*

$$\Lambda_{12} = Y_{23} + bY_{12} + \frac{c}{3}\Pi_{z^3}$$

3. $\Lambda_{13} = \left(ax^2 + (2b+1)xz + z^2\right)\partial_{23} + (bx^2 + cz^2 + 2xz)\partial_{12}$ *is not SRMI for any* a, b, c*. We have,*

$$\Lambda_{13} = Y_{23} + bY_{12} + \Pi_{\frac{c}{3}z^3 + xz^2}$$

Proof. Let us first mention that the specified basic fields Y_1, Y_2, Y_3 have been read in the stabilizers of the considered Poisson tensors, but that we refrain from publishing the often fairly protracted stabilizer-computations. Indeed, once the vector fields Y_i are known, it is easily checked that, in the SRMI cases, they verify the assumptions of Theorem 21. Thus the corresponding Poisson structures are actually SRMI tensors. In order to ascertain that a quadratic Poisson structure Λ is not SRMI, it suffices to prove that $\Lambda \notin J^2(\mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda)$. This will be done thereinafter. All the quoted decompositions can be directly verified. In most instances, the twist is obviously Poisson, so that compatibility follows. In the case of Λ_{10}, the twist $\Lambda_{10,\mathrm{II}} = (y^2 - 2xz)\partial_{23}$ is a non-K-exact Poisson structure. This is a direct consequence of the result $K(\Lambda_{10,\mathrm{II}}) = \vec{\nabla} \wedge \vec{\Lambda}_{10,\mathrm{II}} = -2x\partial_2 - 2y\partial_3 \neq 0$ and the handy formula

$$[P,Q] = (-1)^p \delta(P \wedge Q) - \delta(P) \wedge Q - (-1)^p P \wedge \delta(Q), \forall P \in \mathscr{X}^p(M), Q \in \mathscr{X}^q(M).$$

The statement regarding the dimension of stabilizer $\mathfrak{g}_\Lambda$ is obvious in view of the following main part of this proof.

Denote by E_{ij} ($i,j \in \{1,2,3\}$) the canonical basis of $\mathrm{gl}(3,\mathbb{R})$.

- For Λ_4, if $(a,b) \neq (0,0)$, stabilizer $\mathfrak{g}_{\Lambda_4}$ and the image $J^2(\mathfrak{g}_{\Lambda_4} \wedge \mathfrak{g}_{\Lambda_4})$ are generated by

$$\left(\frac{1}{2}E_{11} + E_{22}, \frac{1}{2}E_{11} + E_{33}\right) \quad \text{and} \quad yz\partial_{23} - \frac{1}{2}xz\partial_{31} - \frac{1}{2}xy\partial_{12},$$

  respectively. Hence, Λ_4 is not SRMI.

- For Λ_8, if $(a,b) \neq (0,0)$, the generators of $\mathfrak{g}_{\Lambda_8}$ and $J^2(\mathfrak{g}_{\Lambda_8} \wedge \mathfrak{g}_{\Lambda_8})$ are

$$(E_{11} + E_{22} + E_{33}, E_{12} - E_{21}) \quad \text{and} \quad -xz\partial_{23} - yz\partial_{31} + (x^2 + y^2)\partial_{12}.$$

  So Λ_8 is not SRMI.

- For Λ_{10}, if $a \neq -\frac{1}{3}$, the generators are

$$(E_{11} + E_{22} + E_{33}, E_{12} + E_{23}) \quad \text{and} \quad (y^2 - xz)\partial_{23} - xy\partial_{31} + x^2\partial_{12}.$$

- For Λ_{11}, $c \neq 0$, Λ_{12}, $c \neq 0$, and Λ_{13}, the generators are

$$(E_{11} + E_{22} + E_{33}, E_{12}, E_{32}) \quad \text{and} \quad (-xz\partial_{23} + x^2\partial_{12}, z^2\partial_{23} - xz\partial_{12}). \blacksquare$$

Remarks.

- For $\Lambda = \Lambda_i$, $i \in \{11,12,13\}$, $c \neq 0$ if $i \in \{11,12\}$, the dimension of the stabilizer is $\dim \mathfrak{g}_\Lambda = 3$, whereas $J\mathfrak{g}_\Lambda \wedge J\mathfrak{g}_\Lambda \wedge J\mathfrak{g}_\Lambda = \{0\}$. Hence, if the dimension of the stabilizer coincides with the dimension of the space, the Poisson structure is not necessarily a SRMI tensor.

- For Λ_{10} e.g., the decomposition proved in [LX92] yields

$$\Lambda_{10} = -\frac{1}{3}Y_{12} + \Pi_{\frac{c}{3}z^3 + xz^2 + (b+\frac{1}{3})x^2z + \frac{a}{3}x^3}.$$

6.3 Poisson cohomology of quadratic structures in a finite-dimensional vector space

6.3.1 Koszul homology and cohomology

Let $\wedge = \wedge_n\langle\vec{\eta}\rangle$ be the Grassmann algebra on $n \in \mathbb{N}_0$ generators $\vec{\eta} = (\eta_1, \ldots, \eta_n)$, i.e. the algebra generated over a field $\mathbb{F}$ of characteristic 0 (in this work $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) by generators $\eta_1, \ldots, \eta_n$ subject to the anticommutation relations $\eta_k\eta_\ell + \eta_\ell\eta_k = 0$, $k, \ell \in \{1, \ldots, n\}$. Set $\wedge = \oplus_{p=0}^{n} \wedge^p$, with obvious notations, and let $\vec{h} = (h_1, \ldots, h_n)$ be dual generators: $i_{h_k}\eta_\ell = \delta_{k\ell}$. We also need the creation operator $e_{\eta_k} : \wedge \ni \omega \to \eta_k\,\omega \in \wedge$ and the annihilation operator $i_{h_k} : \wedge \ni \omega \to i_{h_k}\omega \in \wedge$, where the interior product is defined as usual. Eventually, we denote by E a vector space over $\mathbb{F}$ and by $\vec{X} = (X_1, \ldots, X_n)$ an n-tuple of commuting linear operators on E.

Definition 40. *The complex*

$$0 \to E \otimes_{\mathbb{F}} \wedge^n \to E \otimes_{\mathbb{F}} \wedge^{n-1} \to \ldots \to E \otimes_{\mathbb{F}} \wedge^1 \to E \to 0,$$

with differential $\kappa_{\vec{X}} = \sum_{k=1}^{n} X_k \otimes i_{h_k}$, is the Koszul chain complex (K_*-complex) $K_*(\vec{X}, E)$ associated with $\vec{X}$ on E. The Koszul homology group is denoted by $KH_*(\vec{X}, E)$.

Definition 41. *The complex*

$$0 \to E \to E \otimes_{\mathbb{F}} \wedge^1 \to \ldots \to E \otimes_{\mathbb{F}} \wedge^{n-1} \to E \otimes_{\mathbb{F}} \wedge^n \to 0,$$

with differential $\mathscr{K}_{\vec{X}} = \sum_{k=1}^{n} X_k \otimes e_{\eta_k}$, is the Koszul cochain complex (K^*-complex) $K^*(\vec{X}, E)$ associated with $\vec{X}$ on E. We denote by $KH^*(\vec{X}, E)$ the corresponding Koszul cohomology group.

Observe that commutation of the X_k and anticommutation of the i_{h_k} (resp. the e_{η_k}) entail that $\kappa_{\vec{X}}$ (resp. $\mathscr{K}_{\vec{X}}$) actually squares to 0.

Example 1. It is easily checked that, if we choose $\mathbb{F} = \mathbb{R}$, $E = C^\infty(\mathbb{R}^3)$, $\eta_k = dx_k$ (resp. $\eta_k = \partial_k = \partial_{x_k}$ and $h_k = dx_k$), and $X_k = \partial_k$ ($k \in \{1, 2, 3\}$, x_1, x_2, x_3 canonical coordinates of $\mathbb{R}^3$), the K^*-complex (resp. the K_*-complex) is nothing but the de Rham complex $(\Omega(\mathbb{R}^3), d)$ (resp. its dual version $(\mathscr{X}(\mathbb{R}^3), \delta)$, see above). Note that, if we identify the subspaces $\Omega^k(\mathbb{R}^3)$ of homogeneous forms with the corresponding spaces of components E, E^3, E^3, E, this K^*-complex reads

$$0 \to E \overset{\mathscr{K}=\vec{\nabla}(.)}{\to} E^3 \overset{\mathscr{K}=\vec{\nabla}\wedge(.)}{\to} E^3 \overset{\mathscr{K}=\vec{\nabla}\cdot(.)}{\to} E \to 0, \tag{6.2}$$

with self-explaining notations.

Example 2. For $\mathbb{F} = \mathbb{R}$, $E = \mathscr{S}\mathbb{R}^{3*} = \mathbb{R}[x_1, x_2, x_3]$, $\eta_k = \partial_k$, $X_k = \mathfrak{m}_{P_k}$ ($k \in \{1,2,3\}$, $P_k \in E^{d_k}$, $d_k \in \mathbb{N}$, $\mathfrak{m}_{P_k} : E \ni Q \to P_k Q \in E$), the chain spaces of the K_*-complex are the spaces of homogeneous polyvector fields on $\mathbb{R}^3$ with polynomial coefficients, and anew identification with the corresponding spaces E, E^3, E^3, E of components, allows to write this K_*-complex in the form

$$0 \to E \overset{\kappa=(.)\vec{P}}{\to} E^3 \overset{\kappa=(.)\wedge\vec{P}}{\to} E^3 \overset{\kappa=(.)\cdot\vec{P}}{\to} E \to 0, \tag{6.3}$$

where $\vec{P} = (P_1, P_2, P_3)$.

Remarks.

- Of course, the Koszul cohomology and homology complexes defined in Example 1 are exact, expect that $KH^0(\vec{\partial}, C^\infty(\mathbb{R}^3)) \simeq KH_3(\vec{\partial}, C^\infty(\mathbb{R}^3)) \simeq \mathbb{R}$.
- Let us recall that an R–regular sequence on a module M over a commutative unit ring R, is a sequence $(r_1, \ldots, r_d) \in R^d$, such that r_k is not a zero divisor on the quotient $M/\langle r_1, \ldots, r_{k-1}\rangle M$, $k \in \{1, \ldots, d\}$, and $M/\langle r_1, \ldots, r_d\rangle M \neq 0$. In particular, $x_1, \ldots, x_d$ is a (maximal length) regular sequence on the polynomial ring $R = \mathbb{F}[x_1, \ldots, x_d]$ (so that this ring has depth d).

  It is well-known that the K_*-complex described in Example 2 is exact, except for surjectivity of $\kappa = (.)\cdot\vec{P}$, if sequence $\vec{P} = (P_1, P_2, P_3)$ is regular for $\mathbb{R}[x_1, x_2, x_3]$. For instance, if $\vec{P} = \vec{\nabla} p$, where p is a homogeneous polynomial with an isolated singularity at the origin, sequence $\vec{P}$ is regular, see [Pic05].

6.3.2 Poisson cohomology in dimension 3

Set $E := C^\infty(\mathbb{R}^3)$ and identify—as above—the spaces of homogeneous multivector fields in $\mathbb{R}^3$, with the corresponding component spaces: $\mathscr{X}^0(\mathbb{R}^3) \simeq \mathscr{X}^3(\mathbb{R}^3) \simeq E$ and $\mathscr{X}^1(\mathbb{R}^3) \simeq \mathscr{X}^2(\mathbb{R}^3) \simeq E^3$.

Let $\vec{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3) \in E^3$ be a Poisson tensor and $f \in E, \vec{X} \in E^3, \vec{B} \in E^3, T \in E$ a 0-, 1-, 2-, and 3-cochain of the Poisson complex. The following formulæ for the Poisson coboundary operator $\partial_{\vec{\Lambda}}$ can be obtained by straightforward computations:

$$\begin{aligned}
\partial^0_{\vec{\Lambda}} f &= \vec{\nabla} f \wedge \vec{\Lambda}, \\
\partial^1_{\vec{\Lambda}} \vec{X} &= (\vec{\nabla}\cdot\vec{X})\vec{\Lambda} - \vec{\nabla}(\vec{X}\cdot\vec{\Lambda}) + \vec{X} \wedge (\vec{\nabla}\wedge\vec{\Lambda}), \\
\partial^2_{\vec{\Lambda}} \vec{B} &= -(\vec{\nabla}\wedge\vec{B})\cdot\vec{\Lambda} - \vec{B}\cdot(\vec{\nabla}\wedge\vec{\Lambda}), \\
\partial^3_{\vec{\Lambda}} T &= 0.
\end{aligned}$$

If we denote the differential detailed in Equation (6.2) (resp. in Equation (6.3) if $\vec{P} = \vec{\Lambda}$, in Equation (6.3) if $\vec{P} = \vec{\nabla} \wedge \vec{\Lambda}$) by $\mathscr{K}$ (resp. κ', κ''), we get

$$\partial^0_{\vec{\Lambda}} = \kappa'\mathscr{K}, \partial^1_{\vec{\Lambda}} = \kappa'\mathscr{K} - \mathscr{K}\kappa' + \kappa'', \partial^2_{\vec{\Lambda}} = -\kappa'\mathscr{K} - \kappa'', \partial^3_{\vec{\Lambda}} = 0. \tag{6.4}$$

As aforementioned, investigations are confined in this chapter to quadratic Poisson tensors and polynomial (or formal) Poisson cochains. If structure $\vec{\Lambda}$ is K-exact, i.e., in view of notations due to the elimination of the module basis of multi-vector fields, $\vec{\Lambda} = \vec{\nabla} p \ \ (p \in \mathscr{S}^3\mathbb{R}^{3*}) \Leftrightarrow \vec{\nabla} \wedge \vec{\Lambda} = 0$, homology operator κ'' vanishes. If, moreover, p has an isolated singularity (IS), not only the K^*-complex associated with $\mathscr{K}$ is exact up to injectivity of $\mathscr{K} = \vec{\nabla}(.)$, but also the K_*-complex associated with κ' is, see above, acyclic up to surjectivity of $\kappa' = (.) \cdot \vec{\Lambda}$. In [Pic05], the author has computed inter alia the Poisson cohomology for a weight-homogeneous polynomial p with an IS.

Below, we describe a generic cohomological technique for SRMI Poisson tensors in a finite-dimensional vector space. This approach extends Formulæ (6.4) to dimension n and reduces simultaneously the Poisson coboundary operator ∂_Λ to a single Koszul differential.

6.3.3 Poisson cohomology in dimension n

We denote by L the matrix of maximal minors of a matrix $\ell \in \mathrm{gl}(n, \mathbb{R}^{n*})$ (or of a matrix with entries in a field $\mathbb{F}$ of non-zero characteristic), so L_{ij} is the minor of ℓ obtained by cancellation of line i and column j. More generally, if $\nu = \{1, \ldots, n\}$, $\mathbf{i} = (i_1, \ldots, i_m) \in \nu^m$ ($i_1 < \ldots < i_m$, $m \in \{1, \ldots, n\}$), we denote by $\mathbf{I} = (I_1, \ldots, I_{n-m})$ the complement of $\mathbf{i}$ in ν. If $\mathbf{j} = (j_1, \ldots, j_m)$ is an m-tuple similar to $\mathbf{i}$, we denote by $L_{\mathbf{ij}}$ the minor of ℓ obtained by cancellation of the lines $\mathbf{i}$ and the columns $\mathbf{j}$, and by $L^{\mathbf{ij}}$ the minor of ℓ at the intersections of lines $\mathbf{i}$ and columns $\mathbf{j}$. Hence, $L_{\mathbf{ij}} = L^{\mathbf{IJ}}$ and $L_{\mathbf{IJ}} = L^{\mathbf{ij}}$. Moreover, $D = \det \ell \in \mathscr{S}^n\mathbb{R}^{n*}$ is the determinant of ℓ, $\mathscr{L}$ stands for the matrix of maximal minors of $L \in \mathrm{gl}(n, \mathscr{S}^{n-1}\mathbb{R}^{n*})$, and we apply the just introduced notations $\mathscr{L}_{\mathbf{ij}}$ and $\mathscr{L}^{\mathbf{ij}}$ also to $\mathscr{L}$. Eventually, as already mentioned above, $\mathbf{L}$ denotes the matrix of algebraic maximal minors of ℓ.

Remark. In the following, we systematically assume that $D \neq 0$, i.e. that polynomial D does not vanish everywhere.

Lemma 8. *For any $m \in \{1, \ldots, n-1\}$ and for any $\mathbf{i} = (i_1, \ldots, i_m)$, $\mathbf{j} = (j_1, \ldots, j_m)$ as above, we have*

$$\mathscr{L}_{\mathbf{ij}} = D^{n-m-1} L^{\mathbf{ij}} \text{ and } \mathscr{L}^{\mathbf{ij}} = D^{m-1} L_{\mathbf{ij}}.$$

The first (resp. second) equation also holds for $m = 0$ (resp. $m = n$). In this case it just means that $\det L = D^{n-1}$.

Proof. Of course, the second statement is nothing but a reformulation of the first. We prove the first assertion by induction on m. For $m = n-1$, the assertion is obvious. Indeed, the both sides coincide with the element L_{IJ} of L at the intersection of the line I and the column J. Assume now that the equation holds true for $2 \le m \le n-1$ and take any $\mathbf{i} = (i_1, \ldots, i_{m-1})$ and $\mathbf{j} = (j_1, \ldots, j_{m-1})$ of length $m-1$. Let i_m be an (arbitrary) element of $(n-m+1)$-tuple $\mathbf{I}$. We will also have to consider the m-tuple $\underline{\mathbf{i}} = (i_1, \ldots, i_m, \ldots, i_{m-1})$, where the elements have of course been written in the natural order $i_1 < \ldots < i_m < \ldots < i_{m-1}$. The rank of i_m inside $\mathbf{I}$ and $\underline{\mathbf{i}}$ will be denoted be $r_{\mathbf{I}}(i_m)$ and $r_{\underline{\mathbf{i}}}(i_m)$ respectively. Using these notations, we get

$$\mathscr{L}_{\mathbf{ij}} = \mathscr{L}^{\mathbf{IJ}} = \sum_{j_m \in \mathbf{J}} (-1)^{r_{\mathbf{I}}(i_m) + r_{\mathbf{J}}(j_m)} L_{i_m j_m} \mathscr{L}_{\underline{\mathbf{i}}\underline{\mathbf{j}}}.$$

Applying the induction assumption, we see that $\mathscr{L}_{\underline{\mathbf{i}}\underline{\mathbf{j}}} = D^{n-m-1} L^{\underline{\mathbf{i}}\underline{\mathbf{j}}}$, so that

$$\mathscr{L}_{\mathbf{ij}} = D^{n-m-1} \sum_{j_m \in \nu} (-1)^{r_{\mathbf{I}}(i_m) + r_{\mathbf{J}}(j_m)} L_{i_m j_m} \sum_{\sigma \in \mathscr{P}(\underline{\mathbf{j}})} \operatorname{sign} \sigma \, \ell_{i_1 \sigma_{j_1}} \ldots \ell_{i_m \sigma_{j_m}} \ldots \ell_{i_{m-1} \sigma_{j_{m-1}}},$$

where $\mathscr{P}(\underline{\mathbf{j}})$ is the permutation group of $\underline{\mathbf{j}}$, and where the first sum could be extended to all $j_m \in \nu$, as for $j_m \in \mathbf{j}$ the last determinant vanishes. It is clear that we obtain all the permutations σ of $\underline{\mathbf{j}}$, if we assign j_m to i_p ($p \in \{1, \ldots, m\}$) and, for each choice of p, all the permutations $\mu \in \mathscr{P}(\mathbf{j})$ to the remaining subscripts i_q. Observe that the signature of the permutation σ that associates j_m with i_p and permutes $\mathbf{j}$ by μ, is $\operatorname{sign} \sigma = (-1)^{r_{\underline{\mathbf{i}}}(i_p) - r_{\underline{\mathbf{j}}}(j_m)} \operatorname{sign} \mu$. Hence, we get

$$\mathscr{L}_{\mathbf{ij}} = D^{n-m-1} \textstyle\sum_{p=1}^{m} \sum_{\mu \in \mathscr{P}(\mathbf{j})} \operatorname{sign} \mu \, \ell_{i_1 \mu_{j_1}} \ldots \widehat{\ell_{i_p j_m}} \ldots \ell_{i_{m-1} \mu_{j_{m-1}}} \sum_{j_m \in \nu} (-1)^{r_{\mathbf{I}}(i_m) + r_{\underline{\mathbf{i}}}(i_p) + r_{\mathbf{J}}(j_m) - r_{\underline{\mathbf{j}}}(j_m)} \ell_{i_p j_m} L_{i_m j_m}.$$

Remark now that the exponent of -1 can be replaced by

$$r_{\underline{\mathbf{i}}}(i_p) + r_{\underline{\mathbf{i}}}(i_m) + r_{\underline{\mathbf{i}}}(i_m) + r_{\mathbf{I}}(i_m) + r_{\underline{\mathbf{j}}}(j_m) + r_{\mathbf{J}}(j_m) \sim r_{\underline{\mathbf{i}}}(i_p) + r_{\underline{\mathbf{i}}}(i_m) + i_m + j_m.$$

Thus, the last sum reads $(-1)^{r_{\underline{\mathbf{i}}}(i_p) + r_{\underline{\mathbf{i}}}(i_m)} \sum_{j_m \in \nu} (-1)^{i_m + j_m} \ell_{i_p j_m} L_{i_m j_m}$. If $p \neq m$, this sum vanishes, and if $p = m$ it coincides with determinant D. Eventually, we find

$$\mathscr{L}_{\mathbf{ij}} = D^{n-m} \sum_{\mu \in \mathscr{P}(\mathbf{j})} \operatorname{sign} \mu \, \ell_{i_1 \mu_{j_1}} \ldots \widehat{\ell_{i_m j_m}} \ldots \ell_{i_{m-1} \mu_{j_{m-1}}} = D^{n-m} L^{\mathbf{ij}}. \quad \blacksquare$$

Definition 42. *Let $Y_i = \sum_r \ell_{ir}\partial_r$ be n linear vector fields in $\mathbb{R}^n$. Set*

$$\mathscr{R} = \oplus_{p=0}^{n} \mathscr{R}^p = \oplus_{p=0}^{n} \mathbb{R}[[x_1,\ldots,x_n]] \otimes \wedge_n^p \langle \vec{\partial} \rangle$$

and

$$\mathscr{P} = \oplus_{p=0}^{n} \mathscr{P}^p = D^{-1} \oplus_{p=0}^{n} \mathbb{R}[[x_1,\ldots,x_n]] \otimes \wedge_n^p \langle \vec{Y} \rangle,$$

where $D = \det \ell$ and where $\wedge_n^p \langle \vec{\partial} \rangle$ and $\wedge_n^p \langle \vec{Y} \rangle$ are the terms of degree p of the Grassmann algebras on generators $\vec{\partial} = (\partial_1,\ldots,\partial_n)$ and $\vec{Y} = (Y_1,\ldots,Y_n)$ respectively. Space $\mathscr{R}$ (resp. $\mathscr{P}$) is the space of real *formal Poisson cochains (resp.* potential *formal Poisson cochains).*

Remark. The space of polyvector fields $Y_{\mathbf{k}} = Y_{k_1\ldots k_p} = Y_{k_1} \wedge \ldots \wedge Y_{k_p}$ $(k_1 < \ldots < k_p, p \in \{0,\ldots,n\})$ with coefficients in the quotients by D of formal power series in $(x_1,\ldots,x_n)$, is a concrete model of space $\mathscr{P}$. Indeed, observe first that these spaces are bigraded by the "exterior degree" p and the (total) "polynomial degree", say r. If such a polyvector field vanishes, its homogeneous terms $D^{-1}\sum_{\mathbf{k}} P^{\mathbf{k}r} Y_{\mathbf{k}}$ $(P^{\mathbf{k}r} \in \mathscr{S}^r\mathbb{R}^{n*})$ vanish. If we decompose the Y_i $(i \in \{1,\ldots,n\})$ in the natural basis ∂_i, we immediately see that the sums $\sum_{\mathbf{k}} L^{\mathbf{ki}} P^{\mathbf{k}r}$ vanish for all $\mathbf{i} = (i_1,\ldots,i_p)$ $(i_1 < \ldots < i_p)$. Since these sums can be viewed as the product of a matrix with polynomial entries and the column made up by the $P^{\cdot r}$, the column vanishes outside the vanishing set V of the homogeneous polynomial determinant of this matrix. As the complement of (the conic closed) subset V of $\mathbb{R}^n$ is dense in $\mathbb{R}^n$, the polynomials $P^{\mathbf{k}r}$ vanish everywhere.

Theorem 23. *(i) There is a canonical non surjective injection $i : \mathscr{R} \to \mathscr{P}$ from $\mathscr{R}$ into $\mathscr{P}$.*
(ii) A homogeneous potential cochain $D^{-1}\sum_{\mathbf{k}} P^{\mathbf{k}r} Y_{\mathbf{k}}$ [of bidegree (p,r)] is real if and only if the $[n!/p!(n-p)!]$ homogeneous polynomials $\sum_{\mathbf{k}} L^{\mathbf{ki}} P^{\mathbf{k}r}$ [of degree $p+r$] are divisible by D (for $p = 0$ this condition means that P^r be divisible by D).

Proof. Take a real cochain $C^p = \sum_{\mathbf{i}} \varsigma^{\mathbf{i}} \partial_{\mathbf{i}} \in \mathscr{R}^p$, where, as above, $\mathbf{i} = (i_1,\ldots,i_p)$, $i_1 < \ldots < i_p$. As $\partial_j = D^{-1}\sum_k \mathbf{L}_{kj} Y_k$, we get

$$\begin{aligned}
\partial_{\mathbf{i}} &= D^{-p} \sum_{k_1,\ldots,k_p} \mathbf{L}_{k_1 i_1} \ldots \mathbf{L}_{k_p i_p} Y_{k_1\ldots k_p} \\
&= D^{-p} \sum_{k_1<\ldots<k_p} \left(\sum_{\sigma \in \mathscr{P}(\mathbf{k})} \operatorname{sign}\sigma\, \mathbf{L}_{\sigma_{k_1} i_1} \ldots \mathbf{L}_{\sigma_{k_p} i_p} \right) Y_{k_1\ldots k_p}.
\end{aligned}$$

If $|\mathbf{i}| = \sum_{j=1}^{p} i_j$, it follows from Lemma 8, that the determinant in the above bracket is given by

$$(-1)^{|\mathbf{i}|+|\mathbf{k}|} \mathscr{L}^{\mathbf{ki}} = (-1)^{|\mathbf{i}|+|\mathbf{k}|} D^{p-1} L_{\mathbf{ki}},$$

so that

$$C^p = D^{-1}\sum_{\mathbf{k}}\left(\sum_{\mathbf{i}}(-1)^{|\mathbf{i}|+|\mathbf{k}|}L_{\mathbf{ki}}\,\varsigma^{\mathbf{i}}\right)Y_{\mathbf{k}},$$

where the RHS is in $\mathscr{P}^p$.

Point (ii) is a direct consequence of the preceding remark. ∎

Remark. In view of this theorem, the bigrading $\mathscr{P} = \oplus_{p=0}^{n}\oplus_{r=0}^{\infty}\mathscr{P}^{pr}$, defined on $\mathscr{P}$ by the exterior degree and the polynomial degree, induces a bigrading $\mathscr{R} = \oplus_{p=0}^{n}\oplus_{r=0}^{\infty}\mathscr{R}^{pr}$ on $\mathscr{R}$.

Consider now a quadratic Poisson tensor Λ in $\mathbb{R}^n$. In the following, we *assume* that Λ is SRMI, and more precisely that there are n mutually commuting linear vector fields $Y_i = \sum_{r=1}^{n}\ell_{ir}\partial_r$, $\ell \in \mathrm{gl}(n,\mathbb{R}^{n*})$, such that $D = \det\ell \neq 0$ and

$$\Lambda = \sum_{i<j}\alpha^{ij}Y_{ij} \quad (\alpha^{ij}\in\mathbb{R}).$$

Proposition 37. *The determinant $D = \det\ell \in \mathscr{S}^n\mathbb{R}^{n*}\setminus\{0\}$ of ℓ is the unique joint eigenvector of the Y_i with eigenvalues $\operatorname{div}Y_i \in \mathbb{R}$, i.e., D is, up to multiplication by nonzero constants, the unique nonzero polynomial of $\mathbb{R}^n$ that verifies*

$$Y_iD = (\operatorname{div}Y_i)D, \forall i\in\{1,\ldots,n\}.$$

Moreover, if $D = D_1D_2$, where $D_1 \in \mathscr{S}^{n_1}\mathbb{R}^{n*}$ and $D_2 \in \mathscr{S}^{n_2}\mathbb{R}^{n*}$ $(n_1+n_2=n)$ are two polynomials without common divisor, these factors D_1 and D_2 are also joint eigenvectors. If λ_i and μ_i denote their eigenvalues, we have $\lambda_i+\mu_i = \operatorname{div}Y_i$.

Proof. Set $Y_i = \sum_r \ell_{ir}\partial_r = \sum_{rs}a_{ir}^{s}x_s\partial_r$, $a_{ir}^{s}\in\mathbb{R}$. Note first that $Y_i(\ell_{jr}) = \sum_t a_{jr}^{t}\ell_{it}$, and that $[Y_i,Y_j]=0$ means $Y_i(\ell_{jr}) = Y_j(\ell_{ir})$, for all $i,j,r\in\{1,\ldots,n\}$. If $\mathscr{P}_n$ denotes the permutation group of $\{1,\ldots,n\}$, we then get

$$\begin{aligned}
Y_iD &= \sum_{k=1}^{n}\sum_{\sigma\in\mathscr{P}_n}\operatorname{sign}\sigma\;\ell_{\sigma_1 1}\ldots Y_i(\ell_{\sigma_k k})\ldots\ell_{\sigma_n n}\\
&= \sum_{k=1}^{n}\sum_{\sigma\in\mathscr{P}_n}\operatorname{sign}\sigma\;\ell_{\sigma_1 1}\ldots Y_{\sigma_k}(\ell_{ik})\ldots\ell_{\sigma_n n}\\
&= \sum_{k,t=1}^{n}a_{ik}^{t}\sum_{\sigma\in\mathscr{P}_n}\operatorname{sign}\sigma\;\ell_{\sigma_1 1}\ldots\ell_{\sigma_k t}\ldots\ell_{\sigma_n n}.
\end{aligned}$$

This last sum vanishes if $k \neq t$ since two columns coincide in this determinant. Eventually, we have

$$Y_i D = \left(\sum_k a_{ik}^k\right) D = (\operatorname{div} Y_i)\, D.$$

As for uniqueness, suppose that there is another polynomial $P \in \mathscr{S}\mathbb{R}^{n*} \setminus \{0\}$, such that $Y_i P = (\operatorname{div} Y_i) P$, for all $i \in \{1, \ldots, n\}$. Then $Y_i\,(P/D) = 0$ in $Z = \{x \in \mathbb{R}^n, D(x) \neq 0\}$ and the same reasoning as in the proof of Theorem 21 allows concluding that there exists $\alpha \in \mathbb{R}^*$ such that $P = \alpha D$.

The assertion concerning the factorization of D is easily understood. Indeed, since $((\operatorname{div} Y_i) D_1 - Y_i D_1) D_2 = D_1 (Y_i D_2)$ and as the polynomials D_1 and D_2 have no common divisor, $Y_i D_2 = P D_2$ and $(\operatorname{div} Y_i) D_1 - Y_i D_1 = Q D_1$, where $P = Q$ is a polynomial. Looking at degrees, we immediately see that $P = Q$ is necessarily constant. ∎

Remark. Observe that the eigenvalues $\operatorname{div} Y_i$, $i \in \{1, \ldots, n\}$, cannot vanish simultaneously. Indeed, in this case, polynomial $D \in \mathscr{S}^n \mathbb{R}^{n*} \setminus \{0\}$, $n \in \mathbb{N}^*$, vanishes everywhere.

Definition 43. *The complex*

$$0 \to \mathscr{R}^0 \to \mathscr{R}^1 \to \ldots \to \mathscr{R}^n \to 0$$

with differential $\partial_\Lambda = [\Lambda, .]$, is the formal complex of Poisson tensor $\Lambda \in \mathscr{S}^2 \mathbb{R}^{n*} \otimes \wedge^2 \mathbb{R}^n$. We denote the corresponding cohomology groups by $LH^*(\mathscr{R}, \Lambda)$.

The next theorem shows that if the cochains $C \in \mathscr{R}$ are read as $C = iC \in \mathscr{P}$, the Poisson differential assumes a simplified shape.

Theorem 24. *Set $\Lambda = \sum_{i<j} \alpha^{ij} Y_{ij}$, $\alpha^{ji} = -\alpha^{ij}$, and $X_i = \sum_{j \neq i} \alpha^{ij} Y_j$.*
(i) Let

$$C = D^{-1} \sum_{\mathbf{k}} P^{\mathbf{k}r} Y_{\mathbf{k}} \in \mathscr{P}^{pr}$$

be a homogeneous potential cochain. The Poisson coboundary of C is given by

$$\partial_\Lambda C = \sum_{\mathbf{k}i} X_i \big(D^{-1} P^{\mathbf{k}r}\big)\, Y_i \wedge Y_{\mathbf{k}} = D^{-1} \sum_{\mathbf{k}i} (X_i - \delta_i \operatorname{id})(P^{\mathbf{k}r})\, Y_i \wedge Y_{\mathbf{k}} \in \mathscr{P}^{p+1,r}, \quad (6.5)$$

where $\delta_i = \operatorname{div} X_i \in \mathbb{R}$.

(ii) The Poisson coboundary operator ∂_Λ endows $\mathscr{P}$ with a differential complex structure, and preserves the polynomial degree r. This Poisson complex of Λ over $\mathscr{P}$ contains the Poisson complex $(\mathscr{R},\partial_\Lambda)$ of Λ over $\mathscr{R}$ as a differential subcomplex.

Proof. Note first that if $C = f\mathbf{Y}$, where f a function and $\mathbf{Y}$ a wedge product of vector fields Y_k, we get

$$\partial_\Lambda(f\mathbf{Y}) = [\Lambda, f\mathbf{Y}] = [\Lambda, f] \wedge \mathbf{Y}, \tag{6.6}$$

since the Y_k are mutually commuting. However,

$$[\Lambda, f] = \sum_{i<j} \alpha^{ij}\left((Y_j f)Y_i - (Y_i f)Y_j\right) = \sum_i \Big(\sum_{j\neq i} \alpha^{ij} Y_j f\Big)Y_i = \sum_i (X_i f)Y_i. \tag{6.7}$$

When combining Equations (6.6) and (6.7), we get the first part of Equation (6.5), whereas its second part is the consequence of Proposition 37. ∎

Corollary 5. *The Poisson cohomology groups of Λ over $\mathscr{R}$ and $\mathscr{P}$ are bigraded, i.e.*

$$LH(\mathscr{R},\Lambda) = \oplus_{r=0}^{\infty} \oplus_{p=0}^{n} LH^{pr}(\mathscr{R},\Lambda) \quad \textit{and} \quad LH(\mathscr{P},\Lambda) = \oplus_{r=0}^{\infty} \oplus_{p=0}^{n} LH^{pr}(\mathscr{P},\Lambda),$$

where for instance $LH^{pr}(\mathscr{P},\Lambda)$ is defined by

$$LH^{pr}(\mathscr{P},\Lambda) = \ker(\partial_\Lambda : \mathscr{P}^{pr} \to \mathscr{P}^{p+1,r}) / \operatorname{im}(\partial_\Lambda : \mathscr{P}^{p-1,r} \to \mathscr{P}^{pr}).$$

In the following we deal with the terms $LP^{*r}(\mathscr{P},\Lambda) = \oplus_{p=0}^{n} LP^{pr}(\mathscr{P},\Lambda)$ of the Poisson cohomology over $\mathscr{P}$ and with the corresponding part of Poisson cohomology the subcomplex $\mathscr{R}$.

Theorem 25. *Let E_r be the real finite-dimensional vector space $\mathscr{S}^r\mathbb{R}^{n*}$, and let $\vec{X}_\delta := (X_1 - \delta_1 \,\mathrm{id}, \ldots, X_n - \delta_n \,\mathrm{id})$, $\delta_i = \operatorname{div} X_i$, be the n-tuple of the commuting linear operators $X_i - \delta_i \,\mathrm{id}$ on E_r defined in Theorem 24. The Poisson cohomology space $LH^{*r}(\mathscr{P},\Lambda)$ of Λ over $\mathscr{P}$ coincides with the Koszul cohomology space $KH^*(\vec{X}_\delta, E_r)$ associated with $\vec{X}_\delta$ on E_r:*

$$LH^{*r}(\mathscr{P},\Lambda) \simeq KH^*(\vec{X}_\delta, E_r).$$

Proof. Direct consequence of result $\partial_\Lambda = \sum_i (X_i - \delta_i \,\mathrm{id}) \otimes e_{Y_i}$ proved in Theorem 24. ∎

In order to study the Poisson cohomology group $LH^{\cdot r}(\mathscr{R},\Lambda)$ of the quadratic Poisson tensor Λ over the formal cochain space $\mathscr{R}$, we introduce a long cohomology exact sequence.

Let $\mathscr{S}^{pr}$ be a complementary vector subspace of $\mathscr{R}^{pr}$ in $\mathscr{P}^{pr}$: $\mathscr{P}^{pr}=\mathscr{R}^{pr}\oplus\mathscr{S}^{pr}$. Space $\mathscr{S}=\oplus_{r=0}^{\infty}\oplus_{p=0}^{n}\mathscr{S}^{pr}$ can easily be promoted into the category of differential spaces. Indeed, denote by $p_{\mathscr{R}}$ and $p_{\mathscr{S}}$ the projections of $\mathscr{P}$ onto $\mathscr{R}$ and $\mathscr{S}$ respectively, and set for any $s\in\mathscr{S}$,

$$\phi s=p_{\mathscr{R}}\partial_\Lambda s,\tilde{\partial}_\Lambda s=p_{\mathscr{S}}\partial_\Lambda s.$$

Proposition 38. *(i) The endomorphism* $\tilde{\partial}_\Lambda\in\mathrm{End}_{\mathbb{R}}\mathscr{S}$ *is a differential on* $\mathscr{S}$*, which has weight* $(1,0)$ *with respect to the bigrading of* $\mathscr{S}$*, i.e.* $\tilde{\partial}_\Lambda:\mathscr{S}^{pr}\to\mathscr{S}^{p+1,r}$.
(ii) The linear map $\phi\in\mathrm{Hom}_{\mathbb{R}}(\mathscr{S},\mathscr{R})$ *is an anti-homomorphism of differential spaces from* $(\mathscr{S},\tilde{\partial}_\Lambda)$ *into* $(\mathscr{R},\partial_\Lambda)$*. Its weight with respect to the bidegree is* $(1,0)$*, i.e.* $\phi:\mathscr{S}^{pr}\to\mathscr{R}^{p+1,r}$.
(iii) The sequence $0\to\mathscr{R}\xrightarrow{i}\mathscr{P}\xrightarrow{p_{\mathscr{S}}}\mathscr{S}\to 0$ *is a short exact sequence of homomorphisms of differential spaces, which preserve the bidegree. It induces an exact triangle in cohomology, whose connecting homomorphism* $\phi_\sharp$ *is canonically implemented by* ϕ*. If* $LH^{pr}(\mathscr{S},\tilde{\Lambda})$ *denotes the degree* (p,r) *term of the cohomology space of the complex* $(\mathscr{S},\tilde{\partial}_\Lambda)$*, we have* $\phi_\sharp:LH^{pr}(\mathscr{S},\tilde{\Lambda})\to LH^{p+1,r}(\mathscr{R},\Lambda)$.
(iv) The sequence

$$\begin{aligned}
&0\to LH^{0r}(\mathscr{R},\Lambda)\xrightarrow{i_\sharp}\dots\\
&\quad\xrightarrow{\phi_\sharp}LH^{pr}(\mathscr{R},\Lambda)\xrightarrow{i_\sharp}LH^{pr}(\mathscr{P},\Lambda)\xrightarrow{(p_{\mathscr{S}})_\sharp}LH^{pr}(\mathscr{S},\tilde{\Lambda})\xrightarrow{\phi_\sharp}LH^{p+1,r}(\mathscr{R},\Lambda)\xrightarrow{i_\sharp}\dots\\
&\qquad\qquad\xrightarrow{\phi_\sharp}LH^{nr}(\mathscr{R},\Lambda)\xrightarrow{i_\sharp}LH^{nr}(\mathscr{P},\Lambda)\xrightarrow{(p_{\mathscr{S}})_\sharp}LH^{nr}(\mathscr{S},\tilde{\Lambda})\to 0
\end{aligned}$$

is a long exact cohomology sequence of vector space homomorphisms.
(v) If $\ker^{pr}\phi_\sharp$ *and* $\mathrm{im}^{p+1,r}\phi_\sharp$ *denote the kernel and the image of the restricted map* $\phi_\sharp:LH^{pr}(\mathscr{S},\tilde{\Lambda})\to LH^{p+1,r}(\mathscr{R},\Lambda)$*, we have*

$$LH^{pr}(\mathscr{R},\Lambda)\simeq LH^{p-1,r}(\mathscr{S},\tilde{\Lambda})/\ker^{p-1,r}\phi_\sharp\oplus LH^{pr}(\mathscr{P},\Lambda)/\ker^{pr}\phi_\sharp. \tag{6.8}$$

Proof. Statements (i) and (ii) are direct consequences of equation $\partial_\Lambda^2=0$. For (iii), we only need check that linear map $\phi_\sharp$ coincides with the connecting homomorphism, what is obvious. Eventually, assertion (v) is a corollary of exactness of the long cohomology sequence. ∎

We now identify the $\mathscr{S}$-cohomology with a relative cohomology. Several concepts of relative cohomology can be met in literature. Below, we use the following definition.

Definition 44. *Let V be a vector space endowed with a differential ∂, and let W be a ∂-closed subspace of V. Denote by $\overline{\partial}$ the differential canonically induced by ∂ on the quotient space V/W. The cohomology of the differential space $(V/W,\overline{\partial})$ is called the relative cohomology of (V,W,∂). It is denoted by $H(V,W,\partial)$.*

Proposition 39. *The cohomology induced by ∂_Λ on $\mathscr{S}$ (i.e. the cohomology of differential space $(\mathscr{S},\tilde{\partial}_\Lambda))$ coincides with the relative cohomology of $(\mathscr{P},\mathscr{R},\Lambda)$ (i.e. the cohomology of space $(\mathscr{P}/\mathscr{R},\overline{\partial}_\Lambda))$:*

$$LH(\mathscr{S},\tilde{\Lambda}) \simeq LH(\mathscr{P},\mathscr{R},\Lambda).$$

Proof. It suffices to note that the vector space isomorphism $\psi : \mathscr{P}/\mathscr{R} \ni [\pi] \to p_{\mathscr{S}}\pi \in \mathscr{S}$ intertwines the differentials $\overline{\partial}_\Lambda$ on $\mathscr{P}/\mathscr{R}$ and $\tilde{\partial}_\Lambda$ on $\mathscr{S}$. ∎

Remark. In view of this proposition it is clear that $\mathscr{S}$-cohomology is independent of the chosen splitting $\mathscr{P} = \mathscr{R} \oplus \mathscr{S}$.

Theorem 26. *The Poisson cohomology groups of a SRMI Poisson tensor Λ, over the space $\mathscr{R}$ of cochains with coefficients in the formal power series, are given by*

$$LH^{pr}(\mathscr{R},\Lambda) \simeq LH^{pr}(\mathscr{P},\Lambda)/\ker^{pr}\phi_\sharp \oplus LH^{p-1,r}(\mathscr{P},\mathscr{R},\Lambda)/\ker^{p-1,r}\phi_\sharp,$$

where the above-introduced notations have been used.

Proof. Reformulation of Equation (6.8) and Proposition 39. ∎

Remark.*This theorem reduces computation of the formal Poisson cohomology groups $LH^{pr}(\mathscr{R},\Lambda)$, basically to the Koszul cohomology groups $LH^{pr}(\mathscr{P},\Lambda) \simeq KH^p(\vec{X}_\delta,E_r)$ associated to the afore-detailed operators $\vec{X}_\delta$ on $E_r = \mathscr{S}^r\mathbb{R}^{n*}$ induced by the considered SRMI tensor, and to the relative cohomology groups $LH^{p-1,r}(\mathscr{P},\mathscr{R},\Lambda)$. It thus highlights the link between Poisson and Koszul cohomology. Let us mention that the authors of [MP06] showed, via explicit computations in $\mathbb{R}^3$, that $\mathscr{P}$-cohomology (now identified as Koszul cohomology) and $\mathscr{S}$-cohomology (or relative cohomology) are less intricate than Poisson cohomology.*

The remark concerning the comparative simplicity of the $\mathscr{P}$-cohomology can be easily understood.

Observe that any SRMI Poisson tensor $\Lambda = \sum_{i<j} \alpha^{ij} Y_{ij}$, $\alpha^{ij} \in \mathbb{R}$, with $Y_i = \sum_r \ell_{ir} \partial_r$ and $\ell_{ir} \in \mathbb{R}^{n*}$, reads, locally in $\{D := \det \ell \neq 0\} \subset \{\Lambda \neq 0\} \subset \mathbb{R}^n$,

$$\Lambda = \sum_{i<j} \alpha^{ij} \partial_{s_i s_j} \ (\alpha^{ij} \in \mathbb{R}),$$

where $(s_1, \ldots, s_n)$ are local coordinates. As the Y_i mutually commute, the statement is a direct consequence of the "straightening theorem for vector fields". For instance, for structure $\Lambda = 2aY_{23} + bY_{31} + aY_{12}$, where $Y_1 = x\partial_1 + y\partial_2$, $Y_2 = x\partial_2 - y\partial_1$, $Y_3 = z\partial_3$, and $D = (x^2 + y^2)z$, see Theorem 22, the local (non-polynomial) coordinate transformation

$$x = e^s \cos\theta, y = e^s \sin\theta, z = -e^{-t}$$

leads to $Y_1 = \partial_s, Y_2 = \partial_\theta, Y_3 = \partial_t$, and $\Lambda = 2a\partial_{\theta t} + b\partial_{ts} + a\partial_{s\theta}$.

Hence, *locally* in a dense open subset of $\mathbb{R}^n$, there are coordinate systems or bases in which tensor Λ has *constant coefficients*. The $\mathscr{P}$-cohomology $LH^{*r}(\mathscr{P}, \Lambda)$ however, is the Poisson cohomology in the extended space $\mathscr{P}^{*r} = D^{-1} \oplus_{p=0}^{n} \mathscr{S}^r \mathbb{R}^{n*} \otimes \wedge_n^p \langle \vec{Y} \rangle$, which admits the *global* basis $\vec{Y} = (Y_1, \ldots, Y_n)$ in which structure Λ has *constant coefficients*. This is what makes $\mathscr{P}$-cohomology particularly convenient.

6.4 Koszul cohomology in a finite-dimensional vector space

In view of the above remark regarding the basic ingredients of Poisson cohomology of SRMI tensors of $\mathbb{R}^n$, we take in this section an interest in the Koszul cohomology space $KH^*(\vec{X}_\lambda, E)$ associated to operators $\vec{X}_\lambda := (X_1 - \lambda_1 \,\mathrm{id}, \ldots, X_n - \lambda_n \,\mathrm{id})$ made up of commuting linear transformations $\vec{X} := (X_1, \ldots, X_n)$ of a finite-dimensional real vector space E and a point $\vec{\lambda} := (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. However, Koszul cohomology is known to be closely connected with Spectral Theory—a fundamental principle of multivariate operator theory is that all essential spectral properties of operators $\vec{X}$ in a complex space should be understood in terms of properties of the Koszul complex induced by $\vec{X}_\lambda$, $\vec{\lambda} \in \mathbb{C}^n$—so that the natural framework for investigations on Koszul cohomology is the complex setting.

Proposition 40. *Let (E, ∂) be a differential space over $\mathbb{R}$, and denote by $(E^{\mathbb{C}}, \partial^{\mathbb{C}})$ its complexification. The complexification $H^{\mathbb{C}}(E, \partial)$ of the cohomology space of (E, ∂) and the cohomology $H(E^{\mathbb{C}}, \partial^{\mathbb{C}})$ of differential space $(E^{\mathbb{C}}, \partial^{\mathbb{C}})$, are canonically isomorphic:*

$$H(E^{\mathbb{C}}, \partial^{\mathbb{C}}) \simeq H^{\mathbb{C}}(E, \partial).$$

Proof. Obvious. ∎

Proposition 41. *If $\vec{X} \in \mathrm{End}_{\mathbb{R}}(E)$ are commuting $\mathbb{R}$-linear transformations of a real vector space E, and if $\vec{X}^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(E^{\mathbb{C}})$ are the commuting corresponding complexified $\mathbb{C}$-linear transformations of the complexification $E^{\mathbb{C}}$ of E, the following isomorphism of complex vector spaces holds:*

$$KH^*(\vec{X}^{\mathbb{C}}, E^{\mathbb{C}}) \simeq KH^{*\mathbb{C}}(\vec{X}, E).$$

Proof. In view of Proposition 40, it suffices to check that the complex $K^*(\vec{X}^{\mathbb{C}}, E^{\mathbb{C}})$ is effectively the complexification of the complex $K^*(\vec{X}, E)$. ∎

This proposition allows deducing our subject for investigation, the Koszul cohomology $KH^*(\vec{X}_\lambda, E)$ (where $\vec{\lambda}$ is a point of $\mathbb{R}^n$ and where $\vec{X}$ is an n-tuple of commuting $\mathbb{R}$-linear operators of a finite-dimensional vector space E over $\mathbb{R}$), from its more natural counterpart over the field of complex numbers.

Below, we use the concept of joint spectrum $\sigma(\vec{X})$ of commuting bounded linear operators $\vec{X} = (X_1, \ldots, X_n)$ on a complex vector space E. There are a number of definitions of such spectra in the literature; the considered spaces E are normed spaces, Banach spaces, or Hilbert spaces. Here we investigate Koszul cohomology in finite dimension and need the following characterizations of the elements of the joint spectrum $\sigma(\vec{X})$ (for a proof, we refer the reader to [BR02]):

Proposition 42. *Let $\vec{X} = (X_1, \ldots, X_n)$ be an n-tuple of commuting operators on a finite-dimensional complex vector space E. Then the following statements are equivalent for any fixed $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$:*

(a) $\vec{\lambda} \in \sigma(\vec{X})$

(b) *There exists a basis in E with respect to which the matrices representing the X_j are all upper-triangular, and there exists an index q $(1 \leq q \leq \dim E)$, such that λ_j is the (q,q) entry of the matrix representing X_j, for $j \in \{1, \ldots, n\}$*

(c) *There exists an index q as in Item (b) for every basis in E with respect to which the matrices representing the X_j are all upper-triangular*

(d) *There exists a nonzero vector x such that $X_j x = \lambda_j x$, $\forall j \in \{1, \ldots, n\}$*

(e) *There do not exist Y_j in the subalgebra of $\mathrm{End}_{\mathbb{C}}(E)$ generated by* id *and $\vec{X}$, such that*

$$\sum_{j=1}^{n} Y_j (X_j - \lambda_j \,\mathrm{id}) = \mathrm{id}$$

In the following, we supply some results regarding Koszul cohomology spaces. We use the same notations as above.

Proposition 43. *Let $\wedge = \wedge_n\langle\vec{\eta}\rangle$ be the exterior algebra on n generators $\vec{\eta}$ over a field $\mathbb{F}$ of characteristic 0, and let $\vec{h}$ be dual generators, i.e. $i_{h_k}\eta_\ell = \partial_{k\ell}$. We then have the homotopy formula*

$$e_{\eta_\ell} i_{h_k} + i_{h_k} e_{\eta_\ell} = \delta_{k\ell}\,\mathrm{id},$$

where i_{h_k} and e_{η_ℓ} are the creation and annihilation operators, respectively.

Proof. Obvious. ∎

Proposition 44. *Let $\vec{\mathscr{X}} \in \mathrm{End}_{\mathbb{F}}^{\times n}(E)$ (resp. $\vec{Y} \in \mathrm{End}_{\mathbb{F}}^{\times n}(E)$) be n commuting linear operators $\vec{\mathscr{X}}$ (resp. $\vec{Y}$) on a vector space E over $\mathbb{F}$. We denote by $\mathscr{K} = \sum_\ell \mathscr{X}_\ell \otimes e_{\eta_\ell}$ (resp. $\kappa = \sum_k Y_k \otimes i_{h_k}$) the corresponding Koszul cohomology (resp. homology) operator. The following homotopy-type result holds:*

$$\mathscr{K}\kappa + \kappa\mathscr{K} = \left(\sum_\ell Y_\ell \mathscr{X}_\ell\right) \otimes \mathrm{id} + \sum_{k\ell} [\mathscr{X}_\ell, Y_k] \otimes e_{\eta_\ell} i_{h_k}.$$

Proof. Direct consequence of Proposition 43. ∎

Proposition 45. *Let $\vec{X} \in \mathrm{End}_{\mathbb{C}}^{\times n}(E)$ be n commuting endomorphisms of a finite-dimensional complex vector space E, and let $\vec{\lambda} \in \mathbb{C}^n$. Consider a splitting*

$$E = E_1 \oplus E_2$$

and denote by $i_j : E_j \to E$ (resp. $p_j : E \to E_j$) the injection of E_j into E (resp. the projection of E onto E_j).

If E_1 is stable under the action of the operators X_ℓ, i.e. $p_2 X_\ell i_1 = 0$, and if $\vec{\lambda}$ is not in the joint spectrum $\sigma(\vec{X}_2)$ of the commuting operators $X_{\ell 2} = p_2 X_\ell i_2 \in \mathrm{End}_{\mathbb{C}}(E_2)$, then any cocycle $C \in E \otimes \wedge$ of the Koszul complex $K^*(\vec{X}_\lambda, E)$, where $\vec{X}_\lambda = \vec{X} - \vec{\lambda}\,\mathrm{id}_E$, is cohomologous to a cocycle $C_1 \in E_1 \otimes \wedge$, with $\wedge = \wedge_n\langle\vec{\eta}\rangle$.

Proof. Observe first that if $q(\vec{X}) \in \mathbb{C}[X_1, \ldots, X_n] \subset \mathrm{End}_{\mathbb{C}}(E)$ denotes a polynomial in the X_ℓ, the compound map $q(\vec{X})_2 = p_2 q(\vec{X}) i_2$ coincides with the (same) polynomial $q(\vec{X}_2) \in \mathrm{End}_{\mathbb{C}}(E_2)$ in the $X_{\ell 2}$ ($\star$). Indeed, due to stability of E_1, we have

$$p_2 X_\ell X_k i_2 = p_2 X_\ell i_1 p_1 X_k i_2 + p_2 X_\ell i_2 p_2 X_k i_2 = X_{\ell 2} X_{k2}.$$

This entails in particular that the $X_{\ell 2}$ commute.

As $\vec{\lambda} \notin \sigma(\vec{X}_2)$, Item (e) in Proposition 42 implies that there are n operators $\vec{Y}_2$ in the subalgebra of $\mathrm{End}_{\mathbb{C}}(E_2)$ generated by $\mathrm{id}_{\mathrm{E}_2}$ and $\vec{X}_2$, such that

$$\sum_\ell Y_{\ell 2}(X_{\ell 2} - \lambda_\ell \,\mathrm{id}_{E_2}) = \mathrm{id}_{E_2}\,. \tag{6.9}$$

Hence, for any ℓ, $Y_{\ell 2} = Q_\ell(\vec{X}_2)$ is a polynomial in the X_{k2}. Set now $Y_\ell = Q_\ell(\vec{X}) \in \mathrm{End}_{\mathbb{C}}(E)$.

If applied to operators $\vec{X}_\lambda$ and $\vec{Y}$, Proposition 44 implies that

$$\left(\sum_\ell Y_\ell(X_\ell - \lambda_\ell \,\mathrm{id}_E)\right) \otimes \mathrm{id}_\wedge + \sum_{k\ell} [X_\ell - \lambda_\ell \,\mathrm{id}_E, Y_k] \otimes e_{\eta_\ell} i_{h_k} = \mathscr{K}\kappa + \kappa\mathscr{K}\,,$$

where $\mathscr{K}$ (resp. κ) is the Koszul cohomology (resp. homology) operator associated with $\vec{X}_\lambda$ (resp. $\vec{Y}$) on E. As Y_k is a polynomial in the commuting endomorphisms X_ℓ, the second term on the LHS of the preceding equation vanishes. Hence, when evaluating both sides on a cocycle $C = e \otimes w$ of cochain complex $K^*(\vec{X}_\lambda, E)$, we get

$$\Big(Q(\vec{X})(e)\Big)\, w = \mathscr{K}\kappa(e \otimes w),$$

where $Q(\vec{X}) = \sum_\ell Y_\ell(X_\ell - \lambda_\ell \,\mathrm{id}_E) = \sum_\ell Q_\ell(\vec{X})(X_\ell - \lambda_\ell \,\mathrm{id}_E)$ is a polynomial in the X_ℓ. Up to factor w, the LHS reads

$$Q(\vec{X})(e) = p_1 Q(\vec{X}) i_1 p_1(e) + p_2 Q(\vec{X}) i_1 p_1(e) + p_1 Q(\vec{X}) i_2 p_2(e) + p_2 Q(\vec{X}) i_2 p_2(e),$$

where the second term of the RHS vanishes, in view of the stability of E_1, and where the last term coincides with $p_2(e)$, in view of Remark ($\star$) and Equation (6.9). Eventually, cocycle $C = e \otimes w$ is cohomologous to cocycle

$$C_1 = C - \mathscr{K}\kappa C = \Big(p_1(e) - p_1 Q(\vec{X}) i_1 p_1(e) - p_1 Q(\vec{X}) i_2 p_2(e)\Big) \otimes w \in E_1 \otimes \wedge. \blacksquare$$

The preceding proposition allows in particular recovering the following well-known result:

Corollary 6. *Consider n commuting endomorphisms $\vec{X} \in \mathrm{End}_{\mathbb{C}}^{\times n}(E)$ of a finite-dimensional complex vector space E, and a point $\vec{\lambda} \in \mathbb{C}^n$. Set $\ker \vec{X}_\lambda := \cap_{\ell=1}^n \ker(X_\ell - \lambda_\ell \,\mathrm{id})$. If $\dim(\ker \vec{X}_\lambda) = 0$, the Koszul cohomology $KH^*(\vec{X}_\lambda, E)$ is trivial, and vice versa.*

Proof. It suffices to note that, due to Proposition 42, the dimensional assumption means that $\vec{\lambda} \notin \sigma(\vec{X})$, and to apply the preceding proposition with $E_1 = 0$. Conversely, if there exists $x \in \ker \vec{X}_\lambda \setminus \{0\}$, then $\mathscr{K}_{\vec{X}_\lambda} x = \sum_{\ell=1}^n (X_\ell - \lambda_\ell \,\mathrm{id})(x)\,\eta_\ell = 0$, so that x is a nonbounding 0-cocycle. ∎

The next consequence of Proposition 45 shows that the Koszul cohomology $KH^*(\vec{X}_\lambda, E)$ is—roughly spoken—made up by joint eigenvectors with eigenvalues λ_ℓ.

Consider n commuting endomorphisms $\vec{X} =: \vec{X}^{(1)} \in \mathrm{End}_{\mathbb{C}}^{\times n}(E)$ of a finite-dimensional complex vector space $E =: E^{(1)} =: F^{(1)}$, and a point $\vec{\lambda} \in \mathbb{C}^n$. For any $a \in \{2, 3, \ldots\}$, if $\ker^{(a-1)} := \ker \vec{X}_\lambda^{(a-1)}$ and $E^{(a)} := E^{(a-1)} / \ker^{(a-1)}$, the

$$X_\ell^{(a)} := \left(X_\ell^{(a-1)}\right)^\sharp,$$

$\ell \in \{1, \ldots, n\}$, defined recursively by $X_\ell^{(a)} = \left(X_\ell^{(a-1)}\right)^\sharp : E^{(a)} \ni [e^{(a-1)}] \to [X_\ell^{(a-1)} e^{(a-1)}] \in E^{(a)}$, are again n commuting (well-defined) operators on a finite-dimensional complex vector space. We iterate this procedure finitely many times, thus obtaining operators $X_\ell^{(a)}$, $a \in \{1, \ldots, s+1\}$, until $\ker^{(s+1)} = \ker \vec{X}_\lambda^{(s+1)} = 0$, or, equivalently,

$$\vec{\lambda} \notin \sigma(\vec{X}^{(s+1)}).$$

In the following, we identify the operators $X_\ell^{(a)}$ with their models that arise from the choices of supplementary subspaces $F^{(a)}$ of $\ker^{(a-1)}$ in $E^{(a-1)} \simeq F^{(a-1)}$, $a \in \{2, \ldots, s+1\}$, so that $E^{(a)} \simeq F^{(a)} \subset E^{(a-1)} \simeq F^{(a-1)}$. If we denote by $i_a : F^{(a)} \to F^{(a-1)}$ the inclusion and by $p_a : F^{(a-1)} \to F^{(a)}$ the canonical projection, the isomorphism $E^{(a)} \simeq F^{(a)}$ is $E^{(a)} \ni [f^{(a-1)}] \leftrightarrow p_a f^{(a-1)} \in F^{(a)}$, and operator $X_\ell^{(a)}$, viewed as endomorphism of $F^{(a)}$, reads

$$X_\ell^{(a)} = p_a X_\ell^{(a-1)} i_a, \tag{6.10}$$

since for any $f^{(a)} \in F^{(a)}$, we have $X_\ell^{(a)} f^{(a)} = X_\ell^{(a)}[f^{(a)}] = [X_\ell^{(a-1)} f^{(a)}] = p_a X_\ell^{(a-1)} i_a f^{(a)}$.

Corollary 7. *Let $\vec{\lambda} \in \mathbb{C}^n$ be a point in $\mathbb{C}^n$, and let $\vec{X} = \vec{X}^{(1)} \in \mathrm{End}_{\mathbb{C}}^{\times n}(E)$ be n commuting endomorphisms of a finite-dimensional complex vector space $E = F^{(1)}$.*

Denote by $\vec{X}^{(a)} \in \mathrm{End}_{\mathbb{C}}^{\times n}(F^{(a)})$, $a \in \{2,\ldots,s\}$, the above-depicted "reduced" operators on supplementary spaces $F^{(a)}$, and denote by $\wedge = \wedge_n\langle\vec{\eta}\rangle$ the Grassmann algebra with n generators $\vec{\eta}$.

Any cocycle

$$C \in E \otimes \wedge$$

of the Koszul complex $K^*(\vec{X}_\lambda, E)$ is cohomologous to a cocycle

$$C_1 \in \left(\ker\vec{X}_\lambda^{(1)} \oplus \ker\vec{X}_\lambda^{(2)} \oplus \ldots \oplus \ker\vec{X}_\lambda^{(s)}\right) \otimes \wedge.$$

Proof. It suffices to apply Proposition 45 to the obvious splitting

$$E = E_1 \oplus E_2 := \left(\bigoplus_{a=1}^{s} \ker^{(a)}\right) \oplus F^{(s+1)}.$$

Indeed, the operators $\vec{X}_2$ considered in Proposition 45 read $X_{\ell 2} = p_{s+1}\ldots p_2 X_\ell i_2 \ldots i_{s+1} = X_\ell^{(s+1)}$, where we used the afore-introduced notations i_a and p_a. Hence, the spectral condition $\vec{\lambda} \notin \sigma(\vec{X}_2)$ is satisfied by definition of s, see above. Moreover, if $k^{(a)} \in \ker^{(a)} \subset F^{(a)}$, $a \in \{1,\ldots,s\}$, we have

$$X_\ell\, k^{(a)} = X_\ell\, i_2 \ldots i_a\, k^{(a)} = p_a \ldots p_2\, X_\ell\, i_2 \ldots i_a\, k^{(a)} + \sum_{b=2}^{a} \pi_b\, p_{b-1} \ldots p_2\, X_\ell\, i_2 \ldots i_a\, k^{(a)}. \tag{6.11}$$

Mapping $\pi_b : F^{(b-1)} \to \ker^{(b-1)}$ is the second projection associated with the decomposition $F^{(b-1)} = F^{(b)} \oplus \ker^{(b-1)}$, so that $\mathrm{id}_{F^{(b-1)}} = p_b + \pi_b$. In order to derive Equation (6.11), we utilized this upshot for $b \in \{2,\ldots,a\}$. The first term of the RHS of Equation (6.11) is $X_\ell^{(a)} k^{(a)} = \lambda_\ell\, k^{(a)} \in \ker^{(a)}$, and the terms characterized by index b are elements of the spaces $\ker^{(b-1)}$. Hence, space $E_1 = \oplus_{a=1}^{s} \ker^{(a)}$ is stable under the action of the X_ℓ and Proposition 45 is applicable. ∎

Corollary 8. *On the conditions of Corollary 7, if for any $\ell \in \{1,\ldots,n\}$, the kernel and the image of the transformation $X_\ell - \lambda_\ell\,\mathrm{id}$ are supplementary in E, then any cocycle $C \in E \otimes \wedge$ of the Koszul complex $K^*(\vec{X}_\lambda, E)$ is cohomologous to a cocycle $C_1 \in \ker\vec{X}_\lambda \otimes \wedge$.*

Proof. It suffices to prove that $s = 1$. If $s \neq 1$, there is a nonzero vector $x \in \ker\vec{X}_\lambda^{(2)} \subset F^{(2)}$. Then, for any $k,\ell \in \{1,\ldots,n\}$, $(X_k - \lambda_k\,\mathrm{id})(X_\ell - \lambda_\ell\,\mathrm{id})x = (X_k - \lambda_k\,\mathrm{id})(p_2 X_\ell i_2 x + \pi_2 X_\ell i_2 x - \lambda_\ell x) = (X_k - \lambda_k\,\mathrm{id})(\pi_2 X_\ell i_2 x) = 0$, as $\pi_2 X_\ell i_2 x \in \ker\vec{X}_\lambda$. Hence, for every ℓ, we have $(X_\ell - \lambda_\ell\,\mathrm{id})x \in \ker\vec{X}_\lambda \cap \mathrm{im}(X_\ell - \lambda_\ell\,\mathrm{id}) = 0$. Eventually, $x \in (\ker\vec{X}_\lambda) \cap F^{(2)} = 0$, a contradiction.

6.5 Koszul cohomology associated with Poisson cohomology

We now come back to the Koszul cohomology implemented by a SRMI tensor of $\mathbb{R}^n$. Let us recall that we deal with a SRMI tensor

$$\Lambda = \sum_{j<k} \alpha^{jk} Y_{jk} \quad (\alpha^{jk} \in \mathbb{R}),$$

where the Y_j are n commuting linear vector fields that verify $Y_{1\ldots n} \neq 0$. The main building block of the Poisson cohomology of such a tensor has been identified as the Koszul cohomology space $KH^*(\vec{X}_\delta, E_r)$ associated to the operators $\vec{X}_\delta = (X_1 - \delta_1 \,\mathrm{id}, \ldots, X_n - \delta_n \,\mathrm{id})$, $X_j = \sum_k \alpha^{jk} Y_k$, $\alpha^{kj} = -\alpha^{jk}$, $\delta_j = \mathrm{div}\, X_j$ on the spaces $E_r = \mathscr{S}^r \mathbb{R}^{n*}$, $r \in \mathbb{N}$. We already pointed out that this cohomology can be deduced from its complex counterpart $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$ (see Proposition 41), which is tightly related with joint eigenvectors and the joint spectrum of $\vec{X}^{\mathbb{C}}$ or $\vec{X}_\delta^{\mathbb{C}}$ (see Corollaries 6 and 7). In this section, we further investigate the Koszul cohomology space $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$. In particular, we reduce the computation of this central part of the Poisson cohomology space $LH^{*r}(\mathscr{R}, \Lambda)$ to essentially a problem of linear algebra, and give a description of the spectrum of the transformations $\vec{X}_\delta^{\mathbb{C}}$.

When dealing with commuting operators on a finite-dimensional complex vector space, it is natural to use an upper-triangular representation of these transformations. The following theorem shows that, for our endomorphisms $\vec{X}_\delta^{\mathbb{C}}$ of the space $E_r^{\mathbb{C}} = \mathscr{S}^r \mathbb{C}^{n*}$ (see below), which has the possibly high (complex) dimension $N = (r+n-1)!/[r!(n-1)!]$ (if e.g. $r = 10$ and $n = 3$, this dimension equals $N = 66$), the problem of finding such a representation $\vec{X}_\delta^{\mathbb{C}} \in \mathrm{gl}(N, \mathbb{C})^{\times n}$ (we denote the operators and their representation by the same symbol) reduces to the quest for an upper-triangular representation $\vec{a} \in \mathrm{gl}(n, \mathbb{C})^{\times n}$ of some commuting transformations $\vec{a}$ of $\mathbb{C}^n$. More precisely, the a_j, $j \in \{1, \ldots, n\}$, are the commuting matrices $a_j = (J^1)^{-1} Y_j \in \mathrm{gl}(n, \mathbb{R})$ that correspond to the commuting linear vector fields Y_j.

Proposition 46. *Any basis of $\mathbb{C}^n$, in which the commuting operators $\vec{a}$ have an upper-triangular representation, naturally induces a basis of $E_r^{\mathbb{C}} = \mathscr{S}^r \mathbb{C}^{n*}$, in which all the transformations $\vec{X}_\delta^{\mathbb{C}}$ are upper-triangular.*

Let us first mention that in the sequel the use of super- and subscripts is dictated by esthetic criteria and not at all by contra- or covariance.

Proof. In the following, we denote by $x=(x_1,\ldots,x_n)$ (resp. $z=(z_1,\ldots,z_n)$) the points of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) as well as their coordinates in the canonical basis $(e_1,\ldots,e_n)$. As usual, we set $Y_k=\sum_m \ell_{km}\partial_{x_m}=\sum_{mp}a_k^{mp}x_p\partial_{x_m}$ and use notations as $x^\beta=x_1^{\beta_1}\ldots x_n^{\beta_n}$, $\beta\in\mathbb{N}^n$.

The complexification $E_r^{\mathbb{C}}$ of

$$E_r=\mathscr{S}^r\mathbb{R}^{n*}=\{P\in C^\infty(\mathbb{R}^n):P(x)=\sum_{|\beta|=r}r_\beta x^\beta\quad(x\in\mathbb{R}^n,r_\beta\in\mathbb{R})\}$$

is

$$E_r\oplus iE_r\simeq E_r^{\mathbb{C}}\simeq\mathscr{S}^r\mathbb{C}^{n*}=\{P\in C^\infty(\mathbb{C}^n):P(z)=\sum_{|\beta|=r}c_\beta z^\beta\quad(z\in\mathbb{C}^n,c_\beta\in\mathbb{C})\}.$$

It is also easily seen that the complexification $Y_k^{\mathbb{C}}\in\mathrm{End}_{\mathbb{C}}(E_r^{\mathbb{C}})$ of $Y_k\in\mathrm{End}_{\mathbb{R}}(E_r)$ is the holomorphic vector field

$$Y_k^{\mathbb{C}}=\sum_{mp}a_k^{mp}z_p\partial_{z_m}\in\mathrm{Vect}^{10}(\mathbb{C}^n)$$

of $\mathbb{C}^n$.

It is well-known that the n commuting matrices $a_j=(J^1)^{-1}Y_j\in\mathrm{gl}(n,\mathbb{R})$ can be reduced simultaneously to upper-triangular matrices by a unitary matrix $U\in\mathrm{U}(n,\mathbb{C})$. Consider any matrix $U\in\mathrm{GL}(n,\mathbb{C})$ (resp. any basis $(e'_1,\ldots,e'_n)$ of $\mathbb{C}^n$), such that the $b_j=U^{-1}a_jU\in\mathrm{gl}(n,\mathbb{C})$ are upper-triangular (resp. in which the transformations $\vec{a}$ are all upper-triangular). Denote by $\mathfrak{z}=(\mathfrak{z}_1,\ldots,\mathfrak{z}_n)$ the components of the vectors $z=\sum_j\mathfrak{z}_je'_j\in\mathbb{C}^n$ in the basis $(e'_1,\ldots,e'_n)$, and let $(\varepsilon'_1,\ldots,\varepsilon'_n)$ be the dual basis of this new basis. If viewed as a basis of the space $E_r^{\mathbb{C}}$ of degree r homogeneous polynomials of $\mathbb{C}^n$, the induced basis $\varepsilon'_{j_1}\vee\ldots\vee\varepsilon'_{j_r}$, $j_1\leq\ldots\leq j_r$, of the space $\mathscr{S}^r\mathbb{C}^{n*}$ of symmetric covariant r-tensors of $\mathbb{C}^n$ reads $\mathfrak{z}^\beta$, $\beta\in\mathbb{N}^n$, $|\beta|=r$.

In order to find the matrices of the operators $\vec{X}_\delta^{\mathbb{C}}$ in this "natural" basis $\mathfrak{z}^\beta$, $\beta\in\mathbb{N}^n$, $|\beta|=r$ of $E_r^{\mathbb{C}}$, we range the vectors $\mathfrak{z}^\beta$ according to the lexicographic order $\prec$ and perform the coordinate change $z=U\mathfrak{z},\partial_z=\widetilde{\partial_{\mathfrak{z}}z}^{-1}\partial_{\mathfrak{z}}$ in the first order linear differential operators $(X_j-\delta_j\,\mathrm{id})^{\mathbb{C}}$. We get

$$\begin{aligned}(X_j-\delta_j\,\mathrm{id})^{\mathbb{C}}&=\sum_k\alpha^{jk}\sum_{m\leq p}b_k^{mp}\mathfrak{z}_p\partial_{\mathfrak{z}_m}-\delta_j\,\mathrm{id}^{\mathbb{C}}\\&=\sum_{km}\alpha^{jk}b_k^{mm}\left(\mathfrak{z}_m\partial_{\mathfrak{z}_m}-\mathrm{id}^{\mathbb{C}}\right)+\sum_k\sum_{m<p}\alpha^{jk}b_k^{mp}\mathfrak{z}_p\partial_{\mathfrak{z}_m},\end{aligned}$$

since $\delta_j = \operatorname{div} X_j = \sum_{km} \alpha^{jk} a_k^{mm} = \sum_{km} \alpha^{jk} b_k^{mm}$. As the image of vector $\mathfrak{z}^\beta$ by operator $(X_j - \delta_j \operatorname{id})^{\mathbb{C}}$ is

$$(X_j - \delta_j \operatorname{id})^{\mathbb{C}} \mathfrak{z}^\beta = \sum_{km} \alpha^{jk} b_k^{mm} (\beta_m - 1) \mathfrak{z}^\beta + \sum_k \sum_{m<p} \alpha^{jk} b_k^{mp} \beta_m \mathfrak{z}^{\beta - e_m + e_p}, \quad (6.12)$$

where $\mathfrak{z}^{\beta - e_m + e_p} \prec \mathfrak{z}^\beta$, the matrices of the commuting operators $(X_j - \delta_j \operatorname{id})^{\mathbb{C}}$, $j \in \{1, \ldots, n\}$, in the basis $\mathfrak{z}^\beta$, $\beta \in \mathbb{N}^n$, $|\beta| = r$, of space $E_r^{\mathbb{C}}$, are all upper-triangular. ∎

The next theorem provides a description of the joint spectrum $\sigma_r(\vec{X}_\delta^{\mathbb{C}})$ of the operators $\vec{X}_\delta^{\mathbb{C}} \in \operatorname{End}_{\mathbb{C}}^{\times n}(E_r^{\mathbb{C}})$.

Let $B \in \operatorname{gl}(n, \mathbb{C})$ be the matrix $B_{jk} = b_j^{kk}$ made up by the diagonals of the matrices b_j, see above.

Theorem 27. *The joint spectrum $\sigma_r(\vec{X}_\delta^{\mathbb{C}})$ of the commuting operators $\vec{X}_\delta^{\mathbb{C}} \in \operatorname{End}_{\mathbb{C}}^{\times n}(E_r^{\mathbb{C}})$ on the finite-dimensional complex vector space $E_r^{\mathbb{C}}$, is given by*

$$\sigma_r(\vec{X}_\delta^{\mathbb{C}}) = \{\alpha B I : I \in (\mathbb{N} \cup \{-1\})^n, |I| = r - n\} \subset \mathbb{C}^n,$$

where $|I| = \sum_j I_j$ denotes the length of I.

Proof. Direct consequence of Proposition 42 and Equation (6.12). ∎

Remark. In Proposition 37, we showed that for all k, $Y_k D = (\operatorname{div} Y_k) D$, where $D = \det \ell \in E_n \subset E_n^{\mathbb{C}}$. It of course follows that for all j, $X_j^{\mathbb{C}} D = X_j D = (\operatorname{div} X_j) D = \delta_j \operatorname{id}^{\mathbb{C}} D$, so that $\vec{0} = (0, \ldots, 0) \in \sigma_n(\vec{X}_\delta^{\mathbb{C}})$. This last upshot is immediately recovered from Theorem 27.

Set $K_r(\vec{X}_\delta^{\mathbb{C}}) = \{I \in \ker(\alpha B) : I \in (\mathbb{N} \cup \{-1\})^n, |I| = r - n\}$. Corollary 6 can then be reformulated as follows.

Corollary 9. *The Koszul cohomology $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$ is acyclic if and only if $K_r(\vec{X}_\delta^{\mathbb{C}}) = \varnothing$.*

Proof. Indeed, $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$ is trivial if and only if $\dim(\ker \vec{X}_\delta^{\mathbb{C}}) = 0$, if and only if $\vec{0} \notin \sigma_r(\vec{X}_\delta^{\mathbb{C}})$, i.e. if and only if $K_r(\vec{X}_\delta^{\mathbb{C}}) = \varnothing$. ∎

We now depict a convenient method that allows finding a basis of the space

$$\ker \vec{X}_\delta^{\mathbb{C}(1)} \oplus \ker \vec{X}_\delta^{\mathbb{C}(2)} \oplus \ldots \oplus \ker \vec{X}_\delta^{\mathbb{C}(s)},$$

which houses the Koszul cohomology $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$, see Corollary 7.

In order to simplify notations, we systematically omit in the following description superscript $\mathbb{C}$. We write e.g. $\vec{X}_\delta, E_r, \ldots$ instead of $\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}}, \ldots$

Consider any basis $(e_1, \ldots, e_N)$ of E_r that generates an upper-triangular representation $T_1, \ldots, T_n$ of the operators $\vec{X}_\delta$. The kernel $\ker \vec{X}_\delta$ is then described by the n triangular systems

$$T_1 Z = 0, \ldots, T_n Z = 0, \tag{6.13}$$

(each one) of N homogeneous linear equations in the N complex unknowns $Z = (Z^1, \ldots, Z^N)$.

As understood before, $\vec{0} \in \sigma_r(\vec{X}_\delta)$ if and only if at least one of the lines $\vec{T}^{\,q} = (T_1^{qq}, \ldots, T_n^{qq})$, $q \in \{1, \ldots, N\}$, is $\vec{0} = (0, \ldots, 0)$. We refer to the number μ of such $\vec{0}$–lines $\vec{T}^{\,q_1}, \ldots, \vec{T}^{\,q_\mu}$, $q_1 < \ldots < q_\mu$, as the multiplicity of $\vec{0}$ in the spectrum $\sigma_r(\vec{X}_\delta)$ (in the considered basis $(e_1, \ldots, e_N)$). Of course, the general solution of System (6.13) is a linear combination $Z = \sum_j c_j K_j$, $c_j \in \mathbb{C}$, of $d = \dim \ker \vec{X}_\delta$ independent vectors $K_j \in \mathbb{C}^N$. Let

$$k_j = K_j^1 e_1 + \ldots + K_j^{q_{\nu_j}} e_{q_{\nu_j}}, \; j \in \{1, \ldots, d\}, \tag{6.14}$$

be the corresponding basis of $\ker \vec{X}_\delta$. It can quite easily be seen—just "solve" System (6.13) and start imagining a configuration that leads to the maximal dimension of the space of solutions—that $d \le \mu$ and that the components $K_j^{q_{\nu_j}} \neq 0$ of the vectors k_j with highest superscript correspond to $\vec{0}$–lines $q_{\nu_1} < \ldots < q_{\nu_d}$.

The N-tuple $(k_1, \ldots, k_d, e_1, \ldots, \widehat{e_{q_{\nu_1}}}, \ldots, \widehat{e_{q_{\nu_d}}}, \ldots, e_N)$ is a basis of E_r, since the determinant in the basis $(e_1, \ldots, e_N)$ of the permuted N-tuple $(e_1, \ldots, k_1, \ldots, k_d, \ldots, e_N)$ equals $K_1^{q_{\nu_1}} \ldots K_d^{q_{\nu_d}} \neq 0$. Observe that the k_j are joint eigenvectors of the $\vec{X}_\delta$ associated with eigenvalue 0. Moreover, in view of Equation (6.14), every vector $e_{q_{\nu_j}}$ can be written in terms of "lower" vectors of the new basis. Hence, the first d columns of the representative matrices $T'_1, \ldots, T'_n$ of the operators $\vec{X}_\delta$ in the new basis vanish, these matrices are again upper-triangular, and the lines $\vec{T}^q$, $q \in \{1, \ldots, N\}$, are unchanged up to permutation. The matrices $T'_\ell + \delta_\ell \,\mathrm{id} \in \mathrm{gl}(N, \mathbb{C})$ correspond to the operators X_ℓ, $\ell \in \{1, \ldots, n\}$, and their lower right submatrices $(T'_\ell + \delta_\ell \,\mathrm{id})^{(2)} \in \mathrm{gl}(N-d, \mathbb{C})$ (resp. $T_\ell'^{(2)}$) correspond to the operators $X_\ell^{(2)}$ (resp. $X_\ell^{(2)} - \delta_\ell \,\mathrm{id}^{(2)}$), see Equation (6.10) and Corollary 7.

In other words, in the basis $(e_1, \ldots, \widehat{e_{q_{\nu_1}}}, \ldots, \widehat{e_{q_{\nu_d}}}, \ldots, e_N)$ of a space $F_r^{(2)}$, see Corollary 7, which is supplementary to $\ker \vec{X}_\delta$ in E_r, the operators $\vec{X}_\delta^{(2)}$ are represented by upper-triangular matrices $T_1'^{(2)},\dots,T_n'^{(2)}$. Thus, the above-detailed procedure can be iterated and the general solution of another packet of n (smaller) triangular systems of linear equations

$$T_1'^{(2)}Z = 0,\dots,T_n'^{(2)}Z = 0, \tag{6.15}$$

provides a basis $k_1^{(2)},\dots,k_{d_2}^{(2)}$ of $\ker \vec{X}_\delta^{(2)}$, et cetera.

Remarks.

- *The solutions of the triangular systems of homogeneous linear equations (6.13), (6.15), ... generate a basis of the locus*

$$\ker \vec{X}_\delta^{\mathbb{C}\,(1)} \oplus \ker \vec{X}_\delta^{\mathbb{C}\,(2)} \oplus \dots \oplus \ker \vec{X}_\delta^{\mathbb{C}\,(s)}$$

  of the Koszul cohomology space $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$.

- Observe that if the $b_\ell = U^{-1}a_\ell U$ have been computed, the upper-triangular matrix representations $T_1,\dots,T_n$ of the transformations $\vec{X}_\delta^{\mathbb{C}}$ in the corresponding basis $\mathfrak{z}^\beta$, $\beta \in \mathbb{N}^n$, $|\beta| = r$, of $E_r^{\mathbb{C}}$ are known, see Equation (6.12), and explicit computations can actually be performed.

- As the multiplicity of $\vec{0}$ in the spectrum of the endomorphisms $\vec{X}_\delta^{\mathbb{C}\,(2)}$ is $\mu - d$, and as its multiplicity in the spectrum of the $\vec{X}_\delta^{\mathbb{C}\,(s+1)}$ vanishes, by definition of s, we get

$$\mu = d + d_2 + \dots + d_s = \sum_{j=1}^{s} \dim \ker \vec{X}_\delta^{\mathbb{C}\,(j)}, \tag{6.16}$$

  with self-explaining notations. As the RHS of this equation is independent of the considered basis, the concept of multiplicity of a point $\lambda \in \mathbb{C}^n$ in the joint spectrum of commuting transformations of a finite-dimensional vector space, makes sense. Although this result might be well-known, we could not find it anywhere in literature.

Example 1. Consider structure Λ_2 of the DHC, see Theorem 22, and assume that $a \neq 0, b = 0$. It is easily checked that the matrix

$$U = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix}$$

transforms the above-mentioned matrices a_ℓ simultaneously into upper-triangular matrices b_ℓ. A short computation yields that the space $K_{3t}(\vec{X}_\delta^{\mathbb{C}})$, $t \in \mathbb{N}$, contains the unique point $I_t = (t-1, t-1, t-1)$, so that the multiplicity μ of $\vec{0}$ in the joint spectrum $\sigma_{3t}(\vec{X}_\delta^{\mathbb{C}})$ equals 1, see proof of Theorem 27. It follows that the Koszul cohomology spaces $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_{3t}^{\mathbb{C}})$ are not trivial, see Corollary 9. Furthermore, since the matrices b_ℓ are in fact diagonal in this example, Equation (6.12) entails that $\mathfrak{z}_1^t\mathfrak{z}_2^t\mathfrak{z}_3^t$ belongs to the kernel $\ker_{3t}\vec{X}_\delta^{\mathbb{C}}$ of operators $\vec{X}_\delta^{\mathbb{C}}$ in space $E_{3t}^{\mathbb{C}}$. If we take into account Equation (6.16), we see that $\ker_{3t}\vec{X}_\delta^{\mathbb{C}} = \mathbb{C}\mathfrak{z}_1^t\mathfrak{z}_2^t\mathfrak{z}_3^t$ and that the reduced operators $\vec{X}_\delta^{\mathbb{C}(j)}$, $j \in \{2, \ldots, s\}$, do not exist, i.e. that $s = 1$. Hence, and since the change to canonical coordinates is $z = U\mathfrak{z}$, see proof of Proposition 46, the cohomology space $KH^p(\vec{X}_\delta^{\mathbb{C}}, E_{3t}^{\mathbb{C}})$, $p \in \{0,1,2,3\}$, $t \in \mathbb{N}$, is located inside

$$\mathfrak{z}_1^t\mathfrak{z}_2^t\mathfrak{z}_3^t \bigoplus_{j_1<\ldots<j_p} \mathbb{C}Y_{j_1\ldots j_p} = (z_1^2+z_2^2)^t z_3^t \bigoplus_{j_1<\ldots<j_p} \mathbb{C}Y_{j_1\ldots j_p}.$$

This rather easily obtained upshot is in accordance with the results of [MP06] (modulo slight changes in definitions and notations [e.g. the roles of parameters a and b are exchanged]).

Example 2. For structure Λ_3 of the DHC and parameter value $a = 0$, depending on the value of r, the multiplicity of $\vec{0}$ in the spectrum $\sigma_r(\vec{X}_\delta^{\mathbb{C}})$ equals 0 or 1— and computations are similar to those of the preceding example—, except in the case $r = 3$, which generates multiplicity 3. Since for Λ_3 the matrices a_ℓ are lower-triangular, a coordinate change $z \leftrightarrow \mathfrak{z}$ is not necessary and it can easily be seen that we have $s = 3$ and

$$\ker_3\vec{X}_\delta^{\mathbb{C}} = \mathbb{C}z_1^2z_3, \ker_3\vec{X}_\delta^{\mathbb{C}(2)} = \mathbb{C}z_1z_2z_3, \ker_3\vec{X}_\delta^{\mathbb{C}(3)} = \mathbb{C}z_2^2z_3.$$

The corresponding cohomological upshots are part of the computation of the Poisson cohomology of Λ_3 that we detail in the next section.

Remark. Remember that the operators X_i are defined by $X_i = \sum_j \alpha^{ij}Y_j$, with $\alpha^{ji} = -\alpha^{ij}$. Hence, matrix $\alpha \in \mathrm{gl}(n, \mathbb{R})$ is skew-symmetric, and $\det\alpha$ vanishes for odd n. Of course, the corresponding non-trivial linear combination $\sum_i c_i\alpha^{i*} = 0$, induces a non-trivial combination $\sum_i c_iX_i = 0$ of the X_i (and the $X_i - \delta_i\,\mathrm{id}$), which is significant in computations. In the even dimensional ($n = 2m, m \in \{2,3,\ldots\}$) maximal rank ($\mathrm{rk}\,\alpha = n$) case, the Koszul cohomology $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$ has the following simple description. If (in even dimension n) $\det\alpha \neq 0$, then

$$\bigoplus_{r\in\mathbb{N}} KH^0(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}}) = \mathbb{C}\,\mathscr{D},$$

where $\mathscr{D}$ denotes the complex clone of $\det \ell$, and, for any $r \neq n$ and any $p \in \{1,\ldots,n\}$, the cohomology space $KH^p(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$ vanishes. We do not detail the proof that is, roughly, along the lines of Proposition 37. If $r = n$, the situation is more complicated and new elements of $\ker \vec{X}_\delta^{\mathbb{C}\,(1)} \oplus \ker \vec{X}_\delta^{\mathbb{C}\,(2)} \oplus \ldots \oplus \ker \vec{X}_\delta^{\mathbb{C}\,(s)}$ may enter the play.

6.6 Cohomology spaces of structures Λ_3 and Λ_9

We already pointed out that the Poisson cohomology (or $\mathscr{R}$-cohomology) of SRMI tensors can be deduced from a Koszul cohomology ($\mathscr{P}$-cohomology) and a relative cohomology ($\mathscr{S}$-cohomology), see Theorem 26, Theorem 25, and Proposition 39.

The involved Koszul cohomology has been studied in the last section. We particularized our upshots by means of (pertinent) examples, see Examples 1 and 2, Section 6.5.

Within the cohomology computations of SRMI tensors of the DHC, $\mathscr{S}$-cohomology has so far been determined “by hand”. In the majority of cases, the Poisson cohomology operator respects, in addition to the degrees p and r, a partial polynomial degree k (e.g. the coboundary operator associated with Λ_3 respects the partial degree in $x = x_1, y = x_2$), so that we can decompose space $\mathscr{S}^{pr}$ into smaller spaces $\mathscr{S}^p_{kr}$ (made up by the elements of $\mathscr{S}^{pr}$ that have partial degree k), see [MP06]. The cohomology operator of structure Λ_9 however, does not respect any additional degree. The $\mathscr{S}$-cohomology of Λ_9 is therefore quite intricate.

Theorem 26 leads to the following cohomological upshots for structures Λ_3 and Λ_9. No proofs will be given (for a description of an application of the technique, see [MP06]).

Theorem 28. *If $a \neq 0$, the cohomology spaces of structure Λ_3 are*

$$LH^{0*}(\mathscr{R}, \Lambda_3) = \mathbb{R},$$

$$LH^{1*}(\mathscr{R}, \Lambda_3) = \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Y_3,$$

$$LH^{2*}(\mathscr{R}, \Lambda_3) = \mathbb{R}Y_{23} \oplus \mathbb{R}Y_{31} \oplus \mathbb{R}(2yz\partial_{31} + y^2\partial_{12}),$$

$$LH^{3*}(\mathscr{R}, \Lambda_3) = \mathbb{R}\,\partial_{123} \oplus \mathbb{R}\,y^2 z\,\partial_{123},$$

where the Y_i are those defined in Theorem 22.

Theorem 29. *If $a \neq 0$, the cohomology spaces of structure Λ_9 are*

$$LH^{0*}(\mathscr{R}, \Lambda_9) = \mathbb{R},$$

$$LH^{1*}(\mathscr{R},\Lambda_9) = \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Y_3,$$
$$LH^{3*}(\mathscr{R},\Lambda_9) = \oplus_{r\in\mathbb{N}}\mathbb{R}z^r\partial_{123},$$

and

$$LH^{2*}(\mathscr{R},\Lambda_9) = \oplus_{r\in\mathbb{N}}H^2_r,$$

where

$$\begin{aligned}
&H^2_0 = \mathbb{R}\partial_{23}, \quad H^2_1 = \mathbb{R}C^0_1, \quad H^2_3 = \mathbb{R}C^2_1,\\
&H^2_2 = \mathbb{R}x^2\partial_{23} + \mathbb{R}xz(\partial_{23} - 2^{-1}\partial_{31}) + \mathbb{R}(xz\partial_{12} - z^2\partial_{23})\\
&\qquad + \mathbb{R}(yz\partial_{12} + (-27a^2x^2 - 9axz + 3ay^2 - z^2)\partial_{31}),\\
&H^2_{r+1} = \mathbb{R}C^r_1 + \mathbb{R}C^r_2, \quad r \geq 3,
\end{aligned}$$

with

$$\begin{aligned}
C^r_1 \;=\; & -a(xz^r + ry^2z^{r-1})\partial_{12} + (9a^2xy^r + a(3r-1)(r+1)^{-1}z^{r+1})\partial_{23}\\
& + ayz^r\partial_{31}
\end{aligned}$$

and

$$\begin{aligned}
C^r_2 \;=\; & (-a(r-2)y^4z^{r-3} + y^2z^{r-1})\partial_{12}\\
& + (6a(r-1)^{-1}xyz^{r-1} - ay^3z^{r-2} - r^{-1}yz^r)\partial_{31}\\
& + (9a^2xy^2z^{r-2} - 9ar^{-1}xz^r + 3a(r-3)(r-1)^{-1}y^2z^{r-1}\\
& \qquad\qquad - 3(r-1)r^{-1}(r+1)^{-1}z^{r+1})\partial_{23},
\end{aligned}$$

where the Y_i are those defined in Theorem 22 (and where the terms that contain a power of x, y, or z with a negative exponent are ignored).

6.7 Cohomological phenomena

Let us outline the most important cohomological phenomena.

Consider a SRMI Poisson structure $\Lambda = \sum_{i<j} \alpha^{ij} Y_{ij}$.

It is easily checked that the curl vector field of Λ, see Section 6.2, is given by $K(\Lambda) = \sum_i \delta_i Y_i$, $\delta_i = \operatorname{div} X_i$, $X_i = \sum_j \alpha^{ij} Y_j$. Consequently, K-exactness is (in $\mathbb{R}^n$, $n \geq 3$) equivalent with divergence-freeness. Note now that the 0–cohomology space $LH^{0*}(\mathscr{R},\Lambda)$ of Λ, or space $\mathrm{Cas}(\Lambda)$ of Casimirs of Λ, coincides with the kernel $\ker\vec{X}$, see Equation 6.5. Hence, in view of Proposition 37, for a K-exact tensor, $D^p = (\det \ell)^p$ is a joint eigenvector of the X_i with eigenvalues $p\,\delta_i = 0$, i.e. $D^p \in \ker \vec{X}$. It follows that, for K-exact SRMI Poisson tensors,

$$\oplus_{p\in\mathbb{N}} \mathbb{R}D^p \subset LH^{0*}(\mathscr{R},\Lambda) = \mathrm{Cas}(\Lambda).$$

As for the 1–cohomology space $LH^{1*}(\mathscr{R},\Lambda)$, let us first remark that the stabilizer $\mathfrak{g}_\Lambda$, viewed as a Lie subalgebra of linear vector fields $\mathscr{X}_0^1(\mathbb{R}^n)$, is made up by 1-cocycles (by definition) that do not bound (degree argument), i.e.

$$\mathfrak{g}_\Lambda \subset LH^{1*}(\mathscr{R},\Lambda).$$

Moreover, as Poisson cohomology is an associative graded commutative algebra, the classes of the cocycles in

$$\mathrm{Cas}(\Lambda) \otimes \wedge^p \mathfrak{g}_\Lambda,$$

$0 \le p \le n$, are "preferential" Poisson cohomology classes. Such classes massively appear in the Poisson cohomology of SRMI tensors of the DHC, see [MP06], and of twisted SRMI tensors, see Chapter 5.

However, two other types of classes systematically appear in Poisson cohomology.

1. The classes of type I originate from $\mathscr{P}$-cohomology. In fact, roughly spoken, the locus $\ker \vec{X}_\delta^{\mathbb{C}(1)} \oplus \ker \vec{X}_\delta^{\mathbb{C}(2)} \oplus \ldots \oplus \ker \vec{X}_\delta^{\mathbb{C}(s)}$ of the Koszul cohomology associated with the considered Poisson cohomology generates in some cases nonbounding cocycles in $\mathscr{R}$-cohomology. For instance, for structure Λ_7, the rational functions $D'^{\frac{\gamma}{2}} z^{-1}$, $D' = x^2 + y^2$, $\gamma \in 2\mathbb{N}^*$, induce the classes $D'^{\frac{\gamma}{2}} z^{-1} Y_3$, $Y_3 = z\partial_3$, in space $LH^{1*}(\mathscr{R},\Lambda_7)$.
2. The classes of type II are due to $\mathscr{S}$-cohomology. Indeed, let $\mathfrak{s}$ be a cochain in space $\mathscr{S}$, which is supplementary to $\mathscr{R}$ in $\mathscr{P}$. It happens that $\partial_\Lambda \mathfrak{s} \in \mathscr{R}$. Then, $\partial_\Lambda \mathfrak{s}$—a coboundary of a cochain from the outside of $\mathscr{R}$—is typically a nonbounding cocycle in $\mathscr{R}$.

We refer to these two types of cohomology classes as "singular classes", since some of their coefficients are polynomials on the singular locus of the considered Poisson tensor.

Let us finally briefly comment on the impact of Poisson- and K-exactness on the structure of Poisson cohomology. If tensor Λ, or part of this tensor, is Poisson-exact, see Section 6.2, some elements of space $\wedge^2 \mathfrak{g}_\Lambda$ may be bounding cocycles.

For instance, part Y_{12} of structure Λ_3 of the DHC is Poisson-exact and disappears in the second cohomology space, see Theorem 28. Hence, Poisson-exactness impoverishes Poisson cohomology. In view of the above remark on Casimir functions and the observations made in earlier works, we know that K-exactness significantly enriches the cohomology. Therefore, richness of Poisson cohomology depends in some sense on the distance of the Poisson tensor to Poisson- and K-exactness.

Bibliography

[AP07] Ammar M, Poncin N , *Formal Poisson cohomology of twisted r-matrix induced structures*, Israel J. Math., **165** (2008), 381-411

[ACG08] Ammar M, Chloup V, Gutt S, *universal star products*, Lett. Math. Phys., **84** (2008), 199-215

[AGMP07] Ammar M, Kass G, Masmoudi M, Poncin N, *Strongly r-matrix induced tensors, Koszul cohomology, and arbitrary-dimensional quadratic Poisson cohomology*, ESI and ArXiv preprints

[AMM02] Arnal D, Manchon D, Masmoudi M, *Choix des signes pour la formalité de M. Kontsevich*, Pacific J. Math., **203** (2002), no. 1, 23-66

[Bal96] Balavoine D, *Déformations et rigidité géométrique des algèbres de Leibniz*, Comm. Alg., **24** (1996), 1017-1034

[Bal97] Balavoine D, *Deformations of algebras over a quadratic operad*, Contemp. Math., **202** (1997), 207-234

[BFFLS78] Bayen F, Flato M, Frondsal C, Lichnerowicz A, Sternheimer D, *Deformation theory and quantization*, Ann. Phys., **111** (1978), no. 1, 61-110

[BR02] Bolotnikov V, Rodman R, *Finite-dimensional backward shift invariant subspaces of Arveson spaces*, Lin. Alg. and its Applic., **349** (2002), 265-282

[CE56] Cartan H, Eilenberg S, *Homological Algebra*, Princeton Landmarks Math. (1956), Princeton University Press

[CF01] Cattaneo A S, Felder G, *On the globalization of Kontsevich's star product and the perturbative Poisson sigma model*, Progr. Theoret. Phys. Suppl., **144** (2001), no. 144, 38-53

[CFT02] Cattaneo A S, Felder G, Tommassini L, *From local to global deformation quantization of Poisson manifolds*, Duke Math. J., **115** (2002), no. 2, 359-352

[Cle85] McCleary J, *User's Guide to Spectral Sequences*, Math. Lec. Ser., **12** (1985), Publish or Perish

[Dol05] Dolgushev V, *Covariant and equivariant formality theorems*, Adv. Math., **191** (2005), no. 1, 147-177

[DH91] Dufour J P, Haraki A, *Rotationnels et structures de Poisson quadratiques*, C.R.A.S Paris, **312** (1991), no. 1, 137-140

[DT97] Daletskii L, Takhtajan A L, *Leibniz and Lie algebra structures for Nambu algebra*, Lett. Math. Phys., **39(2)** (1997), no. 2, 127-141

[Fed94] Fedosov B-V, *a simple geometrical construction of deformation quantization*, J. Diff. Geom., **40** (1994), no. 2, 213-238

[FP02] Fialowski A, Penkava M, *Deformation theory of infinity algebras*, J. Algebra, **255** (2002), no. 1, 59-88

[Fil85] Filippov V T, *n-ary Lie algebras*, Sibirskii Math. J., **24** (1985), 126-140

[FLS76] Flato M, Lichnerowicz A, Sternheimer D, *Crochet de Moyal-Vey et quantification*, C. R. Acad. Sci. Paris Sér., A-B **283** (1976), no. 1, Aii, A19-A24

[GT96] El Galiou M, Tihami A, *Star-Product of Quadratic Poisson Structures*, Tokyo J. Math., **19** (1996), no. 2, 475-498

[GK71] Gelfand I, Kazhdan D, *Some problems of differential geometry and the calculation of cohomologies of Lie algebras of vector fields*, Soviet Math. Dokl., **12** (1971), no. 5, 1367-1370

[Ger63] Gerstenhaber M, *The cohomology structure of an associative ring*, Annals of Mathematics, **78** (1963), 267-288

[Ger64] Gerstenhaber M, *On the deformation of rings and algebras*, Annals of Mathematics, **79** (1964), 59-103

[Gin99] Ginzburg V L, *Equivariant Poisson cohomology and a spectral sequence associated with a moment map*, Internat. J. Math., **10** (1999), no. 8, 977-1010

[God52] Godement R, *Théorie des faisceaux*, Publ. Inst. Math. Strasbourg, **XIII** (1952), Hermann

[GM01] Grabowski J, Marmo G, *Jacobi structures revisited*, J. Phys. A: Math. Gen., **34** (2001), no. 49, 10975-10990

[GM03] Grabowski J, Marmo G, *The graded Jacobi algebras and (co)homology*, J. Phys., **36** (2003), no. 1, 161–181

[GMP93] Grabowski J, Marmo G, Perelomov A-M, *Poisson structures: towards a classification*, Modern Phys. Lett. A, **8** (1993), no. 18, 1719-1733

[GR03] Gutt S, Rawnsley J, *Natural star products on symplectic manifolds and quantum moment maps*, Lett. in Math. Phys., **66** (2003), no. 1-2, 123-139

[GR99] Gutt S, Rawnsley J, *Equivalence of star products on a symplectic manifold; an introduction to Deligne's Čech cohomology classes*, Journ. of Geom. Phys., **29** (1999), no. 4, 347-392

[Hoc45] Hochschild G, *On the cohomology groups of an associative algebra*, Annals of Mathematics, **46** (1945), 58-67.

[HKR62] Hochschild G, Kostant B and Rosenberg A, *Differential forms on regular affine algebras*, Trans. Amer. Math. Soc., **02** (1962), 383-408.

[ILLMP01] Ibáñez R, de León M, López B, Marrero J-C, Padrón E, *Duality and modular class of a Nambu-Poisson structure*, J. Phys., **34** (2001), no. 17, 3623-3650

[Kon03] Kontsevich M, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216

[Kra88] Krasil'schchik I S, *Schouten brackets and canonical algebras*, Lecture Notes Math., **1334** (1988), 79-110

[Kra91] Krasil'shchik I S, *Supercanonical algebras and Schouten brackets*, Mat. Zametki, **49**(1) (1991), no. 1, 70-76, Math. Notes, **49** (1991), no. 1-2, 50-54

[Kra99] Krasil'shchik I S, *Lectures on Linear Differential Operators over Commutative Algebras*, Diffiety Institute Preprint Series, **1/99** (1999), http://diffiety.ac.ru

[LS93] Lada T, Stasheff J, *Introduction to SH Lie algebras for physicists*, Internat. J. Theoret. Phys., **32** (1993), no. 7, 1087-1103

[LMS91] Lecomte P.A.B., Michor P W, Schicketanz H, *The multigraded Nijenhuis-Richardson algebra, its universal property and applications*, J. Pure Appl. Algebra, **77** (1992), no. 1, 87-102

[LLMP99] de Léon M, López B, Marrero J-C, Padrón E, *Lichnerowicz-Jacobi cohomology and homology of Jacobi manifolds: modular class and duality*, arXiv:math/9910079

[LLMP03] de Léon M, López B, Marrero J-C, Padrón E, *On the computation of the Lichnerowicz-Jacobi cohomology*, J. Geom. Phys., **44** (2003), no. 4, 507-522p

[Lic77] Lichnerowicz A, *Les variétés de Poisson et leurs algèbres de Lie associees*, J. Diff. Geom. **12** (1977), no. 2, 253-300

[Lic78] Lichnerowicz A, *Les variétés de Jacobi et leurs algèbres de Lie associées*, J. Math. pures et appl., **57** (1978), no. 4, 453-488

[LX92] Liu Z-J, Xu P, *On quadratic Poisson structures*, Lett. Math. Phys., **26** (1992), no. 1, 33-42

[Lod93] Loday J-L, *Une version non commutative des algèbres de Lie: Les algèbres de Leibniz*, Ens. Math., **39** (1993), no. 3-4, 269-293

[Lod95] Loday J-L, *Cup-product for Leibniz cohomology and dual Leibniz algebras*, Math. Scand., **77** (1995), no. 2, 189-196

[MSS02] Markl M, Shnider M, Stasheff J, *Operads in Algebra, Topology and Physics*, AMS Mathematical Surveys and Monographs **96**, 2002

[MMR02] Manchon D, Masmoudi M, Roux A, *On Quantization of Quadratic Poisson Structures*, Comm. in Math. Phys., **225** (2002), no. 1, 121-130

[MP06] Masmoudi M, Poncin N, *On a general approach to the formal cohomology of quadratic Poisson structures*, J. Pure Appl. Algebra, **208** (2007), no. 3, 887-904

[Mon02,1] Monnier P, *Poisson cohomology in dimension two*, Isr. J. Math., **129** (2002), 189-207

[Mon02,2] Monnier P, *Formal Poisson cohomology of quadratic Poisson structures*, Lett. Math. Phys., **59** (2002), no. 3, 253-267

[MV97] Michor P, Vinogradov A-M, *n-ary Lie and associative algebras*, Rend. Sem. Mat. Univ. Politec. Torino, **54** (1996), no. 4, 373-392

[Nak97] Nakanishi N, *Poisson cohomology of plane quadratic Poisson structures*, Publ. Res. Inst. Math. Sci., **33** (1997), no. 1, 73-89

[Nam73] Nambu Y, *Generalized Hamiltonian dynamics*, Phys. Rev., **D7** (1973), 2405-2412

[NR67] Nijenhuis A, Richardson R, *Deformation of Lie algebra structures*, J. Math. Mech., **17** (1967), 89-105

[Pen01] Penkava M, *Infinity Algebras, Cohomology and Cyclic Cohomology, and Infinitesimal Deformations*, arXiv:math/0111088

[Pic05] Pichereau A, *Cohomologie de Poisson en dimension trois*, C. R. Acad. Sci. Paris, Sér. I, **340** (2005), no. 2, 151-154

[RV02] Roger C, Vanhaecke P *Poisson cohomology of the affine plane*, J. Algebra, **251** (2002), no. 1, 448-460

[Rot05] Rotkiewicz M, *Cohomology rings of n-Lie algebras*, Extrac. Math., **20** (2005), no. 3, 219-232

[Roy02] Roytenberg D, *Poisson cohomology of $SU(2)$-covariant "necklace" Poisson structures on S^2*, J. Nonlinear Math. Phys., **9** (2002), no. 3, 347-356

[SS85] Schlessinger M, Stasheff J, *The Lie algebra structure of tangent cohomology and deformation theory*, Journal of Pure and Applied Algebra, **38** (1985), no. 2-3, 313-322

[Sta63] Stasheff J, *On the homotopy associativity of H-spaces I*, Transactions of the AMS, **108** (1963), 275-292

[Sta93] Stasheff J, *The intrinsic bracket on the deformation complex of an associative algebra*, J. Pure Appl. Algebra, **89** (1993), no. 1-2, 231-235

[Tay70] Taylor J L, *A joint spectrum for several commuting operators*, J. Funct. Anal., **6** (1970), 172-191

[Vai73] Vaisman I, *Cohomology and Differential Forms*, Marcel Dekker, Inc., New York (1973)

[Vai90] Vaisman I, *Remarks on the Lichnerowicz-Poisson cohomology*, Ann. Inst. Four., **40** (1990), no. 4, 951-963

[Vai94] Vaisman I, *Lectures on the Geometry of Poisson manifolds*, Progress in Math., **118** (1994), Birkhäuser, Basel

[Vai05] Vaisman I, *Poisson structures on foliated manifolds*, Trav. Math., **XVI** (2005), 139-161

[Vey75] Vey J, *Déformation du crochet de Poisson sur une variété symplectique*, Comment. Math. Helv., **50** (1975), no. 4, 421-454

[VV98] Vinogradov A M, Vinogradov M M, *On multiple generalizations of Lie algebras and Poisson manifolds*, Cont. Math., **219** (1998), 273-287

[VV01] Vinogradov A M, Vinogradov M M, *Graded multiple analogs of Lie algebras*, Diffiety Institute Preprint Series **8**

[Xu92] Xu P, *Poisson cohomology of regular Poisson manifolds*, Ann. Inst. Fourier, **42** (1992), no. 4, 967-988

[Ye06] Yekutieli A, *Continuous and twisted L_∞ morphisms*, J. Pure Appl. Algebra, **207** (2006), no. 3, 575-606