Symbolic dynamics for the geodesic flow on locally symmetric good orbifolds of rank one

Dissertation

zur Erlangung des Doktorgrades
der Fakultät für Elektrotechnik, Informatik und Mathematik
der Universität Paderborn
vorgelegt von

Anke D. Pohl

Januar 2009

Betreuer und Gutachter:	Prof. Dr. Joachim Hilgert, Universität Paderborn
Gutachter:	Prof. Dr. Manfred Einsiedler, The Ohio State University, Eidgenössische Technische Hochschule Zürich
	Prof. Dr. Dieter Mayer, Technische Universität Clausthal

Tag der mündlichen Prüfung: 19. März 2009

Abstract

We present a geometric construction of cross sections for the geodesic flow on a huge class of good orbifolds with the hyperbolic plane as covering manifold. Further we realize first steps towards a generalization to other locally symmetric good orbifolds of rank one.

In the first part of this thesis we consider an arbitrary rank one Riemannian symmetric space D of noncompact type and a group Γ of isometries of D. Under weak requirements on Γ we prove the existence of isometric fundamental regions.

In the second part we specialize to orbifolds of the form $\Gamma\backslash H$ where H is the hyperbolic plane and Γ is a geometrically finite subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Further we require that ∞ is a cuspidal point of Γ and that Γ satisfies an additional weak (and easy to check) condition concerning the structure of the set of isometric spheres of Γ. We construct cross sections for the geodesic flow on $\Gamma\backslash H$ for which the associated discrete dynamical systems are conjugate to discrete dynamical systems on the finite part $\mathbb{R}$ of the geodesic boundary of H. The isometric fundamental regions from the first part play a crucial rôle in this construction. The boundary discrete dynamical systems are of continued fraction type. In turn, the transfer operators produced from them have a particularly simple structure.

For each of these cross sections there is a natural labeling in terms of certain elements of Γ. The arising coding sequences of unit tangent vectors belonging to the cross section can be reconstructed from the endpoints of the associated geodesics. In some situations, the arising symbolic dynamics has a generating function for its future part. In this case, the generating function is also of continued fraction type.

Zusammenfassung

Wir stellen eine geometrische Konstruktion von Poincaré-Schnitten für den geodätischen Fluss auf einer großen Klasse von guten Orbifolds, deren Überdeckungsmannigfaltigkeit die hyperbolische Ebene ist, vor. Des Weiteren führen wir erste Schritte bzgl. einer Verallgemeinerung dieser Konstruktion auf andere lokalsymmetrische gute Orbifolds vom Rang 1 durch.

Im ersten Teil dieser Dissertation betrachten wir einen beliebigen riemannschen symmetrischen Raum D nichtkompakten Typs vom Rang 1 und eine Gruppe Γ von Isometrien auf D. Wir beweisen die Existenz isometrischer Fundamentalbereiche für Γ in D unter der Voraussetzung, dass Γ einige schwache Bedingungen erfüllt.

Im zweiten Teil beschränken wir uns auf die Betrachtung von Orbifolds der Form $\Gamma\backslash H$, wobei H die hyperbolische Ebene und Γ eine geometrisch-endliche Untergruppe von $\mathrm{PSL}(2,\mathbb{R})$ ist. Weiterhin fordern wir, dass ∞ ein Spitzenpunkt von Γ ist und dass Γ eine schwache (und einfach zu überprüfende) Bedingung erfüllt, die die Struktur der Menge der isometrischen Sphären von Γ betrifft. Wir konstruieren Poincaré-Schnitte für den geodätischen Fluss auf $\Gamma\backslash H$, deren assoziierte diskrete dynamische Systeme zu diskreten dynamischen Systemen auf dem endlichen Anteil $\mathbb{R}$ des geodätischen Randes von H konjugiert sind. Die isometrischen Fundamentalbereiche aus dem ersten Teil haben eine tragende Rolle in dieser Konstruktion. Die diskreten dynamischen Systeme auf dem Rand sind verallgemeinerte Kettenbruchabbildungen. Daher haben die zu ihnen gebildeten Transferoperatoren eine besonders einfache Struktur.

Jeder dieser Poincaré-Schnitte läßt eine natürliche Markierung mit Hilfe gewisser Elemente aus Γ zu. Die auftretenden Kodierungssequenzen der Einheitstangentialvektoren im Poincaré-Schnitt können aus den Endpunkten der zugeordneten Geodäten zurückgewonnen werden. In manchen Situationen besitzt die so konstruierte symbolische Dynamik eine erzeugende Funktion für ihren Zukunftsteil. In diesem Fall ist die erzeugende Funktion ebenfalls eine verallgemeinerte Kettenbruchabbildung.

Acknowledgements

First and foremost my thanks go to my advisor Joachim Hilgert for many things: for the continuous support and advice in many aspects, for the freedom to follow my ideas and for the financial support of many summer school, conference, and research visits at various places. Moreover, I thank him for establishing the IRTG, which provided a friendly, inspiring and international environment. Concerning this environment, I would like to thank the many friends and colleagues in Paderborn, Metz and other places who contributed to it. In particular, I thank Dieter Mayer for his abiding interest in my work and a pleasant stay in Clausthal. Finally, my thanks go to Manfred Einsiedler for his offer to be a referee for this thesis.

Contents

Introduction

Symbolic dynamics is the subfield of dynamical systems which is concerned with the construction and investigation of discretizations in space and time and symbolic representations of flows on locally symmetric spaces, or more generally, on orbifolds, and conversely, with finding a geometric interpretation of a given discrete dynamical system. The idea of symbolic dynamics goes back to work of Hadamard [Had98] in 1898. In the following fourty years, this idea was developed by Artin [Art24], Martin [Mar34], Myrberg [Myr31], Nielsen [Nie27], Robbins [Rob37], Morse and Hedlund [MH38] (among others). Since then symbolic dynamics evolved into a rapidly growing field with manifold influence and applications to other fields in mathematics as well as to physics, computer science and engineering.

A relatively recent relation between classical and quantum physics is provided by the combination of the work of Series [Ser85], Mayer [May91], and Lewis and Zagier [LZ01], which we will describe in the following. Suppose that H denotes the hyperbolic plane and consider the geodesic flow on the modular surface $\mathrm{PSL}(2,\mathbb{Z})\backslash H$. Series [Ser85] geometrically constructed an amazingly simple cross section[1] for this flow. Its associated discrete dynamical system is naturally related to a symbolic dynamics on $\mathbb{R}$. The Gauß map is a generating function for the future part of this symbolic dynamics. In [May91], Mayer investigated the transfer operator[2] $\mathcal{L}_\beta$ with parameter β of the Gauß map. His work and that of Lewis and Zagier [LZ01] have shown that there is an isomorphism between the space of Maass cusp forms for $\mathrm{PSL}(2,\mathbb{Z})$ with eigenvalue $\beta(1-\beta)$ and the space of real-analytic eigenfunctions of $\mathcal{L}_\beta$ that have eigenvalue ± 1 and satisfy certain growth conditions. A major step in the proof of this isomorphism is to show that these eigenfunctions of $\mathcal{L}_\beta$ satisfy the Lewis equation

$$f(x) = f(x+1) + (x+1)^{-2\beta} f\big(\tfrac{x}{x+1}\big),$$

more precisely, that they are period functions. Then Lewis and Zagier establish an (explicit) isomorphism between the space of Maass cusp forms and the space of period functions. In the language of quantum physics, Maass cusp forms are eigenstates of the Schrödinger operator for a free particle moving on the modular surface.

To date, a complete generalization of the Lewis-Zagier isomorphism could only be achieved for finite index subgroups of the modular group (see [DH07]). Chang provides symbolic dynamics for these groups and discusses the arising transfer operators in his dissertation [Cha04] (see also [CM01]). The symbolic dynamics

[1] The concepts from symbolic dynamics are explained in Sec. 5.

[2] The notion of transfer operator is introduced in Sec. 7.

for such a subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ is given as a covering of the symbolic dynamics for $\mathrm{PSL}(2,\mathbb{Z})$. It seems reasonable to expect that there is an isomorphism, as in the case of the modular group, between certain spaces of eigenfunctions of the transfer operator in [Cha04] and certain spaces of solutions of the functional equation used in [DH07].

A striking feature of the cross section in these examples is that the arising discrete dynamical system not only has a very simple structure but is also naturally conjugate to a discrete dynamical system of continued fraction type on the finite part $\mathbb{R}$ of the geodesic boundary of H. In other words, the discrete dynamical system on the boundary is locally given by the action of certain elements of $\mathrm{PSL}(2,\mathbb{Z})$ resp. the finite index subgroup. The existing examples suggest that, in order to achieve an extension of this kind of relation between the geodesic flow on the orbifold $\Gamma\backslash H$ and Maass cusp forms for Γ for a wider class of subgroups Γ of $\mathrm{PSL}(2,\mathbb{R})$, a cross section for the geodesic flow has to be constructed in a geometric way, and its associated discrete dynamical system has to be conjugate to a discrete dynamical system of continued fraction type on parts of the boundary of H.

It turns out that both existing methods for the construction of cross sections and symbolic dynamics for the geodesic flow on the orbifold $\Gamma\backslash H$ are not well adapted for this task. The *geometric coding* consists in choosing a fundamental domain for Γ in H with side pairing and taking the sequences of sides cut by a geodesic as coding sequences. The cross section is a set of unit tangent vectors based at the boundary of the fundamental domain, more precisely, based at the image of this boundary under the canonical projection $\pi\colon H \to \Gamma\backslash H$. In general, it is very difficult, if not impossible, to find a conjugate dynamical system on the geodesic boundary of H. In contrast, the *arithmetic coding* starts with a discrete dynamical system or symbolic dynamics related to Γ on parts of the boundary of H and asks for a cross section of the geodesic flow on $\Gamma\backslash H$ that reproduces this system. Usually, writing down such a cross section is a non-trivial task. For an arbitrary discrete dynamical system (even if of continued fraction type), the symbolic dynamics does not reflect the geometry of the geodesic flow. Moreover, arithmetic coding is a group-by-group analysis and not a uniform method. A good overview of geometric and arithmetic coding is the survey article [KU07].

In this thesis we develop a method for the construction of cross sections which do satisfy the demands mentioned above. This method can be applied to a large class of subgroups Γ of $\mathrm{PSL}(2,\mathbb{R})$ acting on H. More precisely, Γ has to be a geometrically finite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of which ∞ is a cuspidal point and which satisfies an additional condition concerning the structure of the set of isometric spheres of Γ. The cusps of Γ, in particular the cusp $\pi(\infty)$, play a particular rôle, for which reason we call our method “cusp expansion”.

The starting point of this method is the set of isometric spheres of Γ, more precisely, a subset of “relevant” isometric spheres. Once one knows the relevant isometric spheres of Γ, each step in the construction is constructive and consists of a finite number of elementary operations. The cross section has a natural labeling by the elements of a certain finite set L of Γ. The discrete dynamical system associated to the cross section is conjugate to a discrete dynamical system on a subset of $\mathbb{R} \times \mathbb{R}$. The boundary discrete dynamical system is locally given by the action of the elements of L, and hence directly related to the symbolic dynamics arising from the natural labeling. In turn, the coding sequence of a geodesic on $\Gamma\backslash H$ (more precisely, of a unit tangent vector in $\Gamma\backslash SH$) can be reconstructed from the endpoints of a corresponding geodesic on H without reconstructing this geodesic. Further, the transfer operator of the boundary discrete dynamical system is of a particularly simple structure: it is a finite sum of a certain action of the elements of L. The method is uniform for all admissible groups Γ. Some steps in the construction involve choices. To some extent these choices allow to control properties of the symbolic dynamics and the transfer operator.

If Γ is the modular group $\mathrm{PSL}(2, \mathbb{Z})$, then the arising transfer operator, more precisely its future part, is the two-term operator

$$\mathcal{L}_\beta f(x) = f(x+1) + (x+1)^{-2\beta} f\left(\tfrac{x}{x+1}\right)$$

and therefore the functional equation

$$f(x) = \mathcal{L}_\beta f(x)$$

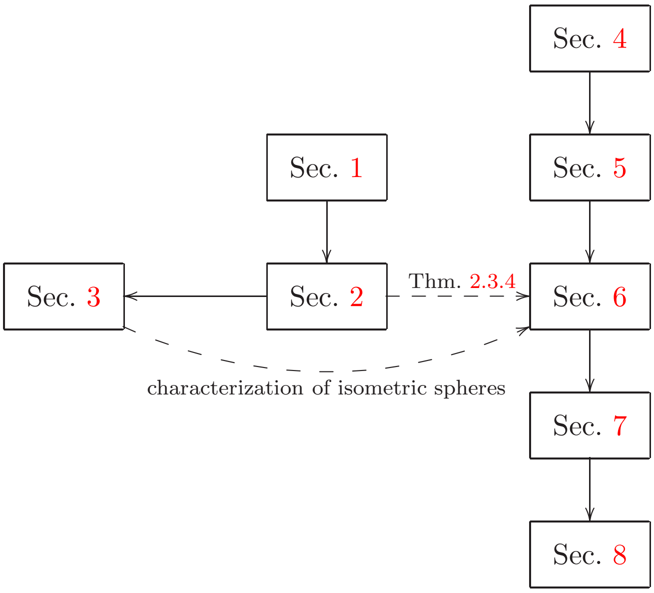
is exactly the one used by Lewis and Zagier in their proof of the isomorphism between the space of Maass cusp forms for $\mathrm{PSL}(2, \mathbb{Z})$ and the space of period functions. A more detailed description of cusp expansion is given in the introduction to Part II, the case of the modular group is worked out in Sec. 8. The method of cusp expansion for a simple class of sample groups is published in [HP08].

It is desirable to generalize the method of cusp expansion to other classes of locally symmetric good orbifolds of rank one and noncompact type. These are orbifolds of the form $\Gamma\backslash D$, where D is a rank one Riemannian symmetric space of noncompact type and Γ is a subgroup of $\mathrm{Isom}(D)$. We will realize the first steps towards this generalization. In the series [CDKR91], [CDKR98], [KR05], [KR], Cowling, Dooley, Korányi and Ricci resp. Korányi and Ricci provide a classification-free construction of all rank one Riemannian symmetric spaces of noncompact type. Using their work we will provide a uniform definition of the notion of isometric spheres, which essentially subsumes the existing definitions in literature for real, complex and quaternionic hyperbolic spaces and extends it to the Cayley plane. Moreover, we prove the existence of so-called isometric fundamental regions for subgroups Γ of $\mathrm{Isom}(D)$ which satisfy some (weak) conditions.

This thesis is structured as follows: Part I is concerned with the existence of isometric fundamental regions for certain subgroups Γ of $\mathrm{Isom}(D)$, where D is a rank one Riemannian symmetric space of noncompact type. As already mentioned, we will not make use of the classification of these spaces. Therefore, in Sec. 1, we summarize the classification-free constructions from [CDKR91], [CDKR98], and [KR05], [KR]. The proof of the existence of isometric spheres as well as the uniform definitions of isometric spheres and Cygan metric is the content of Sec. 2. Sec. 3 discusses the relation to existing definitions and results in literature. In Part II we specialize to D being the two-dimensional real hyperbolic space H and the geodesic flow on good orbifolds $\Gamma\backslash H$ for a wide class of subgroups Γ of the group of orientation-preserving isometries of H. In Sec. 5 we introduce the necessary notions and concepts from symbolic dynamics. The cusp expansion method for the construction of cross sections and symbolic dynamics is carefully expounded in Sec. 6. Sec. 7 briefly treats the transfer operators associated to the arising symbolic dynamics. Finally, in Sec. 8 we review the cross section constructed by Series and show how it is related to our construction.

A guide for the reader

This document need not be read in strict linear order. Its most important section is Sec. 6 in Part II. If one is willing to accept Thm. 2.3.4 and the characterization of isometric spheres in Prop. 3.3.4 and Lemma 3.5.2, one can read Part II independently from Part I. Moreover, Sec. 3.2 mainly serves the purpose to compare Thm. 2.3.4 to already existing results in literature. Only the statements of Prop. 3.3.4 and Lemma 3.5.2 are needed in Part II. To simplify non-linear reading, there are comprehensive lists of notations and terminology at the end of this document.



Part I.

Isometric fundamental regions

A Riemannian manifold D is called *homogeneous* if its group $\mathrm{Isom}(D)$ of (Riemannian) isometries acts transitively on D. A *Riemannian symmetric space* is a connected, homogeneous Riemannian manifold which has an involutive isometry with at least one isolated fixed point. In the following we give some background on Riemannian symmetric spaces and their connection to Lie groups. The material is taken from [Hel01] and [Ebe96].

Each Riemannian symmetric space D decomposes uniquely into a direct product of Riemannian symmetric spaces (the *de Rham decomposition*)

$$D = \mathbb{R}^n \times D_{1,1} \times \cdots \times D_{1,k} \times D_{2,1} \times \cdots \times D_{2,l}$$

where $n \in \mathbb{N}_0$ is maximal, the spaces $D_{i,j}$ are irreducible (that is, they are not themselves non-trivial Riemannian direct products), each $D_{1,j}$ has nonnegative sectional curvature and each $D_{2,j}$ has nonpositive sectional curvature. The component $\mathbb{R}^n$ is called the *flat factor* of D, the spaces $D_{1,j}$ are the *noncompact factors* and the spaces $D_{2,j}$ are the *compact factors* of D. The space D is said to be of *noncompact type* if its de Rham decomposition only contains noncompact factors.

The *rank* of the Riemannian symmetric space D is the maximal number $r \in \mathbb{N}$ such that D contains an r-dimensional connected, totally geodesic, flat submanifold. We will be interested in rank one Riemannian symmetric spaces of noncompact type.

Riemannian symmetric spaces can be identified with quotients of connected Lie groups by certain closed subgroups. This we explain in the following.

Suppose that D is a Riemannian symmetric space and endow $G := \mathrm{Isom}(D)$ with the compact-open topology. Then there is a unique differential structure on G compatible with the given topology such that G becomes a Lie group. Let p be any point in D. If G_0 denotes the connected component of the identity element in G and if K is the subgroup of G_0 which stabilizes p, then K is a compact subgroup of G_0 and G_0/K is isometric to D.

Conversely suppose that G_0 is a connected Lie group and that K is a closed subgroup of G_0 such that (G_0, K) is a Riemannian symmetric pair, i. e., the group $\mathrm{Ad}_{G_0}(K)$ is compact and there exists an involutive smooth automorphism σ of G_0 such that $\big(K_\sigma\big)_0 \subseteq K \subseteq K_\sigma$, where K_σ is the set of fixed points of σ in G_0. Then there is a Riemannian metric on G_0/K such that G_0/K is a Riemannian symmetric space.

É. Cartan, in his magnificent work on Lie groups and symmetric spaces, obtained a complete classification of all Riemannian symmetric spaces by listing all Riemannian symmetric pairs. According to Cartan's classification, there are only four types of rank one Riemannian symmetric spaces of noncompact type, namely the hyperbolic spaces over the real numbers, over the complex numbers, and over the quaternions, and one exceptional space, often considered to be the two-dimensional hyperbolic space over the Cayley numbers. The classical geometric construction of these spaces is based on the classification. The first three types are easily constructed and handled together (see, e. g., [CG74]). The classical (explicit) construction of the exceptional space is much more involved (see, e. g., [Mos73]) due to the non-associativity of the Cayley numbers.

Despite the fact that these spaces are well known and widely studied, a classification-free construction has been provided only recently by Cowling, Dooley, Korányi and Ricci resp. Korányi and Ricci in [CDKR91] and [CDKR98] resp. in [KR05] and [KR]. Both constructions are amazingly easy to work with. The approach in [CDKR91] and [CDKR98] constructs the spaces from so-called H-type algebras with J^2-condition. The H-type algebras are intimately connected with the restricted root space decomposition of the Lie algebra of the isometry group of the constructed symmetric space. The construction in [KR05] and [KR] reflects the geometric side of the spaces. Its basic objects are so-called J^2C-module structures. However, there is a bijection between H-type algebras with J^2-condition and J^2C-module structures, which means that one can translate effortless insights and advantages from one construction to the other.

In Sec. 1 we present these two constructions of rank one Riemannian symmetric spaces of noncompact type, show how to change between them and discuss their isometry groups. The material of this section is mainly contained in the articles [CDKR91], [CDKR98], [KR05] and [KR]. In Sec. 2 we prove, in a uniform way, the existence of isometric fundamental regions for groups of isometries which satisfy some weak properties. The component "isometric" in the naming of isometric fundamental regions derives from one of its building blocks, namely the common exterior of the so-called isometric spheres of Γ. In our definition, an isometric sphere is a certain sphere w.r.t. Cygan metric.

For real, complex and quaternionic hyperbolic spaces, there already exist several (different) definitions of isometric spheres in the literature. Moreover, for certain subgroups of the isometry group of real and complex hyperbolic spaces, the existence of isometric fundamental regions is already known. In Sec. 3 we will investigate which definitions of isometric spheres are subsumed by our uniform one, and we show that the known isometric fundamental regions are special cases of Theorem 2.3.4. The name "isometric sphere" (more precisely, "isometric circle") goes back to a suggestion of Whittaker (see [For72, Preface]). It is motivated by the fact that the isometric spheres of an isometry g of real hyperbolic space is the set on which g acts as a Euclidean isometry. The first investigation of the existence of isometric fundamental regions goes back to Ford in [For72]. Therefore, isometric fundamental regions are sometimes called Ford fundamental regions in the literature.

Throughout we will use the following notation. If T is a topological space and U a subset of T, then the closure of U is denoted by $\overline{U}$ or $\mathrm{cl}(U)$ and its boundary is denoted by ∂U. Moreover, we write U° for the interior of U. The complement of U in T is denoted by $\complement U$ or $T \smallsetminus U$.

For two arbitrary sets A and B, the complement of B in A is written as $A \smallsetminus B$. If $\sim$ is an equivalence relation on A, then $A/_\sim$ denotes the set of equivalence classes. Likewise, if Γ a group acting on A, then we write A/Γ for the space of right cosets.

1. Classification-free constructions

In this section we briefly expound the two classification-free constructions from [CDKR91] and [CDKR98] resp. [KR05] and [KR] of rank one Riemannian symmetric spaces of noncompact type.

In Sec. 1.1 we introduce the notion of H-type algebras, the J^2-condition and the definition of isomorphisms between H-type algebras. H-type algebras are special Euclidean Lie algebras, for which reason we discuss the relation between isomorphisms of H-type algebras and isomorphisms of H-type algebras considered only as Euclidean Lie algebras. It will turn out that these notions only differ for so-called degenerate H-type algebras. Sec. 1.2 is devoted to the notion of C-module structures, the J^2-condition in this context and the definition of isomorphisms between C-module structures. In Sec. 1.3 we establish the isomorphism between the categories of H-type algebras and C-module structures. The classification-free construction of rank one Riemannian symmetric spaces of noncompact type is the content of Sec. 1.4. From an H-type algebra with J^2-condition we construct the model D, which is essentially the Siegel domain model. The most natural model arising from a J^2C-module structure is the ball model B. Using the bijection between H-type algebras with J^2-condition and J^2C-module structures from Sec. 1.3, we show that the (generalized) Cayley transform provides an isometry between these models. Sec. 1.6 is spend on the discussion of the isometry group and its Bruhat decomposition. Our main focus lies on explicit formulas for the action of group elements on the symmetric space and its geodesic boundary in the language of H-type algebras and in that of J^2C-module structures. To that end we introduce in Sec. 1.5 a map $\beta_2\colon V \times V \to C$ for a J^2C-module structure (C, V, J), which will be seen to be closely related to the inner product and the Lie bracket on V.

All omitted proofs can be found in [CDKR91] or [CDKR98] for statements in the language of H-type algebras, and in [KR05] or [KR] for those in the language of J^2C-modules. As long as no confusion can arise, each inner product is denoted by $\langle\cdot,\cdot\rangle$ and its associated norm by $|\cdot|$.

1.1. H-type algebras and the J^2-condition

A vector space is called *Euclidean* if it is a finite-dimensional real vector space endowed with an inner product. A Lie algebra is called *Euclidean* if, in addition to being a Lie algebra, it is a Euclidean vector space. For a vector space $\mathfrak{v}$ let $\mathrm{End}_{\mathrm{vs}}(\mathfrak{v})$ denote the group and vector space of endomorphisms of $\mathfrak{v}$. If $\mathfrak{v}$ carries additional structures, then the elements of $\mathrm{End}_{\mathrm{vs}}(\mathfrak{v})$ are not required to be compatible with these structures. In particular, if $\mathfrak{v}$ is Euclidean, then $\varphi \in \mathrm{End}_{\mathrm{vs}}(\mathfrak{v})$ need not be orthogonal.

Definition 1.1.1. Let $\mathfrak{n}$ be a Euclidean Lie algebra. Then $\mathfrak{n}$ is said to be an *H-type algebra* if

(H1) there are two subvector spaces $\mathfrak{v}$, $\mathfrak{z}$ of $\mathfrak{n}$ (each of which may be trivial) such that

$$[\mathfrak{n}, \mathfrak{z}] = \{0\}, \quad [\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z},$$

and $\mathfrak{n}$ is the orthogonal direct sum of $\mathfrak{z}$ and $\mathfrak{v}$,

(H2) for all $X \in \mathfrak{v}$, all $Z \in \mathfrak{z}$ we have

$$|J(Z)X| = |Z| \cdot |X|$$

where $J\colon \mathfrak{z} \to \mathrm{End}_{\mathrm{vs}}(\mathfrak{v})$ is the $\mathbb{R}$-linear map defined by

$$\langle J(Z)X, Y\rangle = \langle Z, [X, Y]\rangle$$

for all $X, Y \in \mathfrak{v}$, all $Z \in \mathfrak{z}$. The map J is well-defined and unique by Riesz' Representation Theorem (or its finite-dimensional counterpart).

Alternatively to (H1), one can require that the Euclidean Lie algebra $\mathfrak{n}$ has an orthogonal decomposition into subvector spaces $\mathfrak{v}$, $\mathfrak{z}$ such that $[\mathfrak{v}, \mathfrak{v}] \subseteq \mathfrak{z}$ and $[\mathfrak{v}, \mathfrak{z}] = [\mathfrak{z}, \mathfrak{z}] = \{0\}$. If (H1) holds, then $\mathfrak{n}$ is either abelian or two-step nilpotent. In the first case we call $\mathfrak{n}$ *degenerate*, in the second *non-degenerate*.

The construction of a symmetric space from an H-type algebra $\mathfrak{n}$ (with J^2-condition) depends on the choice of $\mathfrak{z}$ and $\mathfrak{v}$ in the decomposition $\mathfrak{z} \oplus \mathfrak{v}$ of $\mathfrak{n}$. We call $(\mathfrak{z}, \mathfrak{v})$ an *ordered decomposition* of $\mathfrak{n}$. The following lemma shows that the ordered decomposition is unique unless $\mathfrak{n}$ is non-trivial and abelian (which precisely is the reason for calling abelian H-type algebras degenerate). It will turn out that both possible ordered decompositions of a non-degenerate abelian H-type algebra give rise to the same symmetric space, but in different models. Nevertheless, this non-uniqueness calls for a careful notion of isomorphisms of H-type algebras, which we will discuss after the lemma.

We denote the center of a Lie algebra $\mathfrak{g}$ by $Z(\mathfrak{g})$.

Lemma 1.1.2. *Let $\mathfrak{n}$ be an H-type algebra. If $\mathfrak{n}$ is non-abelian or $\mathfrak{n} = \{0\}$, then the ordered decomposition $(\mathfrak{z}, \mathfrak{v})$ of $\mathfrak{n}$ is unique. In this case, we have $\mathfrak{z} = Z(\mathfrak{n})$ and $\mathfrak{v} = Z(\mathfrak{n})^{\perp}$. If $\mathfrak{n}$ is abelian and $\mathfrak{n} \neq \{0\}$, then there are two ordered decompositions of $\mathfrak{n}$, namely $(\mathfrak{z}, \mathfrak{v}) = (Z(\mathfrak{n}), \{0\}) = (\mathfrak{n}, \{0\})$ and $(\mathfrak{z}, \mathfrak{v}) = (\{0\}, \mathfrak{n})$.*

Proof. Suppose that $(\mathfrak{z}, \mathfrak{v})$ is an ordered decomposition of $\mathfrak{n}$. Then $\mathfrak{v}$ is uniquely determined by $\mathfrak{z}$, namely $\mathfrak{v} = \mathfrak{z}^{\perp}$. Since $[\mathfrak{z}, \mathfrak{z}] = \{0\}$, we know that $\mathfrak{z}$ is a subvector space of $Z(\mathfrak{n})$ (even a subalgebra). If $\mathfrak{z} = \{0\}$, then $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z} = \{0\}$. In this case, $\mathfrak{n}$ is abelian and $(\mathfrak{z}, \mathfrak{v}) = (\{0\}, \mathfrak{n})$.

Suppose now that $\mathfrak{z} \neq \{0\}$. We have to prove that $\mathfrak{z} = Z(\mathfrak{n})$. For contradiction assume that $\mathfrak{z} \neq Z(\mathfrak{n})$, hence that $\dim \mathfrak{z} < \dim Z(\mathfrak{n})$. Then there is a non-trivial element $X \in \mathfrak{v} \cap Z(\mathfrak{n})$. Fix some $Z \in \mathfrak{z}$, $Z \neq 0$. For all $Y \in \mathfrak{v}$ it follows that

$$\langle J(Z)X, Y\rangle = \langle Z, [X,Y]\rangle = 0,$$

thus $J(Z)X = 0$. But then

$$|J(Z)X| = 0 \neq |Z| \cdot |X|,$$

which is a contradiction to (H2). Therefore, $\mathfrak{z} = Z(\mathfrak{n})$.

This shows that for non-abelian $\mathfrak{n}$ or for $\mathfrak{n} = \{0\}$, the pair $(\mathfrak{z}, \mathfrak{v}) = (Z(\mathfrak{n}), Z(\mathfrak{n})^{\perp})$ is the only candidate for an ordered decomposition of $\mathfrak{n}$. Because there is an ordered decomposition of $\mathfrak{n}$ by hypothesis, $(Z(\mathfrak{n}), Z(\mathfrak{n})^{\perp})$ is indeed an ordered decomposition of $\mathfrak{n}$. For abelian $\mathfrak{n}$, $\mathfrak{n} \neq \{0\}$, we have the two candidates $(\{0\}, \mathfrak{n})$ and $(\mathfrak{n}, \{0\})$, which clearly satisfy (H1) and (H2). □

Let $\mathfrak{n}$ be a non-trivial abelian H-type algebra. Then $\mathfrak{n}$ admits the two ordered decompositions $(\mathfrak{n}, \{0\})$ and $(\{0\}, \mathfrak{n})$. The isomorphism $\mathrm{id}_{\mathfrak{n}}$ of $\mathfrak{n}$ as a Euclidean Lie algebra does not respect these decompositions. In Sec. 1.3 we will see that preserving the decompositions is essential for the bijection between H-type algebras and C-module structures. Therefore, from now on, we will always consider an H-type algebra $\mathfrak{n}$ as being equipped with a (fixed) ordered decomposition and denote it by $\mathfrak{n} = (\mathfrak{z}, \mathfrak{v}, J)$ or, briefly, by $(\mathfrak{z}, \mathfrak{v}, J)$. Although the map J is determined by $\mathfrak{z}$ and $\mathfrak{v}$, we keep it in the triple to fix a notation for it.

Definition 1.1.3. Let $\mathfrak{n}_j = (\mathfrak{z}_j, \mathfrak{v}_j, J_j)$, $j = 1, 2$, be H-type algebras. An *isomorphism* from $\mathfrak{n}_1$ to $\mathfrak{n}_2$ is a pair (φ, ψ) of isomorphisms of Euclidean vector spaces $\varphi\colon \mathfrak{z}_1 \to \mathfrak{z}_2$ and $\psi\colon \mathfrak{v}_1 \to \mathfrak{v}_2$ such that the diagram

$$\begin{tikzcd}
\mathfrak{z}_1 \times \mathfrak{v}_1 \arrow[r, "J_1"] \arrow[d, "\varphi\times\psi"'] & \mathfrak{v}_1 \arrow[d, "\psi"] \\
\mathfrak{z}_2 \times \mathfrak{v}_2 \arrow[r, "J_2"] & \mathfrak{v}_2
\end{tikzcd}$$

commutes.

The following lemma shows that isomorphism of H-type algebras is a refined notion of isomorphism of Euclidean Lie algebras.

Lemma 1.1.4. *Let $\mathfrak{n}_j = (\mathfrak{z}_j, \mathfrak{v}_j, J_j)$, $j = 1, 2$, be H-type algebras and suppose that $(\varphi, \psi)\colon \mathfrak{n}_1 \to \mathfrak{n}_2$ is an isomorphism. Then $\varphi \times \psi\colon \mathfrak{z}_1 \oplus \mathfrak{v}_1 \to \mathfrak{z}_2 \oplus \mathfrak{v}_2$ is an isomorphism of Euclidean Lie algebras.*

Proof. For all $Z \in \mathfrak{z}_2$ and all $X, Y \in \mathfrak{v}_1$ we have

$$\begin{aligned}
\big\langle Z, [\psi(X), \psi(Y)]\big\rangle &= \big\langle J_2(Z)(\psi(X)), \psi(Y)\big\rangle = \big\langle \psi\big(J_1(\varphi^{-1}(Z))X\big), \psi(Y)\big\rangle \\
&= \big\langle J_1(\varphi^{-1}(Z))X, Y\big\rangle = \big\langle \varphi^{-1}(Z), [X,Y]\big\rangle \\
&= \big\langle Z, \varphi([X,Y])\big\rangle.
\end{aligned}$$

Since $\langle\cdot,\cdot\rangle|_{\mathfrak{z}_2\times\mathfrak{z}_2}$ is non-degenerate, we have $\varphi([X,Y]) = [\psi(X),\psi(Y)]$. Define $\chi := \varphi\times\psi$ and let $Z_1, Z_2 \in \mathfrak{z}_1$, $X_1, X_2 \in \mathfrak{v}_1$. Then

$$[Z_1 + X_1, Z_2 + X_2] = [X_1, X_2]$$

and

$$\begin{aligned}[\chi(Z_1 + X_1), \chi(Z_2 + X_2)] &= [\varphi(Z_1) + \psi(X_1), \varphi(Z_2) + \psi(X_2)]\\ &= [\psi(X_1), \psi(X_2)] = \varphi([X_1, X_2])\\ &= \chi([Z_1 + X_1, Z_2 + X_2]).\end{aligned}$$

Since χ is clearly an isomorphism of Euclidean vector spaces, this completes the proof. □

Remark 1.1.5. Suppose that $\mathfrak{n}_1, \mathfrak{n}_2$ are non-degenerate H-type algebras and let $\chi\colon \mathfrak{n}_1 \to \mathfrak{n}_2$ be an isomorphism between $\mathfrak{n}_1$ and $\mathfrak{n}_2$ as Euclidean Lie algebras. Then $\chi(Z(\mathfrak{n}_1)) = Z(\mathfrak{n}_2)$. Hence Lemma 1.1.2 implies that χ is an isomorphism of H-type algebras. In turn, Lemma 1.1.4 shows that the isomorphisms between $\mathfrak{n}_1$ and $\mathfrak{n}_2$ as H-type algebras coincide with the isomorphisms between $\mathfrak{n}_1$ and $\mathfrak{n}_2$ as Euclidean Lie algebras.

For an H-type algebra $(\mathfrak{z}, \mathfrak{v}, J)$ we often write $J_Z X$ instead of $J(Z)X$ and we use the notation $J_{\mathfrak{z}}$ for the set $\{J_Z \mid Z \in \mathfrak{z}\}$.

Definition 1.1.6. An H-type algebra $\mathfrak{n} = (\mathfrak{z}, \mathfrak{v}, J)$ is said to satisfy the *J^2-condition* if

$$\forall\, X \in \mathfrak{v}\ \forall\, Z_1, Z_2 \in \mathfrak{z}\colon \big(\langle Z_1, Z_2\rangle = 0 \ \Rightarrow\ \exists\, Z_3 \in \mathfrak{z}\colon J_{Z_1}J_{Z_2}X = J_{Z_3}X\big). \tag{H3}$$

If $\mathfrak{n}$ is abelian, then (H3) is trivially satisfied.

1.2. C-module structures and the J^2-condition

Definition 1.2.1. A *C-module structure* is a triple (C, V, J) consisting of two Euclidean vector spaces C and V and an $\mathbb{R}$-bilinear map $J\colon C\times V \to V$ satisfying the following properties:

(M1) there exists $e \in C\smallsetminus\{0\}$ such that $J(e, v) = v$ for all $v \in V$,

(M2) for all $\zeta \in C$ and all $v \in V$ we have $|J(\zeta, v)| = |\zeta||v|$.

The cases where $V = \{0\}$ or $C = \mathbb{R}e$ are not excluded, we refer to these as *degenerate*. If $V \neq \{0\}$ and $C \neq \mathbb{R}e$, then the C-module structure (C, V, J) is called *non-degenerate*. For brevity, a C-module structure (C, V, J) is sometimes called a C-module structure[1] on V.

[1]The "C" is "C-module structure" or in "C-module structure on V" does not refer to the Euclidean space C in the triple (C, V, J). Hence, if (C', V, J') satisfies (M1) and (M2), then it is still called a C-module structure on V.

Lemma 1.2.2. *Let (C, V, J) be a C-module structure. If $V \neq \{0\}$, then the element e in (M1) is uniquely determined and of unit length.*

Proof. Let $v \in V \setminus \{0\}$ and suppose that $e, e' \in C \setminus \{0\}$ satisfy $J(e, v) = J(e', v)$. Using (M2) we get

$$0 = |J(e, v) - J(e', v)| = |J(e - e', v)| = |e - e'||v|,$$

hence $|e - e'| = 0$ and therefore $e = e'$. Further $|v| = |J(e, v)| = |e||v|$, and so $|e| = 1$. □

Similar to the situation with H-type algebras, the construction of a symmetric space from a C-module structure (satisfying the J^2-condition) depends on the choice of e in (M1) and uses $|e| = 1$. If (C, V, J) is a C-module structure with $V = \{0\}$, then J vanishes everywhere. Thus every element $a \in C \setminus \{0\}$ satisfies $J(a, \cdot) = \mathrm{id}_V$. In this case, we endow C with a distinguished vector e of unit length and fix it (sometimes) in the notation as (C, e, V, J). The influence of the particular choice of e on the constructed symmetric space is much weaker than that of the different ordered decompositions of degenerate H-type algebras. In fact, the choice of e determines the orthogonal decomposition $C = \mathbb{R}e \oplus C'$ (see below). If e_1, e_2 are two choices for e, then there is an isomorphism between $\mathbb{R}e_1 \oplus C'_1$ and $\mathbb{R}e_2 \oplus C'_2$ as Euclidean vector spaces which respects the decompositions. In turn, (C, e_1, V, J) and (C, e_2, V, J) are isomorphic as C-module structures (see below for the definition of isomorphism).

If (C, V, J) is a non-degenerate C-module structure, then the element e in (M1) is unique by Lemma 1.2.2. For reasons of uniformity, also in this case, we will often use the notation (C, e, V, J) for (C, V, J).

If (C, V, J) is a C-module structure, then we will use $J_\zeta v$ or ζv to abbreviate $J(\zeta, v)$. Further, we set $Cv := \{\zeta v \mid \zeta \in C\}$ for $v \in V$.

Definition 1.2.3. A C-module structure (C, V, J) is said to satisfy the *J^2-condition* if

$$C(Cv) = Cv \quad \text{for all } v \in V. \tag{M3}$$

In this case, V is called[2] a *J^2C-module* and (C, V, J) a *J^2C-module structure.*

Definition 1.2.4. Let (C_1, e_1, V_1, J_1) and (C_2, e_2, V_2, J_2) be C-module structures. An *isomorphism*[3] from (C_1, e_1, V_1, J_1) to (C_2, e_2, V_2, J_2) is a pair (φ, ψ) of isomorphisms of Euclidean vector spaces $\psi \colon V_1 \to V_2$ and $\varphi \colon C_1 \to C_2$ with $\varphi(e_1) = e_2$ such that the diagram

$$\begin{tikzcd} C_1 \times V_1 \arrow[r, "J_1"] \arrow[d, "\varphi\times\psi"'] & V_1 \arrow[d, "\psi"] \\ C_2 \times V_2 \arrow[r, "J_2"] & V_2 \end{tikzcd}$$

[2] As with the "C" in "C-module structure", the "J^2" in "J^2-condition" and "J^2C-module" does not refer to the map J.

[3] In [KR05] and [KR] the condition $\varphi(e_1) = e_2$ is omitted from the definition. However, as discussed, this condition is needed and indeed this stronger notion of isomorphism is used in their work.

commutes.

The following remark shows that the requirement that $\varphi(e_1) = e_2$ in Def. 1.2.4 is relevant only if one of the C-module structures is degenerate.

Remark 1.2.5. If (C_1, e_1, V_1, J_1) and (C_2, e_2, V_2, J_2) are non-degenerate C-module structures and (φ, ψ) is a pair of isomorphisms of Euclidean vector spaces $\psi\colon V_1 \to V_2$ and $\varphi\colon C_1 \to C_2$ such that $J_2 \circ (\varphi \times \psi) = \psi \circ J_1$, then

$$J_2(\varphi(e_1), v) = \psi\big(J_1(e_1, \psi^{-1}(v))\big) = \psi(\psi^{-1}(v)) = v$$

for each $v \in V$. The uniqueness of e_2 then shows that $\varphi(e_1) = e_2$.

For a C-module structure (C, e, V, J) let $C' := e^\perp$ denote the orthogonal complement of $\mathbb{R}e$ in C. For $\zeta = ae + z \in C$ with $a \in \mathbb{R}$ and $z \in C'$ we set $\operatorname{Re}\zeta := a$, the *real part* of ζ, and $\operatorname{Im}\zeta := z$, the *imaginary part*[4] of ζ, and $\overline{\zeta} := ae - z$, the *conjugate* of ζ. Further we use the identification $\mathbb{R}e \to \mathbb{R}$, $ae \mapsto a$ of Euclidean vector spaces.

1.3. Bijection between H-type algebras and C-module structures

Let $\mathfrak{n} = (\mathfrak{z}, \mathfrak{v}, J)$ be an H-type algebra. Endow $\mathbb{R}$ with the standard inner product and consider the Euclidean direct sum $\mathfrak{c} := \mathbb{R} \oplus \mathfrak{z}$. The map $\widetilde{J}\colon \mathfrak{c} \times \mathfrak{v} \to \mathfrak{v}$, defined by

$$\widetilde{J}(t + Z, X) := tX + J_Z X$$

for all $t + Z \in \mathbb{R} \oplus \mathfrak{z}$, $X \in \mathfrak{v}$, is $\mathbb{R}$-bilinear. Since $\langle J_Z X, X\rangle = \langle Z, [X, X]\rangle = 0$, we further find

$$\begin{aligned} |\widetilde{J}(t + Z, X)|^2 &= |tX + J_Z X|^2 = t^2|X|^2 + |J_Z X|^2 \\ &= t^2|X|^2 + |Z|^2|X|^2 = (t^2 + |Z|^2)|X|^2 \\ &= |t + Z|^2|X|^2, \end{aligned}$$

hence $|\widetilde{J}(t + Z, X)| = |t + Z||X|$. Moreover, for each $X \in \mathfrak{v}$ we have

$$\widetilde{J}(1, X) = X.$$

Therefore, $(\mathfrak{c}, \mathfrak{v}, \widetilde{J})$ is a C-module structure with $e = 1$.

The condition (H3) is easily seen to be equivalent to

$$\widetilde{J}(\mathfrak{c})\widetilde{J}(\mathfrak{c})X = \widetilde{J}(\mathfrak{c})X \quad \text{for all } X \in \mathfrak{v}. \tag{H3'}$$

Thus, $(\mathfrak{c}, \mathfrak{v}, \widetilde{J})$ satisfies the J^2-condition if and only if $\mathfrak{n}$ does. This construction provides an assignment of a C-module structure to each H-type algebra (with fixed ordered decomposition).

[4] Note that, in contrast to the usual definition in complex analysis, if $C = \mathbb{C}$ and $\zeta = a + ib \in \mathbb{C}$, then one has here $\operatorname{Im}\zeta = ib$.

Vice versa, let (C, e, V, J) be a C-module structure and let $[\cdot,\cdot]\colon V \times V \to C'$ be the map defined by

$$\langle z, [x,y]\rangle = \langle J(z,x), y\rangle \tag{1.1}$$

for all $z \in \mathbb{C}'$, all $x, y \in V$. Riesz' Representation Theorem (or its finite-dimensional counterpart) shows that $[\cdot,\cdot]$ is well-defined. We extend $[\cdot,\cdot]$ to the Euclidean direct sum $C' \oplus V$ by

$$[z_1 + v_1, z_2 + v_2] := [v_1, v_2]$$

for all $z_j + v_j \in C' \oplus V$. This map is $\mathbb{R}$-bilinear. Since J_z is skew-symmetric for each $z \in C'$ (cf. Sec. 1.4.2), the (extended) map $[\cdot,\cdot]$ is anti-symmetric. Moreover, $[V,V] \subseteq C'$ and $[V, C'] = [C', C'] = \{0\}$ imply the Jacobi identity for $[\cdot,\cdot]$. Thus, $C' \oplus V$ endowed with $[\cdot,\cdot]$ is a Euclidean Lie algebra. Let $J'\colon C' \to \mathrm{End}_{\mathrm{vs}}(V)$ denote the map $J'(z)(v) = J(z,v)$. Then (C', V, J') is an H-type algebra. Using the equivalence of (H3) and (H3'), we see that this H-type algebra satisfies the J^2-condition if and only if (C, V, J) does so.

Using the identification $e = 1$ from Sec. 1.2, the construction of a C-module structure from an H-type algebra

$$(\mathfrak{z}, \mathfrak{v}, J) \mapsto (\mathbb{R} \oplus \mathfrak{z}, 1, \mathfrak{v}, \widetilde{J})$$

and that of an H-type algebra from a C-module structure

$$(C, e, V, J) \mapsto (C', V, J')$$

are inverse to each other. Moreover, one easily sees that these constructions are equivariant under isomorphisms of C-module structures resp. of H-type algebras. More precisely, if $(\varphi, \psi)\colon (\mathfrak{z}_1, \mathfrak{v}_1, J_1) \to (\mathfrak{z}_2, \mathfrak{v}_2, J_2)$ is an isomorphism of H-type algebras, then

$$(\mathrm{id} \times \varphi, \psi)\colon (\mathbb{R} \oplus \mathfrak{z}_1, 1, \mathfrak{v}_1, \widetilde{J}_1) \to (\mathbb{R} \oplus \mathfrak{z}_2, 1, \mathfrak{v}_2, \widetilde{J}_2)$$

is an isomorphism of C-module structures.

Conversely, if $(\varphi, \psi)\colon (C_1, e_1, V_1, J_1) \to (C_2, e_2, V_2, J_2)$ is an isomorphism of C-module structures, then $\varphi(C_1') = C_2'$ and the map

$$(\varphi|_{C_1'}, \psi)\colon (C_1', V_1, J_1') \to (C_2', V_2, J_2')$$

is an isomorphism of H-type algebras.

1.4. Construction of symmetric spaces

1.4.1. The model D

Let $\mathfrak{n} = (\mathfrak{z}, \mathfrak{v}, J)$ be an H-type algebra. Further let $\mathfrak{a}$ be a one-dimensional Euclidean Lie algebra and fix a unit vector H in $\mathfrak{a}$. Then $\mathfrak{a}$ is spanned by H.

We denote by $\mathfrak{s}$ the Euclidean direct sum Lie algebra $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{v}$ with the Lie bracket that is determined by requiring that

$$
\begin{aligned}
[H, X] &= \tfrac{1}{2} X && \text{for all } X \in \mathfrak{v} \\
[H, Z] &= Z && \text{for all } Z \in \mathfrak{z}
\end{aligned}
$$

and that equals the original Lie bracket on $\mathfrak{n}$, when restricted to this Lie algebra.

Let $\exp(\mathfrak{s})$ be the connected, simply connected Lie group with Lie algebra $\mathfrak{s}$. We identify the tangent space to $\exp(\mathfrak{s})$ at the identity with $\mathfrak{s}$. Further we endow $\exp(\mathfrak{s})$ with the left-$\exp(\mathfrak{s})$-invariant Riemannian metric that coincides with the inner product on $\mathfrak{s}$ at the identity of $\exp(\mathfrak{s})$. We parametrize $\exp(\mathfrak{s})$ by

$$
\begin{cases}
\mathbb{R}^+ \times \mathfrak{z} \times \mathfrak{v} & \to \quad \exp(\mathfrak{s}) \\
(t, Z, X) & \mapsto \quad \exp(Z + X) \exp((\log t) H)
\end{cases}
$$

and set $S := \mathbb{R}^+ \times \mathfrak{z} \times \mathfrak{v}$. By requiring this parametrization to be a diffeomorphism and an isometry, S inherits the structure of a connected, simply connected Lie group with Riemannian metric from $\exp(\mathfrak{s})$. The Campbell-Baker-Hausdorff formula for $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ shows that the group operations on $\exp(\mathfrak{s})$ correspond on S to the group operations

$$
\big(t_1, Z_1, X_1\big)\big(t_2, Z_2, X_2\big) = \big(t_1 t_2, Z_1 + t_1 Z_2 + \tfrac{1}{2} t_1^{1/2} [X_1, X_2], X_1 + t_1^{1/2} X_2\big) \tag{1.2}
$$

$$
\big(t, Z, X\big)^{-1} = \big(t^{-1}, -t^{-1} Z, -t^{-1/2} X\big) \tag{1.3}
$$

for all $(t_j, Z_j, X_j), (t, Z, X) \in S$. The differential structure on S coincides with the differential structure on the open subset $\mathbb{R}^+ \times \mathfrak{z} \times \mathfrak{v}$ of some $\mathbb{R}^m$. The manifold S will be important for the definition of the Cygan metric, since it is the symmetric space in horospherical coordinates (see Section 2.2).

Now let

$$
D := \big\{(t, Z, X) \in \mathbb{R} \times \mathfrak{z} \times \mathfrak{v} \,\big|\, t > \tfrac{1}{4}|X|^2\big\}
$$

and consider the bijection

$$
\Theta\colon \begin{cases}
\mathbb{R} \times \mathfrak{z} \times \mathfrak{v} & \to \quad \mathbb{R} \times \mathfrak{z} \times \mathfrak{v} \\
(t, Z, X) & \mapsto \quad \big(t + \tfrac{1}{4}|X|^2, Z, X\big).
\end{cases} \tag{1.4}
$$

Then $\Theta(S) = D$, so that we define the structure of a Riemannian manifold on D by requiring Θ to be an isometry. The differential structure on D is identical to that of D being an open subset of $\mathbb{R}^+ \times \mathfrak{z} \times \mathfrak{v}$. Moreover, Θ induces a simply transitive action of S on D by defining

$$
s \cdot p := \Theta\big(s \Theta^{-1}(p)\big)
$$

for $s \in S$ and $p \in D$. In coordinates $s = (t_s, Z_s, X_s)$ and $p = (t_p, Z_p, X_p)$ this action reads

$$
s \cdot p = \big(t_s t_p + \tfrac{1}{4}|X_s|^2 + \tfrac{1}{2} t_s^{1/2} \langle X_s, X_p\rangle, Z_s + t_s Z_p + \tfrac{1}{2} t_s^{1/2} [X_s, X_p], X_s + t_s^{1/2} X_p\big). \tag{1.5}
$$

Due to the definition of the Riemannian metric, S obviously acts by isometries. We call $o := (1, 0, 0)$ the *base point* of D. The geodesic inversion σ of D at o is given by

$$\sigma(t, Z, X) = \frac{1}{t^2 + |Z|^2}\big(t, -Z, (-t + J_Z)X\big)$$

for all $(t, Z, X) \in D$. Then σ is an isometry, and hence D a symmetric space, if and only if $\mathfrak{n}$ satisfies the J^2-condition. In this case, D has rank one and, if in addition $\mathfrak{n}$ is non-trivial, then D is of noncompact type.

Conversely, let D be a rank one symmetric space of noncompact type. Suppose that $\mathfrak{g}$ is the simple Lie algebra of the Lie group of Riemannian isometries of D. Let ϑ be a Cartan involution of $\mathfrak{g}$, and let $\mathfrak{k}$ and $\mathfrak{p}$ be its ± 1-eigenspaces. Fix a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and choose a vector $H \in \mathfrak{a}$ which spans $\mathfrak{a}$. Then the decomposition of $\mathfrak{g}$ into restricted root spaces is

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus (\mathfrak{a} \oplus \mathfrak{m}) \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha},$$

where

$$\mathfrak{g}_{\beta} := \{X \in \mathfrak{g} \mid [H, X] = \beta(H)X\}$$

for the linear functional $\beta\colon \mathfrak{a} \to \mathbb{R}$ and

$$\mathfrak{m} := \{X \in \mathfrak{k} \mid [H, X] = 0\}.$$

We suppose that H is normalized such that $\alpha(H) = \frac{1}{2}$ and set $p := \dim \mathfrak{g}_{\alpha}$ and $q := \dim \mathfrak{g}_{2\alpha}$. If we endow $\mathfrak{n} := \mathfrak{g}_{2\alpha} \oplus \mathfrak{g}_{\alpha}$ with the inner product

$$\langle X, Y\rangle := -\frac{1}{p + 4q} B(X, \vartheta Y)$$

where B is the Killing form of $\mathfrak{g}$, then $\mathfrak{n} = (\mathfrak{g}_{2\alpha}, \mathfrak{g}_{\alpha}, J)$ is an H-type algebra with J^2-condition. The symmetric space constructed from $(\mathfrak{g}_{2\alpha}, \mathfrak{g}_{\alpha}, J)$ is exactly D. This means that each rank one Riemannian symmetric space of noncompact type arises from the construction above.

Remark 1.4.1. If $\mathfrak{n} = \{0\}$, then the constructed space D is the rank one Euclidean symmetric space $\mathbb{R}$.

1.4.2. The ball model B

Let (C, e, V, J) be a J^2C-module structure and let $W := C \oplus V$ be the Euclidean direct sum of C and V. Consider the unit disc

$$B := \{w \in W \mid |w| < 1\}$$

in W and endow it with the differential structure induced from W. In the following we will define a Riemannian metric on B with respect to which B is a rank one Riemannian symmetric space of noncompact type if $(C, V) \neq (\mathbb{R}e, \{0\})$.

Polarization of the equation in (M2) shows that we have

$$\langle \zeta u, \eta v\rangle + \langle \eta u, \zeta v\rangle = 2\langle \zeta, \eta\rangle \langle u, v\rangle \tag{1.6}$$

for all $\zeta, \eta \in C$ and all $u, v \in V$. For $\eta = e$ and $\zeta = z \in C'$ we get

$$\langle J_z u, v\rangle + \langle J_z^* u, v\rangle = \langle J_z u, v\rangle + \langle u, J_z v\rangle = 2\langle z, e\rangle\langle u, v\rangle = 0.$$

Hence $J_z^* = -J_z$, which shows that J_z is skew-symmetric. For $\zeta = a+z \in \mathbb{R}\oplus C'$ it follows that

$$J_\zeta^* = J_{a+z}^* = J_a^* + J_z^* = J_a - J_z = J_{a-z} = J_{\overline{\zeta}} \tag{1.7}$$

and therefore, for all $u, v \in V$,

$$\langle J_{\overline{\zeta}} J_\zeta u, v\rangle = \tfrac{1}{2}\big(\langle J_\zeta u, v\rangle + \langle u, J_\zeta v\rangle\big) = |\zeta|^2\langle u, v\rangle.$$

Likewise

$$\langle J_\zeta J_{\overline{\zeta}} u, v\rangle = |\overline{\zeta}|^2\langle u, v\rangle = |\zeta|^2\langle u, v\rangle.$$

Hence

$$J_{\overline{\zeta}} J_\zeta = |\zeta|^2 \operatorname{id}_V = J_\zeta J_{\overline{\zeta}}. \tag{1.8}$$

Thus, if we set $\zeta^{-1} := |\zeta|^{-2}\overline{\zeta}$ for $\zeta \in C\setminus\{0\}$, then we have

$$\zeta^{-1}(\zeta v) = v = \zeta(\zeta^{-1} v) \tag{1.9}$$

for all $v \in V$. In Sec. 3.1 we will see that, if $V \neq \{0\}$, there is a multiplication on C such that ζ^{-1} is the inverse of ζ.

Definition 1.4.2. Let $(\zeta, v), (\eta, u) \in W\setminus\{0\}$. Then (ζ, v) is called *equivalent* to (η, u), denoted as $(\zeta, v) \sim (\eta, u)$, if either $\zeta = 0 = \eta$ and $u \in Cv$, or $\zeta \neq 0 \neq \eta$ and $\zeta^{-1}v = \eta^{-1}u$.

Lemma 1.4.3. *The relation $\sim$ is an equivalence relation on $W\setminus\{0\}$.*

Proof. Reflexivity is obvious from the definition. To show symmetry of $\sim$ let $(\zeta, v), (\eta, u) \in W\setminus\{0\}$ such that $(\zeta, v) \sim (\eta, u)$. If $\zeta \neq 0 \neq \eta$, then clearly $(\eta, u) \sim (\zeta, v)$. Suppose that $\zeta = 0 = \eta$. Then $u \in Cv$, say $u = \tau v$. Since $u \neq 0$, we see that $\tau \neq 0$. Equality (1.9) shows that $\tau^{-1}u = \tau^{-1}(\tau v) = v$. Hence $v \in Cu$ and therefore $(\eta, u) \sim (\zeta v)$. This proves symmetry. To show transivity let $(\zeta_1, v_1), (\zeta_2, v_2), (\zeta_3, v_3) \in W\setminus\{0\}$ such that $(\zeta_1, v_1) \sim (\zeta_2, v_2)$ and $(\zeta_2, v_2) \sim (\zeta_3, v_3)$. Suppose first that $\zeta_1 = 0 = \zeta_2$. Then $\zeta_3 = 0$. Further $v_1 \in Cv_2$ and $v_2 \in Cv_3$. Using the J^2-condition we find $v_1 \in C(Cv_3) = Cv_3$. Thus $(\zeta_1, v_1) \sim (\zeta_3, v_3)$. Suppose now that $\zeta_1 \neq 0 \neq \zeta_2$. Then $\zeta_3 \neq 0$ and $\zeta_1^{-1}v_1 = \zeta_2^{-1}v_2 = \zeta_3^{-1}v_3$. Therefore $(\zeta_1, v_1) \sim (\zeta_3, v_3)$. This completes the proof. ◻

For an element $w \in W\setminus\{0\}$ let

$$Cw := \{w' \in W\setminus\{0\} \mid w' \sim w\} \cup \{0\}$$

denote the equivalence class of w together with the element $0 \in W$. Since B is an open subset of the real vector space W, we shall identify the tangent space

T_wB to the point $w \in B$ with W. The Riemannian metric $w \mapsto \langle\cdot,\cdot\rangle_{w-}$ on B is defined by

$$\langle X, Y\rangle_{0-} := 4\langle X, Y\rangle \qquad \text{on } T_0B$$

and, for $w \in B\setminus\{0\}$, by

$$\langle X, Y\rangle_{w-} := \begin{cases} 4\frac{\langle X,Y\rangle}{(1-|w|^2)^2} & \text{if } X, Y \in Cw \\ 4\frac{\langle X,Y\rangle}{1-|w|^2} & \text{if } X, Y \in (Cw)^\perp \\ 0 & \text{if } X \in Cw, Y \in (Cw)^\perp \text{ (or vice versa).} \end{cases} \tag{1.10}$$

1.4.3. The Cayley transform

Let $\mathfrak{n} = (\mathfrak{z}, \mathfrak{v}, J)$ be a non-trivial H-type algebra which satisfies the J^2-condition. Further let $\mathfrak{a}$ be a one-dimensional Euclidean Lie algebra with fixed unit vector H. Suppose that (C, e, V, J) is the J^2C-module structure associated to $(\mathfrak{z}, \mathfrak{v}, J)$ (see Sec. 1.3). We consider the Riemannian symmetric spaces D and B which are constructed in Sec. 1.4.1 resp. 1.4.2.

The *Cayley transform* (in $\mathbb{R} \times \mathfrak{z} \times \mathfrak{v}$-coordinates, see [CDKR98, (2.10a)])

$$\mathcal{C}\colon \begin{cases} \quad B & \to \quad D \\ (t, Z, X) & \mapsto \frac{1}{(1-t)^2+|Z|^2}\big(1 - t^2 - |Z|^2, 2Z, 2(1 - t + J_Z)X\big) \end{cases}$$

is clearly a diffeomorphism from B to D. Its inverse (see [CDKR98, (2.10b)]) is given by

$$\mathcal{C}^{-1}\colon \begin{cases} \quad D & \to \quad B \\ (t, Z, X) & \mapsto \frac{1}{(1+t)^2+|Z|^2}\big(-1 + t^2 + |Z|^2, 2Z, (1 + t - J_Z)X\big). \end{cases}$$

Proposition 1.4.4. *The Cayley transform is an isometry.*

Proof. In [CDKR98], the pullback of the Riemannian metric of D to B via $\mathcal{C}$ is described as follows: Let $p \in B$ and denote the tangent space to B at p by T_pB, which we identify with $\mathbb{R} \times \mathfrak{z} \times \mathfrak{v}$. If $\|\mathfrak{X}\|_p$ denotes the norm on T_pB induced by the pullback of the Riemannian metric on D, then

$$\|\mathfrak{X}\|_0 = 2|X|$$

for all $\mathfrak{X} \in T_0B$. For $p \in B\setminus\{0\}$ one has

$$\|\mathfrak{X}\|_p^2 = \begin{cases} 4\frac{|\mathfrak{X}|^2}{1-|p|^2} & \text{if } \mathfrak{X} \in T_p^{(1)}, \\ 4\frac{|\mathfrak{X}|^2}{(1-|p|^2)^2} & \text{if } \mathfrak{X} \in \mathbb{R}p \oplus T_p^{(2)}, \end{cases}$$

where

$$T_pB = \mathbb{R}p \oplus T_p^{(2)} \oplus T_p^{(1)}$$

is a direct sum which is orthogonal w. r. t. Euclidean metric and Riemannian inner product on T_pB. Thm. 6.8 in [CDKR98] provides the following explicit formulas for $T_p^{(1)}$ and $\mathbb{R}p \oplus T_p^{(2)}$. For $X \in \mathfrak{v}$ we set $\mathfrak{j}(X) := J_{\mathfrak{z}}X$ and define $\mathfrak{k}(X)$ to be the orthogonal complement of $\mathbb{R}X$ in

$$\mathfrak{j}(X)^{\perp} = \{Y \in \mathfrak{v} \mid \forall\, Z \in \mathfrak{z} \colon \langle J_Z X, Y\rangle = 0\}.$$

For $p = (t, Z, X) \in B \smallsetminus \{0\}$ we have

(i) $T_p^{(1)} = \mathfrak{v}$ and $\mathbb{R}p \oplus T_p^{(2)} = \mathbb{R} \oplus \mathfrak{z}$ for $X = 0$,

(ii) $T_p^{(1)} = \mathbb{R} \oplus \mathfrak{z} \oplus \mathfrak{k}(X)$ and $\mathbb{R}p \oplus T_p^{(2)} = \mathfrak{j}(X) \oplus \mathbb{R}X$ for $(t, Z) = (0, 0)$,

(iii) in the remaining cases,

$$T_p^{(1)} = \mathfrak{k} \oplus \bigl\{\bigl(|X|^2 u, |X|^2 W, -(u + J_W)(t - J_Z)W\bigr) \bigm| W \in \mathfrak{z}, u \in \mathbb{R}\bigr\}$$

and

$$\mathbb{R}p \oplus T_p^{(2)} = \bigl\{\bigl((t^2 + |Z|^2)u, (t^2 + |Z|^2)W, (u + J_W)(t - J_Z)X\bigr) \bigm| W \in \mathfrak{z}, u \in \mathbb{R}\bigr\}\,.$$

Note that this subsumes the degenerate cases. On T_0B, the given Riemannian inner product obviously coincides with this one. For $p \in B \smallsetminus \{0\}$, (1.10) implies that it suffices to show that $\mathbb{R}p \oplus T_p^{(2)} = Cp$. Let $p = (\zeta, v) = (t, Z, X) \in B \smallsetminus \{0\}$ (hence $\zeta = (t, Z) \in C = \mathbb{R} \times \mathfrak{z}$ and $v = X \in V = \mathfrak{v}$).

If $v = 0$, then $Cp = C \times \{0\} = C = \mathbb{R} \times \mathfrak{z} = \mathbb{R}p \oplus T_p^{(2)}$. If $\zeta = 0$, then $Cp = Cv = (\mathbb{R} + J_{\mathfrak{z}})X = \mathbb{R}p \oplus T_p^{(2)}$. If $\zeta \neq 0$ and $v \neq 0$, then $(\eta, u) \in Cp$ if and only if $\eta \neq 0$ and $\zeta^{-1}v = \eta^{-1}u$, or $(\eta, u) = 0$. This means that in both cases

$$u = J_\eta J_{\zeta^{-1}} v = |\zeta|^{-2} J_\eta J_{\overline{\zeta}} v = J_{|\zeta|^{-2}} J_{\overline{\zeta}} v.$$

Hence

$$Cp = \Bigl\{\bigl(\eta, J_{|\zeta|^{-2}\eta} J_{\overline{\zeta}} v\bigr) \Bigm| \eta \in C\Bigr\} = \Bigl\{\bigl(|\zeta|^2 \xi, J_\xi J_{\overline{\zeta}} v\bigr) \Bigm| \xi \in C\Bigr\} = \mathbb{R}p \oplus T_p^{(2)}.$$

This completes the proof. □

Remark 1.4.5. Note that for $\mathfrak{n} = \{0\}$ resp. $(C, V) = (\mathbb{R}e, \{0\})$ Prop. 1.4.4 shows that the model B is isometric to the Euclidean symmetric space $\mathbb{R}$.

1.5. The map β_2

Let (C, e, V, J) be a J^2C-module structure. In this section we introduce a map $\beta_2 \colon V \times V \to C$, which will be shown to be C-hermitian, i. e., β_2 is $\mathbb{R}$-bilinear and for all $u, v \in V$ we have $\beta_2(u, v) = \overline{\beta_2(v, u)}$. The map β_2 encodes the inner product and the Lie bracket on V.

We define $\beta_2 \colon V \times V \to C$ by[5]

$$\langle \beta_2(v, u), \zeta\rangle := \langle J_\zeta u, v\rangle \quad \text{for all } \zeta \in C.$$

[5] This is J^* in [KR05].

Lemma 1.5.1. *For $\zeta, \eta \in C$ we have $\langle \eta, \overline{\zeta} \rangle = \langle \overline{\eta}, \zeta \rangle$.*

Proof. Let $\zeta = a + x$ and $\eta = b + y$ ($a, b \in \mathbb{R}$, $x, y \in C'$) be the decompositions of ζ and η w.r.t. $C = \mathbb{R} \oplus C'$. Then

$$\langle \overline{\zeta}, \eta \rangle = \langle a - x, b + y \rangle = \langle a, b \rangle - \langle x, b \rangle + \langle a, y \rangle - \langle x, y \rangle.$$

Since $\mathbb{R}$ and C' are orthogonal, it follows that

$$\begin{aligned} \langle \overline{\zeta}, \eta \rangle &= \langle a, b \rangle - \langle x, y \rangle \\ &= \langle a, b \rangle + \langle x, b \rangle - \langle a, y \rangle - \langle x, y \rangle \\ &= \langle a + x, b - y \rangle \\ &= \langle \zeta, \overline{\eta} \rangle. \end{aligned}$$

This completes the proof. ◻

Proposition 1.5.2. *The map $\beta_2 \colon V \times V \to C$ is $\mathbb{R}$-bilinear. Further we have*

(i) $\beta_2(v, u) = \overline{\beta_2(u, v)}$ *for all $u, v \in V$,*

(ii) $\beta_2(v, v) = \langle v, v \rangle$ *for all $v \in V$.*

Proof. Because $\langle \cdot, \cdot \rangle$ is $\mathbb{R}$-bilinear and J_ζ is $\mathbb{R}$-linear for each $\zeta \in C$, the $\mathbb{R}$-linearity is inherited to β_2. Using (1.7) and Lemma 1.5.1 we find

$$\langle \beta_2(v, u), \zeta \rangle = \langle J_\zeta u, v \rangle = \langle u, J_{\overline{\zeta}} v \rangle = \langle J_{\overline{\zeta}} v, u \rangle = \langle \beta_2(u, v), \overline{\zeta} \rangle = \langle \overline{\beta_2(u, v)}, \zeta \rangle$$

for all $u, v \in V$ and all $\zeta \in C$. Hence $\beta_2(v, u) = \overline{\beta_2(u, v)}$ for all $u, v \in V$, which proves (i).

Finally let $v \in V$. Since $\beta_2(v, v) = \overline{\beta_2(v, v)}$, there is (a unique) $a_v \in \mathbb{R}$ such that $\beta_2(v, v) = a_v$. Moreover,

$$a_v = \langle \beta_2(v, v), e \rangle = \langle ev, v \rangle = \langle v, v \rangle,$$

which shows (ii). ◻

Lemma 1.5.3. *For $v_1, v_2 \in V$ we have*

$$\operatorname{Re} \beta_2(v_1, v_2) = \langle v_1, v_2 \rangle \quad \textit{and} \quad \operatorname{Im} \beta_2(v_1, v_2) = [v_2, v_1].$$

Proof. Let $v_1, v_2 \in V$. By Proposition 1.5.2 we have

$$\begin{aligned} |v_1|^2 + 2\langle v_1, v_2 \rangle + |v_2|^2 = |v_1 + v_2|^2 &= \beta_2(v_1 + v_2, v_1 + v_2) \\ &= |v_1|^2 + \beta_2(v_1, v_2) + \beta_2(v_2, v_1) + |v_2|^2 \\ &= |v_1|^2 + \beta_2(v_1, v_2) + \overline{\beta_2(v_1, v_2)} + |v_2|^2 \\ &= |v_1|^2 + 2 \operatorname{Re} \beta_2(v_1, v_2) + |v_2|^2. \end{aligned}$$

Hence $\operatorname{Re}\beta_2(v_1, v_2) = \langle v_1, v_2\rangle$. To show the second claim note that Prop. 1.5.2 implies that

$$\operatorname{Im}\beta_2(v_1, v_2) = \tfrac{1}{2}\beta_2(v_1, v_2) - \tfrac{1}{2}\overline{\beta_2(v_1, v_2)} = \tfrac{1}{2}\beta_2(v_1, v_2) - \tfrac{1}{2}\beta_2(v_2, v_1).$$

Further note that for $\zeta \in C'$ equality (1.7) reduces to $J_\zeta^* = J_{-\zeta} = -J_\zeta$. For each $\zeta \in C'$ it follows that

$$\begin{aligned}
\langle \zeta, \operatorname{Im}\beta_2(v_1, v_2)\rangle &= \tfrac{1}{2}\langle \zeta, \beta_2(v_1, v_2)\rangle - \tfrac{1}{2}\langle \zeta, \beta_2(v_2, v_1)\rangle \\
&= \tfrac{1}{2}\langle J_\zeta v_2, v_1\rangle - \tfrac{1}{2}\langle J_\zeta v_1, v_2\rangle \\
&= \tfrac{1}{2}\langle J_\zeta v_2, v_1\rangle - \tfrac{1}{2}\langle v_1, J_{\overline{\zeta}} v_2\rangle \\
&= \tfrac{1}{2}\langle J_\zeta v_2, v_1\rangle + \tfrac{1}{2}\langle v_1, J_\zeta v_2\rangle \\
&= \langle J_\zeta v_2, v_1\rangle.
\end{aligned}$$

Recall from (1.1) resp. (H2) that $\langle J_\zeta v_2, v_1\rangle = \langle \zeta, [v_2, v_1]\rangle$ for all $\zeta \in C'$. Therefore, for all $\zeta \in C'$,

$$\langle \zeta, \operatorname{Im}\beta_2(v_1, v_2)\rangle = \langle \zeta, [v_2, v_1]\rangle.$$

Since $\langle\cdot,\cdot\rangle|_{C'\times C'}$ is non-degenerate and $\operatorname{Im}\beta_2(v_1, v_2) \in C'$ and $[v_2, v_1] \in C'$, it follows that $\operatorname{Im}\beta_2(v_1, v_2) = [v_2, v_1]$. □

1.6. The isometry group

Let $\mathfrak{n} = (\mathfrak{z}, \mathfrak{v}, J)$ be a non-trivial H-type algebra which satisfies the J^2-condition, and let $\mathfrak{a}$ be a one-dimensional Euclidean Lie algebra. Construct the Euclidean Lie algebra $\mathfrak{s}$ and the spaces S and D as in Sec. 1.4.1. We denote the to $\mathfrak{n}$ isomorphic J^2C-module structure $(\mathbb{R} \oplus \mathfrak{z}, 1, \mathfrak{v}, \widetilde{J})$ by (C, e, V, J).

Let G denote the full isometry group of D. Suppose that N resp. A are the connected, simply connected Lie groups with Lie algebra $\mathfrak{n}$ resp. $\mathfrak{a}$. Then N and A are subgroups of S, more precisely, S is the semidirect product AN. In the parametrization of S, the groups N and A are given by

$$N = \left\{ n_{(Z,X)} := (1, Z, X) \;\middle|\; (Z, X) \in \mathfrak{z} \times \mathfrak{v} \right\}$$

and

$$A = \left\{ a_t := (t, 0, 0) \;\middle|\; t \in \mathbb{R}^+ \right\}.$$

Let K be the stabilizer of the base point $o = (1, 0, 0)$ in G and let $M := Z_K(A)$ be the centralizer of A in K. Recall the geodesic inversion

$$\sigma(t, Z, x) = \frac{1}{t^2 + |Z|^2}\big(t, -Z, (-t + J_Z)X\big)$$

at the origin o from Sec. 1.4.1.

Theorem 1.6.1 (Thm. 6.4 in [CDKR98]). *The Lie group G has the Bruhat decomposition $MAN \cup N\sigma MAN$.*

Let $\mathcal{X} := \mathbb{R} \times \mathfrak{z} \times \mathfrak{v} \cup \{\infty\}$ be the one-point compactification of $\mathbb{R} \times \mathfrak{z} \times \mathfrak{v}$, where ∞ denotes the point at infinity. A compactification of D is then given by the closure of D in $\mathcal{X}$, namely

$$\begin{aligned}\overline{D}^g &= \left\{(t, Z, X) \in \mathbb{R} \times \mathfrak{z} \times \mathfrak{v} \,\middle|\, t \geq \tfrac{1}{4}|X|^2 \right\} \cup \{\infty\} \\ &= \left\{(\zeta, v) \in C \times V \,\middle|\, \operatorname{Re}\zeta \geq \tfrac{1}{4}|v|^2 \right\} \cup \{\infty\}.\end{aligned}$$

This is precisely the geodesic compactification of D, see [BJ06, Sec. I.2] and [Ebe96, Prop. 1.7.6].

Let $\overline{B}$ be the closed unit ball in $W = \mathbb{R} \times \mathfrak{z} \times \mathfrak{v}$. Then the Cayley transform $\mathcal{C}\colon B \to D$ extends (uniquely) to a homeomorphism $\overline{B} \to \overline{D}^g$. Therefore, [CDKR98, Cor. 6.2] implies the following proposition.

Proposition 1.6.2 (Cor. 6.2 in [CDKR98])**.** *The action of G extends continuously to $\overline{D}^g$.*

The Bruhat decomposition of G implies that the stabilizer G_∞ of ∞ in G equals MAN.

For future purposes we need explicit formulas for the action of the groups M, A and N on $\overline{D}^g$. The action of N and A in $\mathbb{R} \times \mathfrak{z} \times \mathfrak{v}$-coordinates is already given in (1.5). Suppose that $(\zeta, v) \in \overline{D}^g \smallsetminus \{\infty\}$. For $a_t \in A$ we have

$$a_t \infty = \infty \qquad \text{and} \qquad a_t(\zeta, v) = (t\zeta, t^{1/2} v).$$

For $n_{(Z,X)} \in N$ we get $n_{(Z,X)}\infty = \infty$ and

$$n_{(Z,X)}(\zeta, v) = \left(\tfrac{1}{4}|X|^2 + Z + \zeta + \tfrac{1}{2}\beta_2(v, X), X + v\right).$$

In $C \times V$-coordinates, the geodesic inversion σ reads as

$$\sigma\infty = 0, \quad \sigma 0 = \infty,$$

and for $(\zeta, v) \in \overline{D}^g \smallsetminus \{0, \infty\}$,

$$\begin{aligned}\sigma(\zeta, v) &= \frac{1}{(\operatorname{Re}\zeta)^2 + |\operatorname{Im}\zeta|^2}\left(\operatorname{Re}\zeta, -\operatorname{Im}\zeta, J_{-\operatorname{Re}\zeta + \operatorname{Im}\zeta} v\right) \\ &= |\zeta|^{-2}\left(\overline{\zeta}, J_{-\overline{\zeta}} v\right) = \left(\zeta^{-1}, J_{-|\zeta|^{-2}\overline{\zeta}} v\right) \\ &= \left(\zeta^{-1}, -\zeta^{-1} v\right) = \zeta^{-1}(1, -v).\end{aligned}$$

The Cayley transform in $C \times V$-coordinates is

$$\mathcal{C}\colon \left\{\begin{array}{ccl} B & \to & D \\ (\zeta, v) & \mapsto & |1 - \zeta|^{-2}\left(1 + 2\operatorname{Im}\zeta - |\zeta|^2, 2(1 - \overline{\zeta})v\right) \end{array}\right.$$

with

$$\mathcal{C}^{-1}\colon \left\{\begin{array}{ccl} D & \to & B \\ (\eta, u) & \mapsto & |1 + \eta|^{-2}\left(-1 + 2\operatorname{Im}\eta + |\eta|^2, 2(1 + \overline{\eta})u\right). \end{array}\right.$$

The following proposition provides the explicit form of the group M.

Proposition 1.6.3. *The group M equals the group of automorphism of the J^2C-module structure (C, e, V, J), that is, the elements of M are the pairs (φ, ψ) of orthogonal endomorphisms $\varphi\colon C \to C$ with $\varphi(e) = e$ and $\psi\colon V \to V$ such that the diagram*

$$\begin{tikzcd} C \times V \arrow[r, "J"] \arrow[d, "\varphi\times\psi"'] & V \arrow[d, "\psi"] \\ C \times V \arrow[r, "J"] & V \end{tikzcd}$$

commutes. If $(\varphi, \psi) \in M$, then the action on $\overline{D}^g$ is given by $(\varphi, \psi)(\infty) = \infty$ and $(\varphi, \psi)(\zeta, v) = (\varphi(\zeta), \psi(v))$ for $(\zeta, v) \in \overline{D}^g \smallsetminus \{\infty\}$.

Proof. Let $\widetilde{G} := \mathcal{C}^{-1} G \mathcal{C}$. For each $g \in G$ set $\widetilde{g} := \mathcal{C}^{-1} g \mathcal{C}$, and for each subset T of G let $\widetilde{T} := \mathcal{C}^{-1} T \mathcal{C}$ be the corresponding subset of $\widetilde{G}$. Clearly, $\widetilde{G}$ is the full isometry group of B. We will first characterize the centralizer $Z_{\widetilde{K}}(\widetilde{A})$ of $\widetilde{A}$ in $\widetilde{K}$ as a subgroup of $\widetilde{G}$. Let $\widetilde{M}$ denote the automorphism group of (C, e, V, J) and define the action of $(\varphi, \psi) \in \widetilde{M}$ on B by $(\varphi, \psi)(\varphi, v) = (\varphi(\zeta), \psi(v))$. We will show that $\widetilde{M} = Z_{\widetilde{K}}(\widetilde{A})$. By [KR, Prop. 4.1] we have that $\widetilde{M} \subseteq \widetilde{K}$. Let $a_t \in A$ and $(\zeta, v) \in B$. Then one easily calculates that

$$\begin{aligned}
\widetilde{a}_t(\zeta, v) &= \mathcal{C}^{-1} \circ a_t \circ \mathcal{C}(\zeta, v) \\
&= \big| |1-\zeta|^2 + t(1 + 2\operatorname{Im}\zeta - |\zeta|^2) \big|^{-2} \cdot \\
&\quad \cdot \Big(-|1-\zeta|^4 + 4t \operatorname{Im}\zeta + t^2 \left| 1 + 2\operatorname{Im}\zeta - |\zeta|^2 \right|^2, \\
&\qquad 4t^{1/2} |1-\zeta|^2 \big(|1-\zeta|^2 + t(1 - \operatorname{Im}\zeta - |\zeta|^2) \big) \big((1-\overline{\zeta}) v \big) \Big).
\end{aligned}$$

Suppose that $\widetilde{m} = (\varphi, \psi) \in \widetilde{M}$. Since $\varphi(e) = e$, we have that $\varphi(\overline{\zeta}) = \overline{\varphi(\zeta)}$ and $\varphi(\operatorname{Im}\zeta) = \operatorname{Im}\varphi(\zeta)$ for each $\zeta \in C$. Moreover, $|\zeta| = |\varphi(\zeta)|$ for each $\zeta \in C$. Then the first component of $\widetilde{m} \circ \widetilde{a}_t(\zeta, v)$ is given by

$$\begin{aligned}
&\varphi\left(\frac{-|1-\zeta|^4 + 4t\operatorname{Im}\zeta + t^2 \left|1 + 2\operatorname{Im}\zeta - |\zeta|^2\right|^2}{\big||1-\zeta|^2 + t(1 + 2\operatorname{Im}\zeta - |\zeta|^2)\big|^2} \right) = \\
&\qquad = \frac{-|1-\zeta|^4 + 4t\operatorname{Im}\varphi(\zeta) + t^2 \left|1 + 2\operatorname{Im}\zeta - |\zeta|^2\right|^2}{\big||1-\zeta|^2 + t(1 + 2\operatorname{Im}\zeta - |\zeta|^2)\big|^2} \\
&\qquad = \frac{-|1-\varphi(\zeta)|^4 + 4t\operatorname{Im}\varphi(\zeta) + t^2 \left|1 + 2\operatorname{Im}\varphi(\zeta) - |\varphi(\zeta)|^2\right|^2}{\big||1-\varphi(\zeta)|^2 + t(1 + 2\operatorname{Im}\varphi(\zeta) - |\varphi(\zeta)|^2)\big|^2},
\end{aligned}$$

which is the first component of $\widetilde{a}_t \circ \widetilde{m}(\zeta, v)$. The second component of $\widetilde{m} \circ \widetilde{a}_t(\zeta, v)$ reads as

$$\begin{aligned}
&\psi\left(\frac{\big(|1-\zeta|^2 + t(1 - 2\operatorname{Im}\zeta - |\zeta|^2)\big)\big((1-\overline{\zeta})v\big)}{\big||1-\zeta|^2 + t(1 + 2\operatorname{Im}\zeta - |\zeta|^2)\big|^2} \right) = \\
&\qquad = \frac{\varphi\big(|1-\zeta|^2 + t(1 - 2\operatorname{Im}\zeta - |\zeta|^2)\big)\psi\big((1-\overline{\zeta})v\big)}{\big||1-\zeta|^2 + t(1 + 2\operatorname{Im}\zeta - |\zeta|^2)\big|^2}
\end{aligned}$$

$$= \frac{\big(|1-\zeta|^2 + t(1 - 2\operatorname{Im}\varphi(\zeta) - |\zeta|^2)\big)\big(\varphi(1-\overline{\zeta})\psi(v)\big)}{\big||1-\zeta|^2 + t(1+2\operatorname{Im}\zeta - |\zeta|^2)\big|^2}$$
$$= \frac{\big(|1-\varphi(\zeta)|^2 + t(1 - 2\operatorname{Im}\varphi(\zeta) - |\varphi(\zeta)|^2)\big)\big((1-\overline{\varphi(\zeta)})\psi(v)\big)}{\big||1-\varphi(\zeta)|^2 + t(1+2\operatorname{Im}\varphi(\zeta) - |\varphi(\zeta)|^2)\big|^2}.$$

Clearly, this is the second component of $\widetilde{a}_t \circ \widetilde{m}(\zeta, v)$. Hence $\widetilde{m}$ commutes with each $\widetilde{a}_t$. This shows that $\widetilde{M} \subseteq Z_{\widetilde{K}}(\widetilde{A})$.

Conversely, suppose that $\widetilde{m} \in Z_{\widetilde{K}}(\widetilde{A})$. We have to show that $\widetilde{m}(1,0) = (1,0)$. Then [KR, Prop. 4.1] implies that $\widetilde{m} \in \widetilde{M}$. For each $t \in \mathbb{R}^+$, we have

$$\widetilde{a}_t(1,0) = \lim_{n\to\infty} \widetilde{a}_t\big(1 - \tfrac{1}{n}, 0\big) = (1,0).$$

Hence $\widetilde{a}_t(\widetilde{m}(1,0)) = \widetilde{m}(1,0)$. The only points on ∂B that are invariant under all $\widetilde{a}_t$ are $(1,0)$ and $(-1,0)$. Assume for contradiction that $\widetilde{m}(1,0) = (-1,0)$. Since $\widetilde{\sigma} = -\operatorname{id}$, we then get $\widetilde{\sigma} \circ \widetilde{m}(1,0) = (1,0)$. By [KR, Prop. 4.1] we have $\widetilde{\sigma} \circ \widetilde{m} \in \widetilde{M}$. Our previous argument then shows that $\widetilde{\sigma} \circ \widetilde{m}$ commutes with all $\widetilde{a}_t$. Then

$$\widetilde{a}_t \circ \widetilde{\sigma} \circ \widetilde{m} = \widetilde{\sigma} \circ \widetilde{m} \circ \widetilde{a}_t = \widetilde{\sigma} \circ \widetilde{a}_t \circ \widetilde{m},$$

which means that $\widetilde{\sigma}$ commutes with each $\widetilde{a}_t$. This is a contradiction. Hence $\widetilde{m}(1,0) = (1,0)$, which by [KR, Prop. 4.1] shows that $\widetilde{m} \in \widetilde{M}$. Therefore $\widetilde{M} = Z_{\widetilde{K}}(\widetilde{A})$.

Now let $\widetilde{m} = (\varphi, \psi) \in \widetilde{M}$ and set $m := \mathcal{C} \circ \widetilde{m} \circ \mathcal{C}^{-1}$. For $(\eta, u) \in D$ we find

$$\begin{aligned} m(\eta, u) &= \mathcal{C} \circ \widetilde{m} \circ C^{-1}(\eta, u) \\ &= \mathcal{C} \circ \widetilde{m}\left(|1+\eta|^{-2}\big(1 + 2\operatorname{Im}\eta + |\eta|^2, 2(1+\overline{\eta})u\big)\right) \\ &= \mathcal{C}\left(|1+\varphi(\eta)|^{-2}\big(1 + 2\operatorname{Im}\varphi(\eta) + |\varphi(\eta)|^2, 2(1+\overline{\varphi(\eta)})\psi(u)\big)\right) \\ &= (\varphi(\eta), \psi(u)). \end{aligned}$$

The remaining claim follows directly from continuity or, alternatively, from the extension of the Cayley transform to $\overline{B}$. □

Remark 1.6.4. Let $m \in M$ and $a_t \in A$. Then we have, as in each Bruhat decomposition, the equalities $m^{-1} \circ \sigma = \sigma \circ m$ and $a_t^{-1} \circ \sigma = \sigma \circ a_t$ on $\overline{D}^g$.

2. Isometric fundamental regions

Throughout this section let (C, e, V, J) be a J^2C-module structure such that $(C, V) \neq (\mathbb{R}e, \{0\})$ and suppose that $(\mathfrak{z}, \mathfrak{v}, J)$ is the corresponding H-type algebra with J^2-condition. Recall the model D of the rank one Riemannian symmetric space of noncompact type which is constructed from (C, e, V, J) resp. $(\mathfrak{z}, \mathfrak{v}, J)$ in Sec. 1.4.1. Let G denote the full isometry group of D.

The purpose of this section is to prove the existence of so-called isometric fundamental regions for certain subgroups Γ of G. An isometric fundamental region consists of two building blocks. One of them is a fundamental region $\mathcal{F}_\infty$ for the stabilizer group Γ_∞ of ∞, the other one is the common part of the exteriors of all isometric spheres of Γ. Then the isometric fundamental region is the set

$$\mathcal{F} := \mathcal{F}_\infty \cap \bigcap_{g \in \Gamma \smallsetminus \Gamma_\infty} \operatorname{ext} I(g), \tag{2.1}$$

where $\operatorname{ext} I(g)$ denotes the exterior of the isometric sphere $I(g)$ of the element g in $\Gamma \smallsetminus \Gamma_\infty$. The isometric sphere of an element g in $G \smallsetminus G_\infty$ is a sphere in Cygan metric centered at $g^{-1}\infty$. Its radius only depends on the A-part in the Bruhat decomposition of g. The Cygan metric on D, which in fact is a metric on $\overline{D}^g \smallsetminus \{\infty\}$, arises from a group norm. The group norm which we employ here is an extension of the Heisenberg pseudonorm on N to the direct product of the groups $(\mathbb{R}, +)$ and N. The proof that this extension is indeed a group norm needs that the Cauchy-Schwarz Theorem holds for the map β_2, which we show in Sec. 2.1. Moreover, for the definition of the Cygan metric, it is convenient to work with horospherical coordinates on $\overline{D}^g \smallsetminus \{\infty\}$. In Sec. 2.2 we introduce these coordinates, the group norm and the Cygan metric. Further we define the notion of an isometric sphere, its exterior and interior, and prove several properties of isometric spheres, which we need in Sec. 2.3 to show the existence of isometric fundamental regions. The main requirements on the group Γ are that it be of type (O) and that $\Gamma \smallsetminus \Gamma_\infty$ be of type (F) (see Sec. 2.3 for a definition). If $\mathcal{F}$ is the set in (2.1), then Γ being of type (O) implies that $\mathcal{F}$ is open. Lemma 2.2.12 shows that $\mathcal{F}$ does not contain Γ-equivalent points, and the combination of $\Gamma \smallsetminus \Gamma_\infty$ being of type (F), Prop. 2.2.11, Lemma 2.2.16 and Cor. 2.2.15 yields that the Γ-translates of $\overline{\mathcal{F}}$ cover D.

2.1. The Cauchy-Schwarz Theorem for β_2

In this section we show the Cauchy-Schwarz Theorem for β_2. The following two lemmas are needed for its proof.

Lemma 2.1.1. *Let $u, v \in V$. Then $\beta_2(u,v) = 0$ if and only if $v \in (Cu)^\perp$.*

Proof. Let $u, v \in V$. We have $\beta_2(v,u) = 0$ if and only if

$$\langle \beta_2(v,u), \zeta \rangle = 0 \quad \text{for all } \zeta \in C.$$

By the definition of β_2, this holds if and only if

$$\langle \zeta u, v \rangle = 0 \quad \text{for all } \zeta \in C,$$

hence if and only if $v \in (Cu)^\perp$. □

Lemma 2.1.2. *Let $u \in V$ and $\lambda \in C$. Then*

$$\beta_2(\lambda u, u) = |u|^2 \lambda.$$

Proof. The polarization (1.6) shows that for each $\zeta \in C$ we have

$$2\langle \zeta u, \lambda u \rangle = \langle \zeta u, \lambda u \rangle + \langle \lambda u, \zeta u \rangle = 2\langle \zeta, \lambda \rangle |u|^2.$$

Using the definition of β_2, we see that

$$\langle \beta_2(\lambda u, u), \zeta \rangle = \langle \zeta u, \lambda u \rangle = \langle \lambda, \zeta \rangle |u|^2 = \langle |u|^2 \lambda, \zeta \rangle$$

for all $\zeta \in C$. Hence $\beta_2(\lambda u, u) = |u|^2 \lambda$. □

Proposition 2.1.3. *Let $u, v \in V$. Then*

$$|\beta_2(u,v)| \le |u||v|.$$

Equality holds if and only if $v \in Cu$ or $u \in Cv$.

Proof. Let $(v_1, v_2) \in Cu \times (Cu)^\perp$ be the unique pair such that $v = v_1 + v_2$. Using that β_2 is $\mathbb{R}$-bilinear (see Prop. 1.5.2), Lemma 2.1.1 yields

$$\beta_2(u,v) = \beta_2(u,v_1) + \beta_2(u,v_2) = \beta_2(u,v_1).$$

Then Lemma 2.1.2 shows

$$|\beta_2(u,v_1)|^2 = |u|^2 |v_1|^2.$$

Clearly

$$|v_1|^2 \le |v_1|^2 + |v_2|^2 = |v|^2,$$

where equality holds if and only if $v_2 = 0$, hence if and only if $v = v_1 \in Cu$. Thus,

$$|\beta_2(u,v)|^2 = |\beta_2(u,v_1)|^2 = |u|^2|v_1|^2 \le |u|^2|v|^2,$$

where the inequality is an equality if and only if $u = 0$ or $v \in Cu$. This proves the claim. □

2.2. H-coordinates, Cygan metric, and isometric spheres

Consider the closure

$$\overline{S} = \mathbb{R}_0^+ \times \mathfrak{z} \times \mathfrak{v}$$

of S in $\mathbb{R} \times \mathfrak{z} \times \mathfrak{v}$. Then the map Θ from (1.4) induces a bijection between $\overline{S}$ and $\overline{D}^g \smallsetminus \{\infty\}$.

Definition 2.2.1. Let $z = (t, Z, X) \in \overline{D}^g \smallsetminus \{\infty\}$. The *horospherical coordinates* or *H-coordinates* of z relative to the origin o of D are

$$\Theta^{-1}(z) = \left(t - \tfrac{1}{4}|X|^2, Z, X\right) \quad \in \overline{S}.$$

To avoid confusion, the H-coordinates of z will be denoted with a subscript h, that is, $\left(t - \frac{1}{4}|X|^2, Z, X\right)_h$ are the H-coordinates of z.

The following remark gives a geometric characterization of H-coordinates, which explains the naming.

Remark and Definition 2.2.2. Let $z = (t, Z, X) = (\zeta, v) \in \overline{D}^g \smallsetminus \{\infty\}$. The *horosphere* through z with center ∞ is the N-orbit of z. We extend the group A to the set

$$A^+ := A \cup \{a_0\},$$

where $a_0 \colon \overline{D}^g \to \overline{D}^g$ is defined by $a_0\infty := \infty$ and $a_0 z := 0$ for all $z \in \overline{D}^g \smallsetminus \{\infty\}$. Then there is a unique pair $(a_s, n) \in A^+ \times N$ such that

$$n a_s(o) = z.$$

The *height* of z is defined as

$$\mathrm{ht}(z) := t - \tfrac{1}{4}|X|^2 = \operatorname{Re}\zeta - \tfrac{1}{4}|v|^2.$$

Formula (1.5) implies that $s = \mathrm{ht}(z)$ and $n = (1, Z, X)$. Hence the H-coordinates of z are

$$(\mathrm{ht}(z), Z, X)_h = (\mathrm{ht}(z), \operatorname{Im}\zeta, v)_h.$$

This means that the H-coordinates are given by the height of the horosphere on which z lies and the coordinates of z in the canonical parametrization of this horosphere.

The bijection between N and $\mathfrak{z} \times \mathfrak{v}$ induces an inner product on N (which is independent of the height level set).

Definition 2.2.3. Let $(G, \cdot)$ be a group with neutral element 1_G. We call a map $p \colon G \to \mathbb{R}_0^+$ a *group norm* if

(GN1) $p(g) = 0 \Leftrightarrow g = 1_G$,

(GN2) $p(g^{-1}) = p(g)$ for all $g \in G$,

(GN3) $p(gh) \le p(g) + p(h)$ for all $g, h \in G$.

Definition and Remark 2.2.4. If p is a group norm on the group G, then the map $d\colon G\times G\to\mathbb{R}$, $d(g,h):=p(g^{-1}h)$ is a metric on G. It is called the *induced metric from p.*

The *Heisenberg group norm q* (also known as the *Heisenberg pseudonorm*) on each height level set ($\cong N$) is defined by

$$q(Z,X):=\left|\tfrac{1}{4}|X|^2+Z\right|^{1/2}=\left(\tfrac{1}{16}|X|^4+|Z|^2\right)^{1/4}.$$

The equality holds because $\mathfrak{z}$ and $\mathbb{R}\cong\mathfrak{a}$ are orthogonal. The induced metric measures the distance between two elements in some height level set.

To be able to also measure the distance between elements in different height level sets, we extend the Heisenberg pseudonorm to the direct product of the groups $(\mathbb{R},+)$ ("differences between height level sets") and N by

$$p\colon\begin{cases}\mathbb{R}\times N & \to \quad \mathbb{R}\\ (k,Z,X) & \mapsto \quad \left|\tfrac{1}{4}|X|^2+|k|+Z\right|^{1/2}.\end{cases}$$

Obviously, $p|_{\{0\}\times N}=q$.

Proposition 2.2.5. *The map p is a group norm on $\mathbb{R}\times N$.*

Proof. Since $(0,0,0)$ is the neutral element of $\mathbb{R}\times N$, the map p obviously satisfies (GN1). Further for each $g=(k,Z,X)\in\mathbb{R}\times N$ we have

$$\begin{aligned}p(g^{-1})&=p(-k,-Z,-X)=\left|\tfrac{1}{4}|X|^2+|k|-Z\right|^{1/2}=\left(\left(\tfrac{1}{4}|X|^2+|k|\right)^2+|Z|^2\right)^{1/4}\\&=\left|\tfrac{1}{4}|X|^2+|k|+Z\right|^{1/2}=p(g).\end{aligned}$$

This shows (GN2). The triangle equality (GN3) is done in several steps. For each $g=(k,Z,X)\in\mathbb{R}\times N$ it follows

$$\begin{aligned}p(g)&=\left|\tfrac{1}{4}|X|^2+|k|+Z\right|^{1/2}=\left(\left(\tfrac{1}{4}|X|^2+|k|\right)^2+|Z|^2\right)^{1/4}\\&\geq\left(\left(\tfrac{1}{4}|X|^2\right)^2\right)^{1/4}=\tfrac{1}{2}|X|.\end{aligned}$$

Therefore Proposition 2.1.3 implies that for all $(k_j,Z_j,X_j)\in\mathbb{R}\times N$, $j=1,2$, we have

$$\tfrac{1}{4}|\beta_2(X_2,X_1)|\leq\tfrac{|X_1|}{2}\tfrac{|X_2|}{2}\leq p(k_1,Z_1,X_1)p(k_2,Z_2,X_2).$$

Moreover, for all $(Z,X)\in N$ and $k_1,k_2\in\mathbb{R}$ we find

$$\begin{aligned}\left|\tfrac{1}{4}|X|^2+|k_1+k_2|+Z\right|^{1/2}&=\left(\left(\tfrac{1}{4}|X|^2+|k_1+k_2|\right)^2+|Z|^2\right)^{1/4}\\&\leq\left(\left(\tfrac{1}{4}|X|^2+|k_1|+|k_2|\right)^2+|Z|^2\right)^{1/4}\\&=\left|\tfrac{1}{4}|X|^2+|k_1|+|k_2|+Z\right|^{1/2}.\end{aligned}$$

Now let $g_j = (k_j, Z_j, X_j) \in \mathbb{R} \times N$, $j = 1, 2$. Then

$$\begin{aligned}
p(g_1 g_2) &= p\big(k_1 + k_2, Z_1 + Z_2 + \tfrac{1}{2}[X_1, X_2], X_1 + X_2\big) \\
&= \left| \tfrac{1}{4}|X_1 + X_2|^2 + |k_1 + k_2| + Z_1 + Z_2 + \tfrac{1}{2}[X_1, X_2] \right|^{1/2} \\
&\leq \left| \tfrac{1}{4}|X_1 + X_2|^2 + |k_1| + |k_2| + Z_1 + Z_2 + \tfrac{1}{2}[X_1, X_2] \right|^{1/2} \\
&= \left| \tfrac{1}{4}|X_1|^2 + |k_1| + Z_1 + \tfrac{1}{4}|X_2|^2 + |k_2| + Z_2 + \tfrac{1}{2}\left(\langle X_1, X_2\rangle + [X_1, X_2]\right) \right|^{1/2}.
\end{aligned}$$

Lemma 1.5.3 shows that then

$$\begin{aligned}
p(g_1 g_2) &\leq \left| \tfrac{1}{4}|X_1|^2 + |k_1| + Z_1 + \tfrac{1}{4}|X_2|^2 + |k_2| + Z_2 + \tfrac{1}{2}\beta_2(X_2, X_1) \right|^{1/2} \\
&\leq \left[\left| \tfrac{1}{4}|X_1|^2 + |k_1| + Z_1 \right| + \left| \tfrac{1}{4}|X_2|^2 + |k_2| + Z_2 \right| + \tfrac{1}{2}\left|\beta_2(X_2, X_1)\right| \right]^{1/2} \\
&= \left[p(g_1)^2 + p(g_2)^2 + \tfrac{1}{2}|\beta_2(X_2, X_1)| \right]^{1/2} \\
&\leq \left[p(g_1)^2 + p(g_2)^2 + 2p(g_1)p(g_2) \right]^{1/2} \\
&= p(g_1) + p(g_2),
\end{aligned}$$

which completes the proof. □

We define the injective map

$$\kappa\colon \begin{cases} \overline{D}^g \smallsetminus \{\infty\} & \to \quad \mathbb{R} \times N \\ \qquad z & \mapsto \quad \Theta^{-1}(z), \end{cases}$$

where $\kappa(z)$ not only means that we consider the H-coordinates of z, but also that we can perform the group operations of $\mathbb{R} \times N$ on $\kappa(z_1), \kappa(z_2)$ for elements $z_1, z_2 \in \overline{D}^g \smallsetminus \{\infty\}$.

Definition 2.2.6. The *Cygan metric on D* is the metric

$$\varrho\colon \begin{cases} \overline{D}^g \smallsetminus \{\infty\} \times \overline{D}^g \smallsetminus \{\infty\} & \to \quad \mathbb{R} \\ \qquad (g_1, g_2) & \mapsto \quad p\big(\kappa(g_1)^{-1}\kappa(g_2)\big). \end{cases}$$

Since κ is injective, the Cygan metric is in fact a metric on $\overline{D}^g \smallsetminus \{\infty\}$.

Lemma 2.2.7. *Let $z_j = (\zeta_j, v_j) = (k_j, Z_j, X_j)_h$ be elements of $\overline{D}^g \smallsetminus \{\infty\}$. Then the Cygan metric is given by*

$$\begin{aligned}
\varrho(z_1, z_2) &= \left| \tfrac{1}{4}|X_1 - X_2|^2 + |k_1 - k_2| + Z_1 - Z_2 + \tfrac{1}{2}\operatorname{Im}\beta_2(X_2, X_1) \right|^{1/2} \\
&= \left| \tfrac{1}{4}|v_1|^2 + \tfrac{1}{4}|v_2|^2 + |\operatorname{ht}(z_1) - \operatorname{ht}(z_2)| + \operatorname{Im}\zeta_1 - \operatorname{Im}\zeta_2 - \tfrac{1}{2}\beta_2(v_1, v_2) \right|^{1/2}.
\end{aligned}$$

Proof. Lemma 1.5.3 yields

$$\begin{aligned}
\kappa(z_2)^{-1}\kappa(z_1) &= (-k_2, -Z_2, -X_2)(k_1, Z_1, X_1) \\
&= \big(k_1 - k_2, Z_1 - Z_2 + \tfrac{1}{2}[-X_2, X_1], X_1 - X_2\big) \\
&= \big(k_1 - k_2, Z_1 - Z_2 + \tfrac{1}{2}[X_1, X_2], X_1 - X_2\big) \\
&= \big(k_1 - k_2, Z_1 - Z_2 + \tfrac{1}{2}\operatorname{Im}\beta_2(X_2, X_1), X_1 - X_2\big).
\end{aligned}$$

Note that the inversion and multiplication is done in the group $\mathbb{R} \times N$. Using (GN2) for p we get

$$\begin{aligned}\varrho(z_1, z_2) &= p\big(\kappa(z_1)^{-1}\kappa(z_2)\big) = p\big(\kappa(z_2)^{-1}\kappa(z_1)\big)\\ &= p\big(k_1 - k_2, Z_1 - Z_2 + \tfrac{1}{2}\operatorname{Im}\beta_2(X_2, X_1), X_1 - X_2\big)\\ &= \big|\tfrac{1}{4}|X_1 - X_2|^2 + |k_1 - k_2| + Z_1 - Z_2 + \tfrac{1}{2}\operatorname{Im}\beta_2(X_2, X_1)\big|^{1/2}.\end{aligned}$$

This proves the first claimed equality. For the second we note that

$$\begin{aligned}\tfrac{1}{4}|X_1 - X_2|^2 + \tfrac{1}{2}\operatorname{Im}\beta_2(X_2, X_1) &= \tfrac{1}{4}|X_1|^2 + \tfrac{1}{4}|X_2|^2 - \tfrac{1}{2}\langle X_1, X_2\rangle - \tfrac{1}{2}\operatorname{Im}\beta_2(X_1, X_2)\\ &= \tfrac{1}{4}|X_1|^2 + \tfrac{1}{4}|X_2|^2 - \tfrac{1}{2}\beta_2(X_1, X_2),\end{aligned}$$

where the last equality follows from Lemma 1.5.3. Since $X_j = v_j$, $Z_j = \operatorname{Im}\zeta_j$ and $k_j = \operatorname{ht}(z_j)$ for $j = 1, 2$, the second equality holds. $\square$

Definition 2.2.8. Let $g \in G \smallsetminus G_\infty$ and suppose that $g = n_1\sigma m a_t n_2$ with $n_1, n_2 \in N$, $a_t \in A$ and $m \in M$. Then we define $R(g) := t^{-1/4}$. The set

$$I(g) := \big\{z \in D \bigm| \varrho\big(z, g^{-1}\infty\big) = R(g)\big\}$$

is called the *isometric sphere of* g. Further, the set

$$\operatorname{ext} I(g) := \big\{z \in D \bigm| \varrho\big(z, g^{-1}\infty\big) > R(g)\big\}$$

is called the *exterior* of $I(g)$, and

$$\operatorname{int} I(g) := \big\{z \in D \bigm| \varrho\big(z, g^{-1}\infty\big) < R(g)\big\}$$

the *interior* of $I(g)$. The value $R(g)$ is called the *radius* of $I(g)$.

Lemma 2.2.9. *Let $g = n_1\sigma m a_t n_2 \in G \smallsetminus G_\infty$ and $n_2 = (1, Z_{s2}, X_{s2})$. Then*

$$\varrho\big((\zeta, v), g^{-1}\infty\big) = \Big|\tfrac{1}{4}|X_{s2}|^2 + Z_{s2} + \zeta + \tfrac{1}{2}\beta_2(v, X_{s2})\Big|^{1/2}.$$

Further $\varrho\big(\cdot, g^{-1}\infty\big)$ is unbounded on D, and hence $\operatorname{ext} I(g) \neq \emptyset$.

Proof. We have (see Remark 1.6.4)

$$g^{-1} = n_2^{-1}a_t^{-1}m^{-1}\sigma n_1^{-1} = n_2^{-1}\sigma m a_t n_1^{-1}.$$

Since $G_\infty = MAN$, we have

$$g^{-1}\infty = n_2^{-1}\sigma m a_t n_1^{-1}\infty = n_2^{-1}\sigma\infty = n_2^{-1}0 = \big(\tfrac{1}{4}|X_{s2}|^2 - Z_{s2}, -X_{s2}\big).$$

Then Lemma 2.2.7 shows for all $(\zeta, v) \in \overline{D}^g \smallsetminus \{\infty\}$ we have

$$\begin{aligned}\varrho\big((\zeta, v), g^{-1}\infty\big) &= \Big|\tfrac{1}{4}|v|^2 + \tfrac{1}{4}|X_{s2}|^2 + \operatorname{Re}\zeta - \tfrac{1}{4}|v|^2 + \operatorname{Im}\zeta + Z_{s2} + \tfrac{1}{2}\beta_2(v, X_{s2})\Big|^{1/2}\\ &= \Big|\tfrac{1}{4}|X_{s2}|^2 + Z_{s2} + \zeta + \tfrac{1}{2}\beta_2(v, X_{s2})\Big|^{1/2}.\end{aligned}$$

This proves the first claim. Now let $(\zeta, v) \in D$ and $t > 1$ be real. Then

$$\operatorname{ht}\big((t\zeta, v)\big) = t \operatorname{Re}\zeta - \tfrac{1}{4}|v|^2 > \operatorname{Re}\zeta - \tfrac{1}{4}|v|^2 > 0.$$

Hence $(t\zeta, v) \in D$ for all $t > 1$. Further

$$\begin{aligned}\varrho\big((t\zeta, v), g^{-1}\infty\big)^2 &= \big|t\zeta + \tfrac{1}{4}|X_{s2}|^2 + Z_{s2} + \tfrac{1}{2}\beta_2(v, X_{s2})\big| \\ &\geq \Big||t\zeta| - \big|\tfrac{1}{4}|X_{s2}|^2 + Z_{s2} + \tfrac{1}{2}\beta_2(v, X_{s2})\big|\Big| \\ &\geq t|\zeta| - \big|\tfrac{1}{4}|X_{s2}|^2 + Z_{s2} + \tfrac{1}{2}\beta_2(v, X_{s2})\big|,\end{aligned}$$

which converges to ∞ for $t \to \infty$. Hence $\varrho\big(\cdot, g^{-1}\infty\big)$ is unbounded. This completes the proof. $\square$

Lemma 2.2.10. *Let $g \in G \smallsetminus G_\infty$. Then*

(i) $\operatorname{ext} I(g)$ *and* $\operatorname{int} I(g)$ *are open,*

(ii) $\overline{\operatorname{ext} I(g)} = \big\{z \in D \bigm| \varrho\big(z, g^{-1}\infty\big) \geq R(g)\big\} = \complement \operatorname{int} I(g)$,

(iii) $\overline{\operatorname{int} I(g)} = \big\{z \in D \bigm| \varrho\big(z, g^{-1}\infty\big) \leq R(g)\big\} = \complement \operatorname{ext} I(g)$.

(iv) *If* $(\zeta, v) \in \overline{\operatorname{int} I(g)}$, *then* $(\zeta - s, v) \in \operatorname{int} I(g)$ *for each* $s \in (0, \operatorname{Re}\zeta)$.

Proof. Suppose that $g = n_1 \sigma m a_t n_2$ with $n_2 = (1, Z_2, X_2)$. Then $R(g) = t^{-1/4}$ and, by Lemma 2.2.9,

$$\varrho\big((\zeta, v), g^{-1}\infty\big) = \big|\tfrac{1}{4}|X_2|^2 + Z_2 + \zeta + \tfrac{1}{2}\beta_2(v, X_2)\big|^{1/2}$$

for each $(\zeta, v) \in D$. Since $\beta_2(\cdot, X_2)\colon V \to C$ is $\mathbb{R}$-linear (see Prop. 1.5.2) and the topology on C, V resp. D is that of an open subset of $\mathbb{R}^n$ for some n (depending on C, V and D separately), the map

$$f := \begin{cases} D & \to \quad \mathbb{R} \\ (\zeta, v) & \mapsto \quad \big|\tfrac{1}{4}|X_2|^2 + Z_2 + \zeta + \tfrac{1}{2}\beta_2(v, X_2)\big| \end{cases}$$

is continuous. Then

$$\begin{aligned}\operatorname{ext} I(g) &= \Big\{(\zeta, v) \in D \Bigm| \big|\tfrac{1}{4}|X_2|^2 + Z_2 + \zeta + \tfrac{1}{2}\beta_2(v, X_2)\big| > t^{-1/2}\Big\} \\ &= f^{-1}\big((t^{-1/2}, \infty)\big)\end{aligned}$$

and

$$\operatorname{int} I(g) = f^{-1}\big((-\infty, t^{-1/2})\big),$$

which shows that $\operatorname{ext} I(g)$ and $\operatorname{int} I(g)$ are open. Moreover, it follows that $\overline{\operatorname{int} I(g)} \subseteq f^{-1}\big((-\infty, t^{-1/2}]\big)$. To prove the converse inclusion relation, it suffices to show that $f^{-1}(t^{-1/2}) \subseteq \overline{\operatorname{int} I(g)}$. Let $z_0 = (\zeta_0, v_0) \in f^{-1}(t^{-1/2})$. Then

$$\begin{aligned}t^{-1/2} &= \big|\tfrac{1}{4}|X_2|^2 + Z_2 + \zeta_0 + \tfrac{1}{2}\beta_2(v_0, X_2)\big| \\ &= \Big[\big|\tfrac{1}{4}|X_2|^2 + \operatorname{Re}\zeta_0\big|^2 + \big|Z_2 + \operatorname{Im}\zeta_0 + \tfrac{1}{2}\beta_2(v_0, X_2)\big|^2\Big]^{1/2}.\end{aligned}$$

Since $z_0 \in D$, we have $\operatorname{Re}\zeta_0 > 0$. Then for each $s \in (0, \operatorname{Re}\zeta_0)$ it follows

$$\begin{aligned} t^{-1/2} &> \left[\left|\tfrac{1}{4}|X_2|^2 + \operatorname{Re}\zeta_0 - s\right|^2 + \left|Z_2 + \operatorname{Im}\zeta_0 + \tfrac{1}{2}\beta_2(v_0, X_2)\right|^2\right]^{1/2} \\ &= \left|\tfrac{1}{4}|X_2|^2 + Z_2 + \zeta_0 - s + \tfrac{1}{2}\beta_2(v_0, X_2)\right|. \end{aligned}$$

Thus, $(\zeta_0 - s, v_0) \in \operatorname{int} I(g)$ for each $s \in (0, \operatorname{Re}\zeta_0)$ and hence

$$\lim_{s \searrow 0}(\zeta_0 - s, v_0) = (\zeta_0, v_0) \ \in \overline{\operatorname{int} I(g)}.$$

This proves (iii) and (iv). The proof of (ii) is analogous to that of (iii). ☐

Proposition 2.2.11. *We have*

$$\overline{\bigcap_{g \in \Gamma \setminus \Gamma_\infty} \operatorname{ext} I(g)} = \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \overline{\operatorname{ext} I(g)} = D \setminus \bigcup_{g \in \Gamma \setminus \Gamma_\infty} \operatorname{int} I(g).$$

Proof. Lemma 2.2.10(iii) states that $\complement \operatorname{ext} I(g) = \overline{\operatorname{int} I(g)}$ for each $g \in \Gamma \setminus \Gamma_\infty$. Therefore

$$\begin{aligned} \complement\left(\overline{\bigcap_{g \in \Gamma \setminus \Gamma_\infty} \operatorname{ext} I(g)}\right) &= \left(\complement \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \operatorname{ext} I(g)\right)^\circ = \left(\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \complement \operatorname{ext} I(g)\right)^\circ \\ &= \left(\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \overline{\operatorname{int} I(g)}\right)^\circ. \end{aligned}$$

The interior of each isometric sphere is open. Thus

$$\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \operatorname{int} I(g) = \left(\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \operatorname{int} I(g)\right)^\circ$$

and hence

$$\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \operatorname{int} I(g) \subseteq \left(\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \overline{\operatorname{int} I(g)}\right)^\circ.$$

It remains to prove that the converse inclusion relation. To that end pick some $z = (\zeta, v) \in \left(\bigcup\{\overline{\operatorname{int} I(g)} \mid g \in \Gamma \setminus \Gamma_\infty\}\right)^\circ$ and fix $\varepsilon > 0$ such that

$$z_\varepsilon := z + \varepsilon = (\zeta + \varepsilon, v) \in \bigcup_{g \in \Gamma \setminus \Gamma_\infty} \overline{\operatorname{int} I(g)}.$$

Pick $k \in \Gamma \setminus \Gamma_\infty$ such that $z_\varepsilon \in \overline{\operatorname{int} I(k)}$. Then Lemma 2.2.10(iv) shows that $z = z_\varepsilon - \varepsilon \in \operatorname{int} I(k)$. Thus $z \in \bigcup\{\operatorname{int} I(g) \mid g \in \Gamma \setminus \Gamma_\infty\}$. This shows that

$$\left(\bigcup_{g \in \Gamma \setminus \Gamma_\infty} \overline{\operatorname{int} I(g)}\right)^\circ \subseteq \bigcup_{g \in \Gamma \setminus \Gamma_\infty} \operatorname{int} I(g)$$

and hence

$$\complement \bigcup_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{int} I(g) = \overline{\bigcap_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{ext} I(g)}.$$

Finally we have $\complement \operatorname{int} I(g) = \overline{\operatorname{ext} I(g)}$ for each $g \in \Gamma\smallsetminus\Gamma_\infty$ by Lemma 2.2.10(iii). Therefore

$$\overline{\bigcap_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{ext} I(g)} = \complement \bigcup_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{int} I(g) = \bigcap_{g\in\Gamma\setminus\Gamma_\infty} \complement \operatorname{int} I(g) = \bigcap_{g\in\Gamma\smallsetminus\Gamma_\infty} \overline{\operatorname{ext} I(g)}. \qquad \square$$

The next two lemmas use the following explicit expression for the action of $g \in G\smallsetminus G_\infty = N\sigma MAN$ on some element $(\zeta, v) \in \overline{D}^g\smallsetminus\{\infty\}$. Let $g = n_1\sigma m a_t n_2$ with $n_j = (1, Z_{sj}, X_{sj})$ and $m = (\varphi, \psi)$. A lengthy but easy calculation shows that

$$g(\zeta, v) = \Big(\tfrac{1}{4}|X_{s1}|^2 + Z_{s1} + xt^{-1} - \tfrac{1}{2}t^{-1/2}\beta_2(x\psi(X_{s2} + v), X_{s1}), \qquad\qquad (2.2)$$
$$X_{s1} - xt^{-1/2}\psi(X_{s2} + v)\Big)$$

where

$$x := \left[\varphi\left(\tfrac{1}{4}|X_{s2}|^2 + Z_{s2} + \zeta + \tfrac{1}{2}\beta_2(v, X_{s2})\right)\right]^{-1}.$$

Lemma 2.2.12. *If $g \in G\smallsetminus G_\infty$, then g maps $\operatorname{ext} I(g)$ onto $\operatorname{int} I(g^{-1})$ and $I(g)$ onto $I(g^{-1})$.*

Proof. Let $g = n_1\sigma m a_t n_2$ with $n_j = (1, Z_{sj}, X_{sj})$ and $m = (\varphi, \psi)$. Then

$$g^{-1} = n_2^{-1}a_t^{-1}m^{-1}\sigma n_1^{-1} = n_2^{-1}\sigma m a_t n_1^{-1},$$

hence $R(g) = R(g^{-1}) = t^{-1/4}$. It is

$$g^{-1}I(g^{-1}) = \big\{z \in D \bigm| \varrho(gz, g\infty) = R(g^{-1})\big\}.$$

Therefore we will calculate $\varrho(gz, g\infty)$ and compare it to $\varrho\big(z, g^{-1}\infty\big)$. Let (ζ, v) be in D. Set

$$x := \left[\varphi\left(\tfrac{1}{4}|X_{s2}|^2 + Z_{s2} + \zeta + \tfrac{1}{2}\beta_2(v, X_{s2})\right)\right]^{-1}.$$

Then (2.2) and Lemma 2.2.9 show that

$$\begin{aligned}
&\varrho\big(g(\zeta, v), g\infty\big) = \\
&\quad = \Big|\tfrac{1}{4}|X_{s1}|^2 - Z_{s1} + \tfrac{1}{4}|X_{s1}|^2 + Z_{s1} + xt^{-1} - \tfrac{1}{2}t^{-1/2}\beta_2\big(x\psi(X_{s2} + v), X_{s1}\big) \\
&\qquad\quad - \tfrac{1}{2}\beta_2\big(X_{s1} - xt^{-1/2}\psi(X_{s2} + v), X_{s1}\big)\Big|^{1/2} \\
&\quad = \Big|\tfrac{1}{2}|X_{s1}|^2 + xt^{-1} - \tfrac{1}{2}t^{-1/2}\beta_2\big(x\psi(X_{s2} + v), X_{s1}\big) - \tfrac{1}{2}\beta_2(X_{s1}, X_{s1}) \\
&\qquad\quad + \tfrac{1}{2}t^{-1/2}\beta_2\big(x\psi(X_{s2} + v), X_{s1}\big)\Big|^{1/2} \\
&\quad = \big|\tfrac{1}{4}|X_{s1}|^2 + xt^{-1} - \tfrac{1}{2}\beta_2(X_{21}, X_{21})\big|^{1/2}.
\end{aligned}$$

Prop. 1.5.2 states that $\beta_2(X_{21}, X_{21}) = \langle X_{21}, X_{21}\rangle = |X_{21}|^2$. Thus,

$$\varrho\big(g(\zeta, v), g\infty\big) = \left|xt^{-1}\right|^{1/2} = |x|^{1/2}t^{-1/2}.$$

Since $\varphi \in O(C)$, it follows

$$\begin{aligned}|x|^{1/2} &= \left|\varphi\left(\tfrac{1}{4}|X_{s2}|^2 + Z_{s2} + \zeta + \tfrac{1}{2}\beta_2(v, X_{s2})\right)\right|^{-1/2}\\ &= \left|\tfrac{1}{4}|X_{s2}|^2 + Z_{s2} + \zeta + \tfrac{1}{2}\beta_2(v, X_{s2})\right|^{-1/2}\\ &= \varrho\big((\zeta, v), g^{-1}\infty\big)^{-1}.\end{aligned}$$

Therefore

$$\varrho\big((\zeta, v), g^{-1}\infty\big)\varrho\big(g(\zeta, v), g\infty\big) = t^{-1/2}.$$

Hence $(\zeta, v) \in I(g)$ if and only if

$$t^{-1/2} = t^{-1/4}\varrho\big(g(\zeta, v), g\infty\big),$$

which is equivalent to $\varrho\big(g(\zeta, v), g\infty\big) = t^{-1/4}$. This is the case if and only if $g(\zeta, v) \in I(g^{-1})$. Thus, $gI(g) = I(g^{-1})$.

Further, $(\zeta, v) \in \operatorname{ext} I(g)$ if and only if

$$t^{-1/2} > t^{-1/4}\varrho\big(g(\zeta, v), g\infty\big),$$

which is the case if and only if $\varrho\big(g(\zeta, v), g\infty\big) < t^{-1/4}$. This is equivalent to $g(\zeta, v) \in \operatorname{int} I(g^{-1})$. Therefore $g \operatorname{ext} I(g) = \operatorname{int} I(g^{-1})$. $\square$

Lemma 2.2.13. *Let $n = (1, Z, X) \in N$, $a_s \in A$ and $m = (\varphi, \psi) \in M$.*

(i) *If $n' := \big(1, sZ, s^{1/2}X\big)$, then $a_s n = n' a_s$.*

(ii) *For all $u, v \in V$ we have $\varphi\big(\beta_2(v, u)\big) = \beta_2(\psi(v), \psi(u))$.*

(iii) *If $n' := \big(1, \varphi(Z), \psi(X)\big)$, then $mn = n'm$.*

Proof. Formula (1.2) shows

$$a_s n = (s, 0, 0)(1, Z, X) = \big(s, sZ, s^{1/2}X\big) = \big(1, sZ, s^{1/2}X\big)\big(s, 0, 0\big) = n' a_s,$$

which proves (i). To show (ii) let $u, v \in V$. Recall that $\varphi \in O(C)$, $\psi \in O(V)$ and that $\psi\big(J(\eta, w)\big) = J\big(\varphi(\eta), \psi(w)\big)$ for each $(\eta, w) \in C \oplus V$. For each $\zeta \in C$ we have

$$\begin{aligned}\big\langle \varphi\big(\beta_2(v, u)\big), \zeta\big\rangle &= \big\langle \beta_2(v, u), \varphi^{-1}(\zeta)\big\rangle = \big\langle J\big(\varphi^{-1}(\zeta), u\big), v\big\rangle\\ &= \big\langle \psi\big(J\big(\varphi^{-1}(\zeta), u\big)\big), \psi(v)\big\rangle = \langle J(\zeta, \psi(u)), \psi(v)\rangle\\ &= \langle \beta_2(\psi(v), \psi(u)), \zeta\rangle\,.\end{aligned}$$

Thus $\varphi\big(\beta_2(v, u)\big) = \beta_2(\psi(v), \psi(u))$. For the proof of (iii) let $(\zeta, v) \in D$. Then

$$\begin{aligned}m\big(n(\zeta, v)\big) &= m\left(\tfrac{1}{4}|X|^2 + Z + \zeta + \tfrac{1}{2}\beta_2(v, X), X + v\right)\\ &= \left(\varphi\left(\tfrac{1}{4}|X|^2 + Z + \zeta + \tfrac{1}{2}\beta_2(v, X)\right), \psi(X + v)\right)\\ &= \left(\tfrac{1}{4}|\psi(X)|^2 + \varphi(Z) + \varphi(\zeta) + \tfrac{1}{2}\varphi\big(\beta_2(v, X)\big), \psi(X) + \psi(v)\right).\end{aligned}$$

Now (ii) yields

$$\begin{aligned} m\big(n(\zeta,v)\big) &= \big(\tfrac{1}{4}|\psi(X)|^2 + \varphi(Z) + \varphi(\zeta) + \tfrac{1}{2}\beta_2\big(\psi(v),\psi(X)\big), \psi(X)+\psi(v)\big) \\ &= n'\big(\varphi(\zeta),\psi(v)\big) = n'\big(m(\zeta,v)\big). \end{aligned}$$

Hence $mn = n'm$. ☐

Proposition 2.2.14. *Let $g \in G_\infty$ and $h \in G \smallsetminus G_\infty$. Then*

$$\begin{aligned} gI(h) &= I\big(ghg^{-1}\big), \\ g \operatorname{int} I(h) &= \operatorname{int} I\big(ghg^{-1}\big), \\ g \operatorname{ext} I(h) &= \operatorname{ext}\big(ghg^{-1}\big). \end{aligned}$$

Proof. We prove the assertion for the three cases $g \in A$, $g \in N$ and $g \in M$. Since $G_\infty = MAN$, for general $g \in G_\infty$, the claim then follows immediately from composition. Let $h = n_1\sigma m a_t n_2$ with $n_j = (1, Z_j, X_j)$.

Suppose first that $g = a_s$ and let $(\zeta, v) \in D$. For $j \in \{1,2\}$ set $n_j' := \big(1, sZ_j, s^{1/2}X_j\big)$. Then Lemma 2.2.13 implies that

$$ghg^{-1} = a_s n_1 \sigma m a_t n_2 a_s^{-1} = n_1'\sigma m a_s^{-1} a_t a_s^{-1} n_2' = n_1' \sigma m a_{s^{-2}t} n_2'.$$

Therefore $R\big(ghg^{-1}\big) = s^{1/2}t^{-1/4} = s^{1/2}R(h)$. By Lemma 2.2.9 we have

$$\varrho\big((\zeta,v),(ghg^{-1})^{-1}\infty\big) = \left|\tfrac{1}{4}s|X_2|^2 + sZ_2 + \zeta + \tfrac{1}{2}s^{1/2}\beta_2(v,X_2)\right|^{1/2}$$

and

$$\begin{aligned} \varrho\big(a_s^{-1}(\zeta,v), h^{-1}\infty\big) &= \varrho\big((s^{-1}\zeta, s^{-1/2}v), h^{-1}\infty\big) \\ &= \left|\tfrac{1}{4}|X_2|^2 + Z_2 + s^{-1}\zeta + \tfrac{1}{2}s^{-1/2}\beta_2(v,X_2)\right|^{1/2} \\ &= s^{-1/2}\varrho\big((\zeta,v),(ghg^{-1})^{-1}\infty\big). \end{aligned}$$

Then

$$\begin{aligned} gI(h) &= \big\{ gz \in D \,\big|\, \varrho\big(z, h^{-1}\infty\big) = R(h) \big\} \\ &= \big\{ z \in D \,\big|\, \varrho\big(g^{-1}z, h^{-1}\infty\big) = R(h) \big\} \\ &= \Big\{ z \in D \,\Big|\, \varrho\big(z, (ghg^{-1})^{-1}\infty\big) = s^{1/2}R(h) \Big\} \\ &= \big\{ z \in D \,\big|\, \varrho\big(z, (ghg^{-1})^{-1}\infty\big) = R(ghg^{-1}) \big\} \\ &= I\big(ghg^{-1}\big). \end{aligned}$$

Analogously, we see that $g \operatorname{ext} I(h) = \operatorname{ext} I\big(ghg^{-1}\big)$ and $g \operatorname{int} I(h) = \operatorname{int} I\big(ghg^{-1}\big)$. Now suppose that $g = n_3 = (1, Z_3, X_3)$. Then $ghg^{-1} = (n_3 n_1)\sigma m a_t (n_2 n_3^{-1})$ and

$$\begin{aligned} n_2 n_3^{-1} &= \big(1, Z_2 - Z_3 - \tfrac{1}{2}[X_2, X_3], X_2 - X_3\big) \\ &= \big(1, Z_2 - Z_3 - \tfrac{1}{2}\operatorname{Im}\beta_2(X_3, X_2), X_2 - X_3\big), \end{aligned}$$

where the last equality follows from Lemma 1.5.3. Therefore, by Lemmas 2.2.9 and 1.5.3,

$$\begin{aligned}
&\varrho\big((\zeta,v),(n_3hn_3^{-1})^{-1}\infty\big) = \\
&\quad = \Big|\tfrac14|X_2-X_3|^2 + Z_2 - Z_3 - \tfrac12\operatorname{Im}\beta_2(X_3,X_2) + \zeta + \tfrac12\beta_2(v,X_2-X_3)\Big|^{1/2} \\
&\quad = \Big|\tfrac14|X_2|^2 + \tfrac14|X_3|^2 - \tfrac12\langle X_2,X_3\rangle - \tfrac12\operatorname{Im}\beta_2(X_3,X_2) + \\
&\qquad\qquad + Z_2 - Z_3 + \zeta + \tfrac12\beta_2(v,X_2-X_3)\Big|^{1/2} \\
&\quad = \Big|\tfrac14|X_2| + \tfrac14|X_3|^2 - \tfrac12\beta_2(X_3,X_2) + Z_2 - Z_3 + \zeta + \tfrac12\beta_2(v,X_2-X_3)\Big|^{1/2}.
\end{aligned}$$

We have

$$n_3^{-1}(\zeta,v) = \big(\tfrac14|X_3|^2 - Z_3 + \zeta - \tfrac12\beta_2(v,X_3), -X_3+v\big)\,.$$

Hence

$$\begin{aligned}
&\varrho\big(n_3^{-1}(\zeta,v),h^{-1}\infty\big) = \\
&\quad = \Big|\tfrac14|X_2|^2 + Z_2 + \tfrac14|X_3|^2 - Z_3 + \zeta - \tfrac12\beta_2(v,X_3) + \tfrac12\beta_2(-X_3+v,X_2)\Big|^{1/2} \\
&\quad = \Big|\tfrac14|X_2|^2 + \tfrac14|X_3|^2 + Z_2 - Z_3 + \zeta - \tfrac12\beta_2(v,X_3) - \tfrac12\beta_2(X_3,X_2) + \\
&\qquad\qquad + \tfrac12\beta_2(v,X_2)\Big|^{1/2} \\
&\quad = \Big|\tfrac14|X_2|^2 + \tfrac14|X_3|^2 + Z_2 - Z_3 + \zeta - \tfrac12\beta_2(X_3,X_2) + \tfrac12\beta_2(v,X_2-X_3)\Big|^{1/2} \\
&\quad = \varrho\big((\zeta,v),n_3h^{-1}n_3^{-1}\infty\big).
\end{aligned}$$

Since $R(h) = R\big(ghg^{-1}\big)$, the claim follows for this case.

Finally suppose that $g = m_2 = (\varphi,\psi)$ and set $n_j' := \big(1,\varphi(Z_j),\psi(X_j)\big)$ for $j\in\{1,2\}$. Lemma 2.2.13 shows that

$$ghg^{-1} = m_2n_1\sigma ma_tn_2m_2^{-1} = n_1'm_2\sigma ma_tm_2^{-1}n_2' = n_1'\sigma m_2^{-1}mm_2^{-1}a_tn_2'.$$

From this it follows that $R(h) = R\big(m_2hm_2^{-1}\big)$ and further that

$$\varrho\big((\zeta,v),(ghg^{-1})^{-1}\infty\big) = \Big|\tfrac14|\psi(X_2)|^2 + \varphi(Z_2) + \zeta + \tfrac12\beta_2(v,\psi(X_2))\Big|^{1/2}.$$

From Lemma 2.2.13 it follows that

$$\varrho\big((\zeta,v),(ghg^{-1})^{-1}\infty\big) = \Big|\tfrac14|X_2|^2 + \varphi(Z_2) + \zeta + \tfrac12\varphi\big(\beta_2(\psi^{-1}(v),X_2)\big)\Big|^{1/2}.$$

On the other side we calculate

$$\begin{aligned}
\varrho\big(g^{-1}(\zeta,v),h^{-1}\infty\big) &= \varrho\big((\varphi^{-1}(\zeta),\psi^{-1}(v)),h^{-1}\infty\big) \\
&= \Big|\tfrac14|X_2|^2 + Z_2 + \varphi^{-1}(\zeta) + \tfrac12\beta_2(\psi^{-1}(v),X_2)\Big|^{1/2} \\
&= \Big|\tfrac14|X_2|^2 + \varphi(Z_2) + \zeta + \tfrac12\varphi\big(\beta_2(\psi^{-1}(v),X_2)\big)\Big|^{1/2} \\
&= \varrho\big((\zeta,v),gh^{-1}g^{-1}\infty\big).
\end{aligned}$$

Thus, also in this case we have $gI(h) = I\big(ghg^{-1}\big)$, $g\operatorname{ext} I(h) = \operatorname{ext} I\big(ghg^{-1}\big)$ and $g\operatorname{int} I(h) = \operatorname{int} I\big(ghg^{-1}\big)$. This completes the proof. $\square$

An immediate corollary of Lemma 2.2.14 is the following assertion.

Corollary 2.2.15. *Let Γ be a subgroup of G and $g \in \Gamma_\infty$. Then*

$$g \bigcap_{h\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{ext} I(h) = \bigcap_{h\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{ext} I(h)$$

and

$$g \bigcup_{h\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{int} I(h) = \bigcup_{h\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{int} I(h).$$

Lemma 2.2.16. *Let $z \in D$ and $g \in G\smallsetminus G_\infty$. Then*

$$\operatorname{ht}(z) = \left(\frac{\varrho(z, g^{-1}\infty)}{R(g)}\right)^4 \operatorname{ht}(gz).$$

Proof. Suppose that $z = (\zeta, v)$ and $g = n_1\sigma m a_t n_2$ with $n_j = (1, Z_j, X_j)$ and $m = (\varphi, \psi)$. We first evaluate $\operatorname{ht}(gz)$. Set

$$x := \left[\varphi\left(\tfrac14|X_2|^2 + Z_2 + \zeta + \tfrac12\beta_2(v, X_2)\right)\right]^{-1}.$$

By (2.2) we have

$$\begin{aligned}
\operatorname{ht}(gz) &= \operatorname{Re}\left(\tfrac14|X_1|^2 + Z_1 + xt^{-1} - \tfrac12 t^{-1}\beta_2\big(x\psi(X_2+v), X_1\big)\right) \\
&\qquad\qquad\qquad\qquad - \tfrac14\left|X_1 - xt^{-1/2}\psi(X_2+v)\right|^2 \\
&= \tfrac14|X_1|^2 + t^{-1}\operatorname{Re}(x) - \tfrac12 t^{-1/2}\operatorname{Re}\beta_2\big(x\psi(X_2+v), X_1\big) \\
&\qquad\qquad\qquad\qquad - \tfrac14\left|X_1 - xt^{-1/2}\psi(X_2+v)\right|^2 \\
&= \tfrac14|X_1|^2 + t^{-1}\operatorname{Re}(x) - \tfrac12 t^{-1/2}\operatorname{Re}\beta_2\big(x\psi(X_2+v), X_1\big) \\
&\qquad - \tfrac14|X_2|^2 - \tfrac14|x|^2t^{-1}|\psi(X_2+v)|^2 + \tfrac12\left\langle X_1, xt^{-1/2}\psi(X_2+v)\right\rangle.
\end{aligned}$$

Now Lemma 1.5.3 states that

$$t^{-1/2}\operatorname{Re}\beta_2\big(x\psi(X_2+v), X_1\big) = t^{-1/2}\left\langle x\psi(X_2+v), X_1\right\rangle = \left\langle X_1, xt^{-1/2}\psi(X_2+v)\right\rangle.$$

Therefore

$$\operatorname{ht}(gz) = t^{-1}\operatorname{Re}(x) - \tfrac14|x|^2t^{-1}|X_2+v|^2 = t^{-1}|x|^2\left[\operatorname{Re}(|x|^{-2}x) - \tfrac14|X_2+v|^2\right].$$

Since $\operatorname{Re}\varphi(\eta) = \operatorname{Re}\eta$ for each $\eta \in C$, it follows that

$$\begin{aligned}
\operatorname{Re}\left(|x|^{-2}x\right) &= \operatorname{Re}\left(\overline{x}^{-1}\right) = \operatorname{Re}\left(\overline{x^{-1}}\right) = \operatorname{Re}\left(x^{-1}\right) \\
&= \operatorname{Re}\left(\tfrac14|X_2|^2 + Z_2 + \zeta + \tfrac12\beta_2(v, X_2)\right) \\
&= \tfrac14|X_2|^2 + \operatorname{Re}\zeta + \tfrac12\operatorname{Re}\beta_2(v, X_2) \\
&= \tfrac14|X_2|^2 + \operatorname{Re}\zeta + \tfrac12\langle v, X_2\rangle \\
&= \tfrac14|X_2+v|^2 + \operatorname{Re}\zeta - \tfrac14|v|^2.
\end{aligned}$$

Thus,

$$\mathrm{ht}(gz) = t^{-1}|x|^2 \left(\operatorname{Re}\zeta - \tfrac{1}{4}|v|^2\right) = t^{-1}|x|^2\,\mathrm{ht}(z).$$

Lemma 2.2.9 shows that

$$\varrho\big((\zeta, v), g^{-1}\infty\big) = \left|\tfrac{1}{4}|X_2|^2 + Z_2 + \zeta + \tfrac{1}{2}\beta_2(v, X_2)\right|^{1/2} = |x|^{-1/2}.$$

Since $R(g) = t^{-1/4}$, we have

$$\mathrm{ht}(z) = t|x|^{-2}\,\mathrm{ht}(gz) = \left(\frac{\varrho(z, g^{-1}\infty)}{R(g)}\right)^4 \mathrm{ht}(gz).$$

This completes the proof. □

2.3. Fundamental region

A subset Y of a metric space X is a *fundamental region in* X for a group Γ of isometries on X if and only if it satisfies the following properties:

(F1) The set Y is open in X.

(F2) The members of $\{gY \mid g \in \Gamma\}$ are mutually disjoint.

(F3) $X = \bigcup\{g\overline{Y} \mid g \in \Gamma\}$.

If, in addition, Y is connected, then it is a *fundamental domain* for Γ in X.

Remark 2.3.1. There are various (non-equivalent) definitions of fundamental regions and fundamental domains in the literature. The definition above seems to be quite common, it is taken from [Rat06, p. 234]. The group Γ must necessarily be discrete to have a fundamental region [Rat06, Thm 6.5.3].

Definition 2.3.2. A subgroup Γ of G is said to be *of type (O)* if

$$\bigcap_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{ext} I(g) = D \setminus \overline{\bigcup_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{int} I(g)}.$$

Suppose that S is a subset of G and let $\langle S\rangle$ denote the subgroup of G generated by S. Then S is said to be of *of type (F)*, if for each $z \in D$ the maximum of the set

$$\left\{\, \mathrm{ht}(gz) \;\middle|\; g \in \langle S\rangle \right\}$$

exists.

Remark 2.3.3. Let Γ be a subgroup of G and suppose that the set

$$\{\operatorname{int} I(g) \mid g \in \Gamma\smallsetminus\Gamma_\infty\}$$

of interiors of all isometric spheres is locally finite. Then

$$\overline{\bigcup_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{int} I(g)} = \bigcup_{g\in\Gamma\smallsetminus\Gamma_\infty} \overline{\operatorname{int} I(g)}.$$

by [vQ79, Hilfssatz 7.14]. Lemma 2.2.10(iii) implies that

$$\complement\left(\overline{\bigcup_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{int} I(g)}\right) = \complement \bigcup_{g\in\Gamma\smallsetminus\Gamma_\infty} \overline{\operatorname{int} I(g)} = \bigcap_{g\in\Gamma\smallsetminus\Gamma_\infty} \complement\overline{\operatorname{int} I(g)} = \bigcap_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{ext} I(g).$$

Hence, if the set of interiors of isometric spheres is locally finite, then Γ is of type (O).

Theorem 2.3.4. *Let Γ be a subgroup of G of type (O) such that $\Gamma\smallsetminus\Gamma_\infty$ is of type (F). Suppose that $\mathcal{F}_\infty$ is a fundamental region for Γ_∞ in D satisfying*

$$\overline{\mathcal{F}}_\infty \cap \bigcap_{g\in\Gamma\smallsetminus\Gamma_\infty} \overline{\operatorname{ext} I(g)} = \overline{\mathcal{F}_\infty \cap \bigcap_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{ext} I(g)}.$$

Then

$$\mathcal{F} := \mathcal{F}_\infty \cap \bigcap_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{ext} I(g)$$

is a fundamental region for Γ in D. If, in addition, $\mathcal{F}_\infty$ is connected, then $\mathcal{F}$ is a fundamental domain for Γ in D.

Proof. Since Γ is of type (O), the set $\bigcap\{\operatorname{ext} I(g) \mid g \in \Gamma\smallsetminus\Gamma_\infty\}$ is open. Further $\mathcal{F}_\infty$ is open as a fundamental region for Γ_∞ in D. Thus, $\mathcal{F}$ is open. For the proof of (F2) let $z \in \mathcal{F}$ and $g \in \Gamma\smallsetminus\{\mathrm{id}\}$. If $g \in \Gamma_\infty$, then $gz \notin \mathcal{F}_\infty$ since $\mathcal{F}_\infty$ satisfies (F2) for Γ_∞. If $g \in \Gamma\smallsetminus\Gamma_\infty$, then $z \in \operatorname{ext} I(g)$. Lemma 2.2.12 states that $gz \in \operatorname{int} I\big(g^{-1}\big)$ and thus $gz \notin \operatorname{ext} I\big(g^{-1}\big)$. *A fortiori*, $gz \notin \bigcap_{h\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{ext} I(h)$. Therefore, in each case, $gz \notin \mathcal{F}$.

It remains to prove that $D \subseteq \Gamma\cdot\overline{\mathcal{F}}$. To that end let $z \in D$. Since $\Gamma\smallsetminus\Gamma_\infty$ is of type (F), the set $\langle\Gamma\smallsetminus\Gamma_\infty\rangle z$ contains an element of maximal height, say w. This means in particular that for all $h \in \Gamma\smallsetminus\Gamma_\infty$ we have

$$\operatorname{ht}(w) \geq \operatorname{ht}(hw). \tag{2.3}$$

We claim that $w \in \bigcap\big\{\overline{\operatorname{ext} I(g)} \bigm| g \in \Gamma\smallsetminus\Gamma_\infty\big\} =: A$. For contradiction assume that $w \notin A$. From Prop. 2.2.11 it follows that

$$w \in \complement A = \bigcup_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{int} I(g).$$

Hence we find $h \in \Gamma\smallsetminus\Gamma_\infty$ such that $w \in \operatorname{int} I(h)$. Then Lemma 2.2.16 implies that

$$\operatorname{ht}(hw) = \left(\frac{R(h)}{\varrho\big(w, h^{-1}\infty\big)}\right)^4 \operatorname{ht}(w) > \operatorname{ht}(w),$$

which contradicts (2.3). Thus, $w \in A$.

Since $\mathcal{F}_\infty$ satisfies (F3) for Γ_∞, there is $h \in \Gamma_\infty$ such that $hw \in \overline{\mathcal{F}}_\infty$. Cor. 2.2.15 implies that $hw \in A$. Finally

$$hw \in \overline{\mathcal{F}}_\infty \cap A = \overline{\mathcal{F}_\infty \cap \bigcap_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{ext} I(g)} = \overline{\mathcal{F}}.$$

This shows that $\mathcal{F}$ is a fundamental region for Γ in H. For each $g \in \Gamma \setminus \Gamma_\infty$, the set $\operatorname{ext} I(g)$ is connected. Hence, if $\mathcal{F}_\infty$ is connected, then $\mathcal{F}$ is connected. This completes the proof. □

In the proof of Theorem 2.3.4 we proved the following corollary.

Corollary 2.3.5. *Under the hypotheses of Theorem 2.3.4 we have*

$$\Gamma_\infty \cdot \overline{\mathcal{F}} = \overline{\bigcap_{g \in \Gamma \setminus \Gamma_\infty} \operatorname{ext} I(g)}.$$

The next corollary is a special case of Theorem 2.3.4.

Corollary 2.3.6. *Let Γ be a subgroup of G of type (O) and $\Gamma_\infty = \{\mathrm{id}\}$. Further suppose that $\Gamma \setminus \Gamma_\infty$ is of type (F). Then*

$$\mathcal{F} := \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \operatorname{ext} I(g)$$

is a fundamental region for Γ in D.

3. Projective models

The purpose of this section is to show that the existing definitions of isometric spheres and results concerning the existence of isometric fundamental regions in literature are essentially covered by the definitions and results in Sec. 2. The reason for the reservation towards a confirmation to cover all existing definitions and results is twofold: On the one hand the auther cannot guarantee to be aware of all existing results. On the other hand, at least for real hyperbolic plane, the literature contains non-equivalent definitions of isometric spheres. Moreover, the existence results of isometric fundamental regions by Ford are proved for a weaker notion of fundamental region than that used by us. Sec. 3.5 contains a detailed discussion of the latter issues.

Let (C, V, J) be a J^2C-module structure. In Sec. 3.1 we introduce the structure of division algebras on C following [KR05] and [KR]. For C being an associative division algebra, we redo, in Sec. 3.2, the classical projective construction of hyperbolic spaces in terms of the J^2C-module structure. A substantial part of Sec. 3.2 we spend on a detailed study of the relation between the isometry group G of the symmetric space and the natural "matrix" group on the projective space. This investigation will show that the matrix group is isomorphic to a certain subgroup G^{res} of G. In Sec. 3.3 we use these results to provide a characterization of the isometric sphere of $g \in G^{\mathrm{res}}$ via a cocycle. In Sec. 3.4 we prove that a special class of subgroups Γ of G^{res} are of type (O) with $\Gamma \smallsetminus \Gamma_\infty$ being of type (F) and use Theorem 2.3.4 to show the existence of an isometric fundamental domain for Γ. Finally, in Sec. 3.5, we bring together these investigations for a comparison with the existing literature.

3.1. Division algebras induced by J^2C-module structures

Let (C, V, J) be a J^2C-module structure. Suppose that $V \neq \{0\}$ and fix an element $v \in V \smallsetminus \{0\}$. Condition (M3) shows that for each pair $(\zeta, \eta) \in C \times C$ there exists (at least one) element $\tau \in C$ such that

$$J(\zeta, J(\eta, v)) = J(\tau, v).$$

We claim that τ is uniquely determined by (ζ, η) and v. Suppose that we have $\tau_1, \tau_2 \in C$ such that

$$J(\tau_1, v) = J(\zeta, J(\eta, v)) = J(\tau_2, v).$$

Then $J(\tau_1 - \tau_2, v) = 0$ and by (M2)

$$|\tau_1 - \tau_2| \cdot |v| = |J(\tau_1 - \tau_2, v)| = 0.$$

Since $|v| \neq 0$, it follows that $|\tau_1 - \tau_2| = 0$ and hence $\tau_1 = \tau_2$. Hence, each choice $v \in V \setminus \{0\}$ equips C with a multiplication $\cdot_v : C \times C \to C$ via

$$\zeta \cdot_v \eta := \tau \quad \Leftrightarrow \quad J(\zeta, J(\eta, v)) = J(\tau, v).$$

Equation (1.9) shows that the inverse of $\zeta \in C \setminus \{0\}$ is $\zeta^{-1} = |\zeta|^{-2}\overline{\zeta}$, independent of the choice of $v \in V \setminus \{0\}$.

The following properties of $(C, \cdot_v)$ are shown in [KR].

Proposition 3.1.1 (Prop. 1.1 and Cor. 1.5 in [KR])**.**

(i) *For each $v \in V \setminus \{0\}$, the Euclidean vector space C with the multiplication $\cdot_v$ is a normed, not necessarily associative, division algebra.*

(ii) *The multiplication $\cdot_v$ on C is independent of the choice of $v \in V \setminus \{0\}$ if and only if $(C, \cdot_v)$ is associative for one (and hence for all) v.*

We call the J^2C-module structure (C, V, J) *associative* if $(C, \cdot_v)$ is associative for some (and hence each) $v \in V \setminus \{0\}$ and otherwise *non-associative.* If $(C, \cdot_v)$ is associative, we will eliminate the subscript v of the multiplication $\cdot_v$.

Remark 3.1.2. Only associative J^2C-module structures are modules in the sense of [Bou98].

Remark 3.1.3. Suppose that (C, V, J) is an associative J^2C-module structure. Their classification in [KR05, Sec. 4] shows that C is real or complex or quaternionic numbers.

3.2. Projective construction in the associative cases

Throughout let (C, V, J) be an associative J^2C-module structure. The classical projective construction of real, complex and quaternionic hyperbolic n-space starts with a non-degenerate, indefinite C-sesquilinear hermitian form Φ of signature $(n, 1)$ on $E := C^{n+1}$, singles out the set of so-called Φ-negative vectors and considers the subspace of C-projective space of E defined by the set of Φ-negative vectors. Endowed with a certain Riemannian metric, this space becomes a rank one Riemannian symmetric space of noncompact type.

In Sec. 3.2.1 we recall the notion of C-sesquilinear hermitian forms and show the existence of an orthonormal C-basis of V. This allows us to define the C-projective space $P_C(E)$ and its differential structure in Sec. 3.2.2. Here we introduce the notion of Φ-negative, Φ-zero and Φ-positive vectors and prove that the boundary of the projective space $P_C\big(E_-(\Phi)\big)$ of Φ-negative vectors is the projective space of Φ-zero vectors. In Sec. 3.2.3 we specialize to two non-degenerate, indefinite C-sesquilinear hermitian forms Ψ_1 and Ψ_2. The manifold $P_C\big(E_-(\Psi_1)\big)$ is canonically related to the ball model B. The Siegel domain model, which is the model D up to a scaling in the V-coordinate, will be seen to be a natural space of representatives of $P_C\big(E_-(\Psi_2)\big)$. We investigate the relation between C-linear maps on E preserving the form Ψ_j and isometries of the symmetric space B. It will turn out that the projective group of these linear maps is isomorphic to a subgroup of the isometry group of B, which we can explicitly characterize.

3.2.1. C-sesquilinear hermitian forms

Definition 3.2.1. Let M be a (left) C-module and $\Phi\colon M \times M \to C$ a map. Then Φ is said to be a *C-sesquilinear hermitian form* if Φ is C-hermitian and C-linear in the first variable, that is, if the following properties are satisfied:

(SH1) $\Phi(\zeta_1 x_1 + \zeta_2 x_2, y) = \zeta_1 \Phi(x_1, y) + \zeta_2 \Phi(x_2, y)$ for all $\zeta_1, \zeta_2 \in C$ and $x, y \in M$,

(SH2) $\Phi(x, y) = \overline{\Phi(y, x)}$ for all $x, y \in M$.

A C-sesquilinear hermitian form Φ is called *non-degenerate* if $\Psi(m_1, \cdot) = 0$ implies that $m_1 = 0$. It is called *indefinite*, if there exist $m_1, m_2 \in M$ such that $\Phi(m_1, m_1) < 0$ and $\Phi(m_2, m_2) > 0$.

Proposition 3.2.2. *The map $\beta_1\colon C \times C \to C$, $\beta_1(x, y) := x\overline{y}$, is C-sesquilinear hermitian. Further* $\operatorname{Re}\beta_1(x, y) = \langle x, y\rangle$ *for all $x, y \in C$.*

Proof. Obviously, β_1 is $\mathbb{R}$-bilinear. For each $y \in C$, the C-linearity of $\beta_1(\cdot, y)$ is exactly the left-sided distribution law of the division algebra C. To show (SH2) let $x, y \in C$. Then $J_{y\overline{x}} = J_y J_{\overline{x}}$. From (1.7) it follows that

$$J_{\overline{y\overline{x}}} = J^*_{y\overline{x}} = J^*_{\overline{x}} J^*_y = J_x J_{\overline{y}} = J_{x\overline{y}}.$$

Therefore

$$\beta_1(x, y) = x\overline{y} = \overline{y\overline{x}} = \overline{\beta_1(y, x)}.$$

This completes the proof that β_1 is C-sesquilinear hermitian. For each $x \in C$ we have $\beta_1(x, x) = x\overline{x} = |x|^2$ by (1.8). Then polarization of $\beta_1(x, x) = |x|^2$ over $\mathbb{R}$ implies the remaining statement. □

Remark 3.2.3. An immediate consequence of Proposition 3.2.2 is that conjugation and multiplication in C anticommute, i. e., $\overline{x}\,\overline{y} = \overline{yx}$ for all $x, y \in C$. In particular, for a C-sesquilinear hermitian form Φ on the C-module M we have

$$\Phi(m_1, \zeta m_2) = \Phi(m_1, m_2)\overline{\zeta}$$

for all $m_1, m_2 \in M$ and all $\zeta \in C$. Moreover, for each $m \in M$ we have $\Phi(m, m) = \overline{\Phi(m, m)}$ and therefore $\Phi(m, m) \in \mathbb{R}$.

Lemma 3.2.4. *For all $\zeta, \eta, \xi \in C$ we have $\langle \zeta, \xi\eta\rangle = \langle \overline{\xi}\zeta, \eta\rangle$.*

Proof. Let $v \in V \setminus \{0\}$. Using (1.6) and (1.7) we find

$$2\langle \zeta, \xi\eta\rangle |v|^2 = 2\langle \zeta v, \xi\eta v\rangle = 2\langle \overline{\xi}\zeta v, \eta v\rangle = 2\langle \overline{\xi}\zeta, \eta\rangle |v|^2,$$

hence $\langle \zeta, \xi\eta\rangle = \langle \overline{\xi}\zeta, \eta\rangle$. □

Proposition 3.2.5. *The map $\beta_2\colon V \times V \to C$ is C-sesquilinear hermitian.*

Proof. Because of Prop. 1.5.2 it remains to show that β_2 is C-linear in the first variable. Let $v, u \in V$ and $\zeta \in C$. Lemma 3.2.4 and (1.7) yield that for all $\eta \in C$ we have

$$\langle \zeta\beta_2(v,u), \eta\rangle = \langle \beta_2(v,u), \overline{\zeta}\eta\rangle = \langle \overline{\zeta}\eta u, v\rangle = \langle \eta u, \zeta v\rangle = \langle \beta_2(\zeta v, u), \eta\rangle.$$

Hence $\zeta\beta_2(v,u) = \beta_2(\zeta v, u)$. □

Definition 3.2.6. A finite sequence $(v_1, \ldots, v_n)$ in V is called an *orthonormal C-basis* of V if

(CON1) $|v_j| = 1$ for each $j \in \{1, \ldots, n\}$,

(CON2) for each pair $(i,j) \in \{1, \ldots, n\}^2$, $i \neq j$, the sets Cv_i and Cv_j are orthogonal,

(CON3) if we use for each $j \in \{1, \ldots, n\}$ the bijection $C \to Cv_j$, $\zeta \mapsto \zeta v_j$ to equip Cv_j with the structure of an Euclidean vector space and a C-module, then V is isomorphic as C-module and Euclidean vector space to the direct sum $\bigoplus_{j=1}^n Cv_j$ of the Euclidean spaces and C-modules Cv_j, $j = 1, \ldots, n$.

The following lemma states that V is a free C-module.

Lemma 3.2.7. *There is an orthonormal C-basis of V.*

Proof. Clearly, one finds a sequence $(v_1, \ldots, v_n)$ in V which satisfies (CON1) and (CON2) such that V is isomorphic as Euclidean vector space to the direct sum $\bigoplus_{j=1}^n Cv_j$. From Prop. 3.2.5 and 1.5.2 it follows that

$$\psi\colon \begin{cases} V & \to & \bigoplus_{j=1}^n Cv_j \\ v & \mapsto & \sum_{j=1}^n \beta_2(v, v_j)v_j \end{cases}$$

is an isomorphism (of Euclidean vector spaces) from V to $\bigoplus_{j=1}^n Cv_j$. Let $v \in V$, $\eta \in C$ and suppose that v is isomorphic to $(\zeta_1 v_1, \ldots, \zeta_n v_n) \in \bigoplus_{j=1}^n Cv_j$. Then

$$\psi(\eta v) = \sum_{j=1}^n \beta_2(\eta v, v_j)v_j = \sum_{j=1}^n \big(\eta\beta_2(v, v_j)\big)v_j = \eta\Big(\sum_{j=1}^n \beta_2(v, v_j)v_j\Big) = \eta\psi(v).$$

This shows that ψ is indeed an isomorphism of C-modules. □

3.2.2. The C-projective space $P_C(E)$

Let $W := C \oplus V$ be the Euclidean direct sum of C and V and let $E := C \oplus W = C \oplus C \oplus V$ be that of C and W. Consider the map

$$\cdot\colon \begin{cases} C \times E & \to & E \\ (\tau, (\zeta, \eta, v)) & \mapsto & (\tau\zeta, \tau\eta, \tau v). \end{cases}$$

Since (C, V, J) is associative and hence $\sigma(\tau v) = (\sigma\tau)v$ for each $v \in V$ and all $\sigma, \tau \in C$ (or equivalently, since the definition of the product in C does not depend on the choice of $v \in V \setminus \{0\}$, see Prop. 3.1.1), E becomes a C-module. Since V is a free C-module by Lemma 3.2.7, also E is a free C-module.

Two elements z_1, z_2 of $E \setminus \{0\}$ are called *equivalent* ($z_1 \sim z_2$) if there is $\tau \in C$ such that

$$\tau z_1 = z_2.$$

Note that τ is actually in $C \setminus \{0\}$. Then E being a C-module guarantees that $\sim$ is an equivalence relation on $E \setminus \{0\}$. The *C-projective space* $P_C(E)$ of E is defined as the set of equivalence classes of $\sim$,

$$P_C(E) := \big(E \setminus \{0\}\big)/_{\sim},$$

endowed with the induced topology and the differential structure generated by the following (standard) charts: Let $\{v_1, \ldots, v_{n-1}\}$ be an orthonormal C-basis of V. Then E is isomorphic to C^{n+1} both as Euclidean vector space and C-module. For each $j = 1, \ldots, n+1$ the set

$$U_j := \big\{[(\zeta_1, \ldots, \zeta_{n+1})] \in P_C(E) \bigm| \zeta_j \neq 0\big\}$$

is open, and the maps $\varphi_j \colon U_j \to C^n$

$$\varphi_j\big([(\zeta_1, \ldots, \zeta_{n+1})]\big) := \zeta_j^{-1}(\zeta_1, \ldots, \widehat{\zeta_j}, \ldots, \zeta_{n+1})$$

are pairwise compatible in the sense that they are real differentiable (they are not C-differentiable unless C is commutative). Here, $\widehat{\zeta_j}$ means that ζ_j is omitted, hence $(\zeta_1, \ldots, \widehat{\zeta_j}, \ldots, z_{n+1}) = (\zeta_1, \ldots, \zeta_{j-1}, \zeta_{j+1}, \ldots, \zeta_{n+1}) \in C^n$. Obviously, the differential structure is independent from the choice of the orthonormal C-basis of V, and $P_C(E)$ is a real smooth manifold of dimension $n \cdot \dim_{\mathbb{R}} C$.

Let Φ be a C-sesquilinear hermitian form Φ on E. Recall from Remark 3.2.3 that $\Phi(z, z) \in \mathbb{R}$ for each $z \in E$. Suppose that q is its *associated quadratic form*, that is

$$q\colon \begin{cases} E & \to \quad \mathbb{R} \\ z & \mapsto \quad \Phi(z, z). \end{cases}$$

Then we define the following sets:

$$\begin{aligned} E_-(\Phi) &:= q^{-1}\big((-\infty, 0)\big) && \text{the set of } \Phi\text{-}\textit{negative vectors}, \\ E_0(\Phi) &:= q^{-1}(0) \setminus \{0\} && \text{the set of } \Phi\text{-}\textit{zero vectors}, \text{ and} \\ E_+(\Phi) &:= q^{-1}\big((0, \infty)\big) && \text{the set of } \Phi\text{-}\textit{positive vectors}. \end{aligned}$$

We note that for each $\tau \in C$ and $z \in E \setminus \{0\}$ we have

$$q(\tau z) = \Phi(\tau z, \tau z) = \tau \Phi(z, z)\overline{\tau} = \Phi(z, z)\tau\overline{\tau} = q(z)|\tau|^2. \tag{3.1}$$

Lemma 3.2.8. *Let Φ be a C-sesquilinear hermitian form on E. The set $P_C(E)$ equals the disjoint union $P_C\big(E_-(\Phi)\big) \cup P_C\big(E_0(\Phi)\big) \cup P_C\big(E_+(\Phi)\big)$.*

Proof. Clearly, $P_C(E) = P_C\big(E_-(\Phi)\big) \cup P_C\big(E_0(\Phi)\big) \cup P_C\big(E_+(\Phi)\big)$. Hence it remains to prove that this union is disjoint. Suppose that q denotes the quadratic form associated with Φ. If we assume for contradiction that

$$P_C\big(E_-(\Phi)\big) \cap P_C\big(E_0(\Phi)\big) \neq \emptyset,$$

then there are equivalent $z_1, z_2 \in E\setminus\{0\}$ such that $q(z_1) < 0$ and $q(z_2) = 0$. But then there is $\tau \in C\setminus\{0\}$ such that

$$0 = q(z_2) = q(\tau z_1) = |\tau|^2 q(z_1) < 0,$$

which is a contraction. Hence $P_C\big(E_-(\Phi)\big) \cap P_C\big(E_0(\Phi)\big) = \emptyset$. Analogously we see that $P_C\big(E_-(\Phi)\big) \cap P_C\big(E_+(\Phi)\big) = \emptyset$ and $P_C\big(E_+(\Phi)\big) \cap P_C\big(E_0(\Phi)\big) = \emptyset$. $\square$

Let $\mathrm{GL}_C(E)$ be the group of all C-linear invertible maps $E \to E$. For a C-sesquilinear hermitian form Φ on E we set

$$U(\Phi, C) := \big\{ g \in \mathrm{GL}_C(E) \;\big|\; \forall\, z_1, z_2 \in E\colon \Phi(gz_1, gz_2) = \Phi(z_1, z_2) \big\}.$$

This subgroup of $\mathrm{GL}_C(E)$ will be of special importance in the next sections.

Let $\pi\colon E\setminus\{0\} \to P_C(E)$ denote the projection on the equivalence classes. Since $C\setminus\{0\}$ acts homeomorphically on $E\setminus\{0\}$, the projection π is open. Further π is continuous by the definition of the topology on $P_C(E)$.

Proposition 3.2.9. *Let $\Phi\colon E \times E \to C$ be a C-sesquilinear hermitian form, which is indefinite and non-degenerate. Then*

$$\partial P_C\big(E_-(\Phi)\big) = P_C\big(E_0(\Phi)\big).$$

Proof. Let $U \subseteq P_C\big(E_-(\Phi)\big)$. All complements, closures, interiors, and boundaries of subsets of $E\setminus\{0\}$ are taken in $E\setminus\{0\}$. At first we show that

$$\complement\pi\left(\complement\partial\pi^{-1}(U)\right) = \partial U. \tag{3.2}$$

To that end let $M := \pi^{-1}(U)$. Then $\complement M = \pi^{-1}\left(\complement U\right)$. From π being open and continuous it follows that $(\complement U)^\circ = \pi\big((\complement M)^\circ\big)$. This yields

$$\pi\left(\complement\overline{M}\right) = \pi\left(\left(\complement M\right)^\circ\right) = \left(\complement U\right)^\circ = \complement\overline{U},$$

hence $\complement\pi(\complement\overline{M}) = \overline{U}$. Again from π being open and continuous we get $U^\circ = \pi\left(M^\circ\right)$. Therefore

$$\begin{aligned}
\complement\pi\left(\complement\partial\pi^{-1}(U)\right) &= \complement\pi\left(\complement\partial M\right)\\
&= \complement\pi\left(\complement\big(\overline{M} \cap \complement M^\circ\big)\right)\\
&= \complement\left(\pi\left(\complement\overline{M}\right) \cup \pi\left(M^\circ\right)\right)\\
&= \complement\pi\left(\complement\overline{M}\right) \cap \complement\pi\left(M^\circ\right)\\
&= \overline{U} \cap \complement U^\circ = \partial U.
\end{aligned}$$

Let q denote the quadratic form associated to Φ. Then q is smooth. Since Φ is nondegenerate, 0 is a regular value for $q|_{E\setminus\{0\}}$. Hence $E_0(\Phi) = q^{-1}(0)\setminus\{0\}$ is the boundary of the bounded submanifold $q^{-1}((-\infty, 0])\setminus\{0\}$ and therefore also of $E_-(\Phi) = q^{-1}((-\infty, 0))$. Then the statement follows from (3.2) and Lemma 3.2.8 with $U := P_C\big(E_-(\Phi)\big)$. $\square$

3.2.3. The C-hyperbolic spaces $P_C(E_-(\Psi_1))$ and $P_C(E_-(\Psi_2))$

We define two specific non-degenerate, indefinite C-sesquilinear hermitian forms Ψ_1 and Ψ_2 and consider the manifolds $P_C\big(E_-(\Psi_j)\big)$ defined in Sec. 3.2.2. For each space we choose a set of representatives and a Riemannian metric on it such that $P_C\big(E_-(\Psi_1)\big)$ is essentially the ball model from Sec. 1.4.2 and $P_C\big(E_-(\Psi_2)\big)$ becomes the Siegel domain model for hyperbolic space, which is essentially the model D from Sec. 1.4.1. Recall that G denotes the full isometry group of B resp. D and let $Z(\Psi_j, C)$ be the center of $U(\Psi_j, C)$. The purpose of this section is to establish a natural and explicit isomorphism between the quotient group $\mathrm{PU}(\Psi_j, C) := U(\Psi_j, C)/Z(\Psi_j, C)$ and a subgroup G^{res} of G. Moreover, we will explicitly characterize G^{res}. In Sec. 3.3, the isomorphism is used to show that the definition of isometric spheres in literature is subsumed by our definition.

The closed unit ball $\overline{B}$ in $W = C \oplus V$ is a canonical set of representatives for $\overline{P_C\big(E_-(\Psi_1)\big)}$. We define a Riemannian metric on B, which, up to a multiplicative factor of 4, is identical to (1.10). Taking advantage of (C, V, J) being associative, the formula for the Riemannian metric simplifies considerably in comparison with (1.10). Each element of $U(\Psi_1, C)$ induces a map on B. The explicit and easy to handle expression for the Riemannian metric on B allows to show that each induced map is an isometry. Moreover, we can show that precisely the elements of $Z(\Psi_1, C)$ induce the identity on B. Hence $\mathrm{PU}(\Psi_1, C)$ is isomorphic to a subgroup G^{res} of G. For a characterization of this subgroup, we will switch to the manifold $P_C\big(E_-(\Psi_2)\big)$. As before, we choose a natural set H of representatives for $P_C\big(E_-(\Psi_2)\big)$ resp. a set $\overline{H}^g$ of representatives for $\overline{P_C\big(E_-(\Psi_2)\big)}$. There are two canonical ways to define a Riemannian metric on H.

On the one hand, the space H resp. $\overline{H}^g$ is a linear rescaling of D resp. $\overline{D}^g$ in the V-coordinate, which allows to endow H with the Riemannian metric from D. Then $\overline{H}^g$ becomes the geodesic closure of H. On the other hand, there exists an element $T \in \mathrm{GL}_C(E)$ such that $\Psi_2 \circ (T \times T) = \Psi_1$. The map T factors to a diffeomorphism between H and B resp. between $\overline{H}^g$ and $\overline{B}$. Hence we can endow H with the pull-back of the Riemannian metric of B via T as well. Since this diffeomorphism turns out to be the Cayley transform, one easily sees that both canonical Riemannian metrics on H are essentially the same. Then $\mathrm{PU}(\Psi_2, C)$ is isomorphic to G^{res} and the explicit formulas for the action of an element $g \in G$ on D from Sec. 1.6 translate to H without effort, preserving their simple structure. In turn, we are able to characterize the group G^{res} in G, and to give explicit representatives in $U(\Psi_2, C)$ for each element $g \in G^{\mathrm{res}}$. We will see that for real hyperbolic spaces the group G^{res} is all of G, but for complex and quaternionic hyperbolic spaces G^{res} is a strict subgroup of G.

Let $\beta_3\colon W \times W \to C$ be the sum of β_1 and β_2, hence

$$\beta_3\big((\eta_1, v_1), (\eta_2, v_2)\big) := \beta_1(\eta_1, \eta_2) + \beta_2(v_1, v_2).$$

Then we define the maps $\Psi_j\colon E\times E\to C$ $(j=1,2)$ by

$$\begin{aligned}\Psi_1\big((\zeta_1,\eta_1,v_1),(\zeta_2,\eta_2,v_2)\big) &:= -\beta_1(\zeta_1,\zeta_2)+\beta_1(\eta_1,\eta_2)+\beta_2(v_1,v_2)\\ &= -\beta_1(\zeta_1,\zeta_2)+\beta_3\big((\eta_1,v_1),(\eta_2,v_2)\big)\end{aligned}$$

and

$$\Psi_2\big((\zeta_1,\eta_1,v_1),(\zeta_2,\eta_2,v_2)\big) := -\beta_1(\zeta_1,\eta_2)-\beta_1(\eta_1,\zeta_2)+\beta_2(v_1,v_2).$$

Let q_j denote the associated quadratic forms. Then for all $(\zeta,w)=(\zeta,\eta,v)\in E$ we have

$$q_1\big((\zeta,w)\big)=q_1\big((\zeta,\eta,v)\big)=-|\zeta|^2+|\eta|^2+|v|^2=-|\zeta|^2+|w|^2.$$

Moreover, employing Prop 3.2.2, it follows that

$$\begin{aligned}q_2\big((\zeta,\eta,v)\big) &= -\beta_1(\zeta,\eta)-\beta_1(\eta,\zeta)+|v|^2\\ &= -\Big(\beta_1(\zeta,\eta)+\overline{\beta_1(\zeta,\eta)}\Big)+|v|^2\\ &= -2\operatorname{Re}\beta_1(\zeta,\eta)+|v|^2=-2\langle\zeta,\eta\rangle+|v|^2.\end{aligned}$$

Lemma 3.2.10. *For $j=1,2$, the map Ψ_j is a C-sesquilinear hermitian form on E which is non-degenerate and indefinite.*

Proof. Propositions 3.2.2 and 3.2.5 imply that Ψ_j is C-sesquilinear hermitian. Let $(\zeta,\eta,v)\in E$ such that for all $(\sigma,\tau,u)\in E$ we have

$$\Psi_1\big((\zeta,\eta,v),(\sigma,\tau,u)\big)=0.$$

Prop. 3.2.2 and Prop. 1.5.2 show that

$$0=\Psi_1\big((\zeta,\eta,v),(0,\eta,v)\big)=\beta_1(\eta,\eta)+\beta_2(v,v)=|\eta|^2+|v|^2.$$

Hence $(\eta,v)=(0,0)$. Further

$$0=\Psi_1\big((\zeta,\eta,v),(\zeta,0,0)\big)=-\beta_1(\zeta,\zeta)=-|\zeta|^2,$$

and therefore $\zeta=0$. This shows that Ψ_1 is non-degenerate. Suppose now that $(\zeta,\eta,v)\in E$ such that for all $(\sigma,\tau,u)\in E$ we have

$$\Psi_2\big((\zeta,\eta,v),(\sigma,\tau,u)\big)=0.$$

Then

$$0=\Psi_2\big((\zeta,\eta,v),(0,0,v)\big)=|v|^2$$

and therefore $v=0$. Moreover,

$$0=\Psi_2\big((\zeta,\eta,v),(0,\zeta,0)\big)=-|\zeta|^2.$$

Thus $\zeta=0$ and analogously $\eta=0$. Hence Ψ_2 is non-degenerate. Finally, for $v\in V\smallsetminus\{0\}$,

$$\begin{aligned}q_1\big((1,0,0)\big)&=-1 &\quad\text{and}\quad& q_1\big((0,0,v)\big)=|v|^2>0,\\ q_2\big((1,1,0)\big)&=-2 &\quad\text{and}\quad& q_2\big((0,0,v)\big)=|v|^2>0.\end{aligned}$$

Therefore, Ψ_1 and Ψ_2 are indefinite. □

Set of representatives for $\overline{P_C(E_-(\Psi_1))}$

Proposition 3.2.9 shows that for $j \in \{1,2\}$ we have

$$\overline{P_C\big(E_-(\Psi_j)\big)} = P_C\big(E_-(\Psi_j)\big) \cup P_C\big(E_0(\Psi_j)\big).$$

If $z = (\zeta, w) \in E_-(\Psi_1) \cup E_0(\Psi_1)$, then $\zeta \neq 0$. Therefore the element $[z] \in P_C\big(E_-(\Psi_1)\big) \cup P_C\big(E_0(\Psi_1)\big)$ is represented by $(1, \zeta^{-1}w)$, and this is the unique representative of the form $(1, *)$. If $z \in E_-(\Psi_1)$, then

$$0 > q_1\big(1, \zeta^{-1}w\big) = -1 + |\zeta^{-1}w|^2.$$

This shows that $1 > |\zeta^{-1}w|^2$. Conversely, if $w \in W$ with $|w|^2 < 1$, then $[(1,w)]$ is an element of $P_C\big(E_-(\Psi_1)\big)$. Recall the unit ball $B = \{w \in W \mid |w| < 1\}$ in W. Then $P_C\big(E_-(\Psi_1)\big)$ and B are in bijection via the map

$$[(\zeta, w)] \mapsto \zeta^{-1}w.$$

The same argument shows that $\partial P_C\big(E_-(\Psi_1)\big) = P_C\big(E_0(\Psi_1)\big)$ is bijective to ∂B via the same map. We define $\tau_B \colon \overline{P_C\big(E_-(\Psi_1)\big)} \to \overline{B}$ via

$$\tau_B\big([(\zeta, w)]\big) := \zeta^{-1}w.$$

Its inverse is

$$\tau_B^{-1}(w) = [(1,w)].$$

In comparison with Sec. 3.2.2, we see that $\overline{P_C\big(E_-(\Psi_1)\big)}$ is a subset of U_1 and τ_B a restriction of φ_1. Therefore, τ_B is a diffeomorphism between the manifolds $P_C\big(E_-(\Psi_1)\big)$ and B, and also between the manifolds with boundary $\overline{P_C\big(E_-(\Psi_1)\big)}$ and $\overline{B}$.

Riemannian metric on B

Since B is an open subset of the vector space W, we may and shall identify the tangent space at a point of B with W. We define a Riemannian metric $\tilde{\varrho}$ on B by

$$\begin{aligned}\tilde{\varrho}(p)(X,Y) &:= \frac{\langle X, Y\rangle}{1-|p|^2} + \frac{\langle \beta_3(X,p), \beta_3(Y,p)\rangle}{(1-|p|^2)^2} \\ &= \frac{1}{(1-|p|^2)^2}\operatorname{Re}\Big((1-|p|^2)\beta_3(X,Y) + \beta_1\big(\beta_3(X,p), \beta_3(Y,p)\big)\Big)\end{aligned}$$

for all $p \in B$ and all $X, Y \in T_pB = W$. Prop. 3.2.12 below shows that $\tilde{\varrho}$ essentially coincides with the Riemannian metric defined in (1.10).

Lemma 3.2.11. *Let $x, y \in W$. Then $\beta_3(x,y) = 0$ if and only if $y \in (Cx)^\perp$.*

Proof. Let $\zeta, \eta, \xi \in C$. Proposition 3.2.2 yields

$$\langle \zeta\overline{\eta}, \xi\rangle = \operatorname{Re}\beta_1\big(\zeta\overline{\eta}, \xi\big) = \operatorname{Re}\big(\zeta\overline{\eta}\,\overline{\xi}\big) = \operatorname{Re}\big(\zeta\overline{\xi\eta}\big) = \operatorname{Re}\beta_1(\zeta, \xi\eta) = \langle \zeta, \xi\eta\rangle.$$

By Riesz' representation theorem we know that $\beta_3(x, y) = 0$ if and only if

$$\langle \beta_3(x, y), \xi\rangle = 0 \quad \text{for all } \xi \in C.$$

Supposing that $x = (\zeta, v), y = (\eta, u) \in C \oplus V$, it follows from the definitions of β_3, β_1, β_2 and the inner product on W that

$$\begin{aligned}\langle \beta_3(x, y), \xi\rangle &= \langle \beta_1(\zeta, \eta), \xi\rangle + \langle \beta_2(v, u), \xi\rangle = \langle \zeta\overline{\eta}, \xi\rangle + \langle \xi u, v\rangle \\ &= \langle \zeta, \xi\eta\rangle + \langle v, \xi u\rangle = \langle \overline{\xi}\zeta, \eta\rangle + \langle \overline{\xi} v, u\rangle \\ &= \langle \overline{\xi} x, y\rangle.\end{aligned}$$

Hence $\beta_3(x, y) = 0$ if and only if $y \in (Cv)^{\perp}$. □

Proposition 3.2.12. *The map $\widetilde{\varrho}$ coincides, up to a factor of 4, with the Riemannian metric given by (1.10) on B.*

Proof. For $p = 0$ and all $X, Y \in T_0 B$ we have

$$\widetilde{\varrho}(0)(X, Y) = \langle X, Y\rangle = \tfrac{1}{4}\langle X, Y\rangle_{0-}.$$

Suppose that $p = (\zeta, v) \in B \setminus \{0\}$. We claim that the equivalence class Cp (see Sec. 1.4.2) coincides with the C-orbit $C \cdot p$ of p. For $\zeta = 0$ this is obviously true. Suppose that $\zeta \neq 0$. For each $\tau \in C \setminus \{0\}$, we have $\tau p = (\tau\zeta, \tau v)$ and

$$(\tau\zeta)^{-1}(\tau v) = \zeta^{-1}\tau^{-1}\tau v = \zeta^{-1} v.$$

Hence $(\zeta, v) \sim (\tau\zeta, \tau v)$. If $\tau = 0$, then $\tau p = 0 \in Cp$ by definition. Conversely, suppose that $(\eta, u) \in Cp \setminus \{0\}$. Then

$$\eta\zeta^{-1} p = \eta\zeta^{-1}(\zeta, v) = \big(\eta, \eta\zeta^{-1} v\big) = \big(\eta, \eta\eta^{-1} u\big) = (\eta, u).$$

Thus, $(\eta, u) \in C \cdot p$. Clearly, $0 \in Cp \cap C \cdot p$. This shows that $Cp = C \cdot p$.

Now let $p \in B \setminus \{0\}$ and $X, Y \in T_p B$. Suppose first that $X, Y \in (Cp)^{\perp}$. Then Lemma 3.2.11 shows that

$$\widetilde{\varrho}(p)(X, Y) = \frac{\langle X, Y\rangle}{1 - |p|^2} = \tfrac{1}{4}\langle X, Y\rangle_{p-}.$$

Suppose now that $X \in Cp$ and $Y \in (Cp)^{\perp}$ (or vice versa). Then

$$\widetilde{\varrho}(p)(X, Y) = \frac{\langle X, Y\rangle}{1 - |p|^2} = 0 = \tfrac{1}{4}\langle X, Y\rangle_{p-}.$$

Finally suppose that $X, Y \in Cp$. Then $X = \tau_1 p$ and $Y = \tau_2 p$. From Lemma 1.5.3, Prop. 3.2.2 and 3.2.5 it follows that

$$\begin{aligned}\langle X, Y\rangle &= \operatorname{Re}\beta_3(X, Y) = \operatorname{Re}\beta_3(\tau_1 p, \tau_2 p) = \operatorname{Re}\big(\tau_1\beta_3(p, p)\overline{\tau}_2\big) = |p|^2\operatorname{Re}(\tau_1\overline{\tau}_2) \\ &= |p|^2\langle \tau_1, \tau_2\rangle.\end{aligned}$$

Then

$$\begin{aligned}
\widetilde{\varrho}(p)(X,Y) &= \frac{\langle X,Y\rangle}{1-|p|^2} + \frac{\langle \beta_3(\tau_1 p,p),\beta_3(\tau_2 p,p)\rangle}{(1-|p|^2)^2} \\
&= \frac{|p|^2\langle \tau_1,\tau_2\rangle}{1-|p|^2} + \frac{\langle \tau_1\beta_3(p,p),\tau_2\beta_3(p,p)\rangle}{(1-|p|^2)^2} \\
&= \frac{|p|^2\langle \tau_1,\tau_2\rangle}{1-|p|^2} + \frac{|p|^4\langle \tau_1,\tau_2\rangle}{(1-|p|^2)^2} \\
&= \frac{|p|^2\langle \tau_1,\tau_2\rangle}{(1-|p|^2)^2} \\
&= \frac{\langle X,Y\rangle}{(1-|p|^2)^2} = \tfrac{1}{4}\langle X,Y\rangle_{p-}.
\end{aligned}$$

This completes the proof. □

Induced Riemannian isometries on B

Let $\pi_B := \tau_B \circ \pi\colon E_-(\Psi_1) \to B$ and suppose that $g \in U(\Psi_1, C)$. Since g is C-linear, it induces a (unique) map $\widetilde{g}$ on B by requiring the diagram

$$\begin{tikzcd}
E_-(\Psi_1) \arrow[r, "g"] \arrow[d, "\pi_B"'] & E_-(\Psi_1) \arrow[d, "\pi_B"] \\
B \arrow[r, dotted, "\widetilde{g}"] & B.
\end{tikzcd}$$

to commute. The next goal is to show that each such induced map is a Riemannian isometry on B.

To simplify proofs we fix an orthonormal C-basis $\mathcal{B}(V) := \{v_1,\dots,v_{n-1}\}$ for V. Then $\mathcal{B}(E) := \{e_1,e_2,\dots,e_{n+1}\}$, given by

$$\begin{aligned}
&e_1 := (1,0,0), \quad e_2 := (0,1,0)\\
&e_k := (0,0,v_{k-2}) \qquad \text{for } k = 3,\dots,n+1,
\end{aligned}$$

is an orthonormal C-basis for E. In the following we will identify each element in E, W and V, and each map $g \in U(\Psi_1, C)$ with its representative with respect to $\mathcal{B}(E)$. Since E is a left C-module, the representing vector of $z \in E$ is a row, and the application of a map $g \in U(\Psi_1, C)$ to z corresponds to the multiplication of the row vector of z to the matrix of g (and not vice versa, as usually in linear algebra). Further we use the notation z^* for $\overline{z}^\top$.

Lemma 3.2.13. *Let $v = (\zeta_1,\dots,\zeta_{n-1}), u = (\eta_1,\dots,\eta_{n-1}) \in V = C^{n-1}$. Then*

$$\beta_2(v,u) = \sum_{j=1}^{n-1} \zeta_j\overline{\eta_j} = vu^*.$$

Proof. Let $v_j, v_k \in \mathcal{B}(V)$. If $j = k$, then $\beta_2(v_j,v_k) = |v_j|^2 = 1$. If $j \neq k$, then v_j is orthogonal to Cv_k, hence

$$\langle \beta_2(v_j,v_k),\xi\rangle = \langle \xi v_k, v_j\rangle = 0$$

for each $\xi \in C$, hence $\beta_2(v_j, v_k) = 0$. The claim now follows by C-sesquilinearity of β_2. $\square$

Remark 3.2.14. Let $g \in U(\Psi_1, C)$ and suppose that

$$g = \begin{pmatrix} a & b \\ c^\top & A \end{pmatrix}$$

w. r. t. to $\mathcal{B}(E)$, where $a \in C$, $b, c \in C^n$ and $A \in C^{n\times n}$. Then the induced map $\tilde{g}\colon B \to B$ is given by

$$\widetilde{g}(p) = (a + pc^\top)^{-1}(b + pA). \tag{3.3}$$

Further, the representing matrix of Ψ_1 is $\left(\begin{smallmatrix} -1 & \\ & I \end{smallmatrix}\right)$. Since g preserves Ψ_1, we find the conditions

$$\begin{aligned}
\begin{pmatrix} -1 & \\ & I \end{pmatrix} &= \begin{pmatrix} a & b \\ c^\top & A \end{pmatrix} \begin{pmatrix} -1 & \\ & I \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ b^* & A^* \end{pmatrix} \\
&= \begin{pmatrix} -a & b \\ -c^\top & A \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ b^* & A^* \end{pmatrix} \\
&= \begin{pmatrix} -|a|^2 + |b|^2 & -a\overline{c} + bA^* \\ -c^\top \overline{a} + Ab^* & -c^\top \overline{c} + AA^* \end{pmatrix}
\end{aligned} \tag{3.4}$$

on the entries of g.

In Prop. 3.2.15 below we show that the induced map $\widetilde{g}$ of $g \in U(\Psi_1, C)$ is a Riemannian isometry. The proof consists mainly of a long concentrated calculation.

Proposition 3.2.15. *Let $g \in U(\Psi_1, C)$. The induced map on B is a Riemannian isometry.*

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c^\top & A \end{smallmatrix}\right) \in U(\Psi_1, C)$ with $a \in C$, $b, c \in C^n$ and $A \in C^{n\times n}$. Recall the induced map $\widetilde{g}$ from (3.3) and let $p \in B$ and $X, Y \in T_pB$. The derivative of $\tilde{g}$ at p is

$$\begin{aligned}
\tilde{g}'(p)X &= -(a + pc^\top)^{-1} X c^\top (a + pc^\top)^{-1}(b + pA) + (a + pc^\top)^{-1} XA \\
&= (a + pc^\top)^{-1} \left(-Xc^\top \tilde{g}(p) + XA \right).
\end{aligned}$$

Let $\tilde{X} := \tilde{g}'(p)X$ and $\tilde{Y} := \tilde{g}'(p)Y$. We have to show that

$$\tilde{\varrho}\big(\tilde{g}(p)\big)\big(\tilde{X}, \tilde{Y}\big) = \tilde{\varrho}(p)(X, Y),$$

for which we first evaluate some components of

$$\tilde{\varrho}\big(\tilde{g}(p)\big)\big(\tilde{X}, \tilde{Y}\big) = \frac{1}{\big(1 - |\tilde{g}(p)|^2\big)^2} \operatorname{Re}\Big(\big(1 - |\tilde{g}(p)|^2\big)\beta_3\big(\tilde{X}, \tilde{Y}\big) + \beta_3\big(\tilde{X}, \tilde{g}(p)\big)\beta_3\big(\tilde{g}(p), \tilde{Y}\big) \Big).$$

Since g preserves Ψ_1, we have

$$\begin{aligned}
1-|p|^2 &= -\Psi_1\big((1,p),(1,p)\big) \\
&= -\Psi_1\big((1,p)g,(1,p)g\big) = |a+pc^\top|^2-|b+pA|^2 \\
&= |a+pc^\top|^2\left(1-|a+pc^\top|^{-2}|b+pA|^2\right) \\
&= |a+pc^\top|^2\left(1-\left|(a+pc^\top)^{-1}(b+pA)\right|^2\right) \\
&= |a+pc^\top|^2\left(1-|\tilde g(p)|^2\right).
\end{aligned}$$

By the expression for the derivatives and the C-sesquilinearity of β_3 we have

$$\beta_3\big(\tilde X,\tilde Y\big)=\big(a+pc^\top\big)^{-1}\beta_3\big(-Xc^\top\tilde g(p)+XA,-Yc^\top\tilde g(p)+YA\big)\overline{\big(a+pc^\top\big)}^{-1}.$$

Further

$$\begin{aligned}
&\beta_3\big(-Xc^\top\tilde g(p)+XA,-Yc^\top\tilde g(p)+YA\big)= \\
&\quad=\Big[-Xc^\top\tilde g(p)+XA\Big]\Big[-Yc^\top\tilde g(p)+YA\Big]^* \\
&\quad=\Big[-Xc^\top\tilde g(p)+XA\Big]\Big[-\tilde g(p)^*\overline{c}Y^*+A^*Y^*\Big] \\
&\quad=Xc^\top\tilde g(p)\tilde g(p)^*\overline{c}Y^*+XAA^*Y^*-XA\tilde g(p)^*\overline{c}Y^*-Xc^\top\tilde g(p)A^*Y^*.
\end{aligned}$$

Now $AA^*=I+c^\top\overline{c}$ by (3.4) and $\tilde g(p)\tilde g(p)^*=|\tilde g(p)|^2$, hence

$$\begin{aligned}
&\beta_3\big(-Xc^\top\tilde g(p)+XA,-Yc^\top\tilde g(p)+YA\big)= \\
&\qquad\qquad=\big(1+|\tilde g(p)|^2\big)Xc^\top\overline{c}Y^*+XY^*-XA\tilde g(p)^*\overline{c}Y^*-Xc^\top\tilde g(p)A^*Y^*.
\end{aligned}$$

Likewise we have

$$\begin{aligned}
&\beta_3\big(\tilde X,\tilde g(p)\big)\beta_3\big(\tilde g(p),\tilde Y\big)= \\
&\quad=\big(a+pc^\top\big)^{-1}\Big[-Xc^\top\tilde g(p)+XA\Big]\tilde g(p)^*\tilde g(p)\Big[-Yc^\top\tilde g(p)+YA\Big]^*\overline{\big(a+pc^\top\big)}^{-1},
\end{aligned}$$

where

$$\begin{aligned}
&\Big[-Xc^\top\tilde g(p)+XA\Big]\tilde g(p)^*\tilde g(p)\Big[-Yc^\top\tilde g(p)+YA\Big]^*= \\
&\quad=\Big[-Xc^\top\tilde g(p)\tilde g(p)^*\tilde g(p)+XA\tilde g(p)^*\tilde g(p)\Big]\Big[-\tilde g(p)^*\overline{c}Y^*+A^*Y^*\Big] \\
&\quad=|\tilde g(p)|^4Xc^\top\overline{c}Y^*-|\tilde g(p)|^2Xc^\top\tilde g(p)A^*Y^*-|\tilde g(p)|^2XA\tilde g(p)^*\overline{c}Y^*+ \\
&\qquad\qquad+XA\tilde g(p)^*\tilde g(p)A^*Y^*.
\end{aligned}$$

Up to now we have for the term in $\mathrm{Re}(\ldots)$ the following equality

$$\begin{aligned}
&\big(a+pc^\top\big)\Big((1-|\tilde g(p)|^2)\beta_3(\tilde X,\tilde Y)+\beta_3(\tilde X,\tilde g(p))\beta_3(\tilde g(p),\tilde Y)\Big)\overline{\big(a+pc^\top\big)} \\
&\quad=\big(1-|\tilde g(p)|^2\big)\Big[\big(1+|\tilde g(p)|^2\big)Xc^\top\overline{c}Y^*+XY^*-XA\tilde g(p)^*\overline{c}Y^*\Big]- \\
&\qquad-\big(1-|\tilde g(p)|^2\big)\Big[Xc^\top\tilde g(p)A^*Y^*\Big]++|\tilde g(p)|^4Xc^\top\overline{c}Y^*-|\tilde g(p)|^2Xc^\top\tilde g(p)A^*Y^*- \\
&\qquad-|\tilde g(p)|^2XA\tilde g(p)^*\overline{c}Y^*+XA\tilde g(p)^*\tilde g(p)A^*Y^*
\end{aligned}$$

$$
\begin{aligned}
&= Xc^\top \overline{c} Y^* - |\tilde{g}(p)|^4 Xc^\top \overline{c} Y^* + XY^* - |\tilde{g}(p)|^2 XY^* - XA\tilde{g}(p)^* \overline{c} Y^* + \\
&\quad + |\tilde{g}(p)|^2 XA\tilde{g}(p)^* \overline{c} Y^* - Xc^\top \tilde{g}(p) A^* Y^* + |\tilde{g}(p)|^2 Xc^\top \tilde{g}(p) A^* Y^* + \\
&\quad + |\tilde{g}(p)|^4 Xc^\top \overline{c} Y^* - |\tilde{g}(p)|^2 Xc^\top \tilde{g}(p) A^* Y^* - |\tilde{g}(p)|^2 XA\tilde{g}(p)^* \overline{c} Y^* + \\
&\quad + XA\tilde{g}(p)^* \tilde{g}(p) A^* Y^* \\
&= Xc^\top \overline{c} Y^* + XY^* - |\tilde{g}(p)|^2 XY^* - XA\tilde{g}(p)^* \overline{c} Y^* - Xc^\top \tilde{g}(p) A^* Y^* + \\
&\quad + XA\tilde{g}(p)^* \tilde{g}(p) A^* Y^*.
\end{aligned}
$$

We simplify the expression $-A\tilde{g}(p)^*\overline{c} - c^\top \tilde{g}(p) A^* + A\tilde{g}(p)^* \tilde{g}(p) A^*$. From

$$
\tilde{g}(p) = \big(a + pc^\top\big)^{-1} \big(b - pA\big) = |a + pc^\top|^{-2} \big(a + pc^\top\big)^* \big(b + pA\big)
$$

it follows that

$$
\begin{aligned}
A\tilde{g}(p)^* \tilde{g}(p) A^* &= |a + pc^\top|^{-4} A(b + pA)^* \big(a + pc^\top\big)\big(a + pc^\top\big)^* (b + pA) A^* \\
&= |a + pc^\top|^{-2} A\big(b^* + A^* p^*\big)(b + pA) A^* \\
&= |a + pc^\top|^{-2} \big(Ab^* + AA^* p^*\big)\big(bA^* + pAA^*\big) \\
&= |a + pc^\top|^{-2} \left[Ab^* bA^* + AA^* p^* bA^* + Ab^* pAA^* + AA^* p^* pAA^*\right].
\end{aligned}
$$

Using (3.4) we find

$$
\begin{aligned}
&|a + pc^\top|^2 \Big(-A\tilde{g}(p)^* \overline{c} - c^\top \tilde{g}(p) A^* + A\tilde{g}(p)^* \tilde{g}(p) A^*\Big) = \\
&\quad = -A(b - pA)^* \big(a + pc^\top\big)\overline{c} - c^\top \big(a + pc^\top\big)^* (b + pA) A^* + Ab^* bA^* + \\
&\qquad + AA^* p^* bA^* + Ab^* pAA^* + AA^* p^* pAA^* \\
&\quad = -A\big(b^* + A^* p^*\big)\big(a + pc^\top\big)\overline{c} - c^\top \big(\overline{a} + \overline{c} p^*\big)(b + pA) A^* + Ab^* bA^* + \\
&\qquad + AA^* p^* bA^* + Ab^* pAA^* + AA^* p^* pAA^* \\
&\quad = \big(-Ab^* - AA^* p^*\big)\big(a\overline{c} + pc^\top \overline{c}\big) - \big(c^\top \overline{a} + c^\top \overline{c} p^*\big)\big(bA^* + pAA^*\big) + \\
&\qquad + Ab^* bA^* + AA^* p^* bA^* + Ab^* pAA^* + AA^* p^* pAA^* \\
&\quad = -Ab^* a\overline{c} - AA^* p^* a\overline{c} - Ab^* pc^\top \overline{c} - AA^* p^* pc^\top \overline{c} - c^\top \overline{a} bA^* - c^\top \overline{c} p^* bA^* - \\
&\qquad - c^\top \overline{a} pAA^* - c^\top \overline{c} p^* pAA^* + Ab^* bA^* + AA^* p^* bA^* + Ab^* pAA^* + AA^* p^* pAA^* \\
&\quad = Ab^*(-a\overline{c} + bA^*) + AA^* p^*(-a\overline{c} + bA^*) + (-c^\top \overline{a} + Ab^*) pAA^* + \\
&\qquad + AA^* p^* p(-c^\top \overline{c} + AA^*) - Ab^* pc^\top \overline{c} - c^\top \overline{a} bA^* - c^\top \overline{c} p^* bA^* - c^\top \overline{c} p^* pAA^* \\
&\quad = (I + c^\top \overline{c}) p^* p - c^\top \overline{a} pc^\top \overline{c} - c^\top \overline{a} a\overline{c} - c^\top \overline{c} p^* a\overline{c} - c^\top \overline{c} p^* p(I + c^\top \overline{c}) \\
&\quad = p^* p - c^\top \overline{a} pc^\top \overline{c} - |a|^2 c^\top \overline{c} - c^\top \overline{c} p^* a\overline{c} - c^\top \overline{c} p^* pc^\top \overline{c} \\
&\quad = p^* p - c^\top \big(|a|^2 + \overline{a} pc^\top + \overline{c} p^* a + \overline{c} p^* pc^\top\big)\overline{c} \\
&\quad = p^* p - c^\top \big(a + pc^\top\big)^* \big(a + pc^\top\big)\overline{c} \\
&\quad = p^* p - |a + pc^\top|^2 c^\top \overline{c}.
\end{aligned}
$$

Plugging this equality in the above formula it follows

$$\begin{aligned}
&(a+pc^\top)\Big((1-|\tilde g(p)|^2)\beta_3(\tilde X,\tilde Y)+\beta_3(\tilde X,\tilde g(p))\beta_3(\tilde g(p),\tilde Y)\Big)\overline{(a+pc^\top)}=\\
&=|a+pc^\top|^{-2}\Big[|a+pc^\top|^2Xc^\top\bar cY^*+|a+pc^\top|^2(1-|\tilde g(p)|^2)XY^*\\
&\qquad\qquad +Xp^*pY^*-|a+pc^\top|^2Xc^\top\bar cY^*\Big]\\
&=|a+pc^\top|^{-2}[(1-|p|^2)XY^*+Xp^*pY^*]
\end{aligned}$$

Let $\zeta\in C$. Lemma 1.8 shows

$$\begin{aligned}
\operatorname{Re}\left[(a+pc^\top)^{-1}\zeta\overline{(a+pc^\top)}^{-1}\right]&=\operatorname{Re}\beta_1\big((a+pc^\top)^{-1}\zeta,(a+pc^\top)^{-1}\big)\\
&=\left\langle(a+pc^\top)^{-1}\zeta,(a+pc^\top)^{-1}\right\rangle\\
&=\left\langle\zeta,\overline{(a+pc^\top)}^{-1}(a+pc^\top)^{-1}\right\rangle\\
&=|a+pc^\top|^{-2}\operatorname{Re}\zeta.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\tilde\varrho(\tilde g(p))(\tilde X,\tilde Y)&=\frac{|a+pc^\top|^4}{(1-|p|^2)^2}|a+pc^\top|^{-4}\operatorname{Re}\big((1-|p|^2)XY^*+Xp^*pY^*\big)\\
&=\tilde\varrho(p)(X,Y).
\end{aligned}$$

$\square$

In the following we will determine which elements in $U(\Psi_1,C)$ induce the same isometry on B. We denote the center of C by $Z(C)$ and set

$$Z^1(C):=\{a\in Z(C)\mid |a|=1\},$$

which are the central elements in C of unit length. Further we let $Z(\Psi_1,C)$ denote the center of $U(\Psi_1,C)$.

Lemma 3.2.16. *We have*

$$Z(\Psi_1,C)=\{a\,\mathrm{id}_E\mid a\in Z^1(C)\}.$$

Proof. Clearly, $\{a\,\mathrm{id}_E\mid a\in Z^1(C)\}\subseteq Z(\Psi_1,C)$. For the converse inclusion relation let

$$g=\begin{pmatrix}a&b\\c^\top&A\end{pmatrix}\in Z(\Psi_1,C).$$

For each $d\in C$, $|d|=1$, and each matrix $D\in C^{n\times n}$, $DD^*=I$, the matrix

$$h=\begin{pmatrix}d&0\\0&D\end{pmatrix}$$

is in $U(\Psi_1,C)$. So necessarily,

$$\begin{pmatrix}ad&bD\\c^\top d&AD\end{pmatrix}=gh=hg=\begin{pmatrix}da&db\\Dc^\top&DA\end{pmatrix}.$$

The left upper entries show that $a \in Z(C)$. Choosing different values for d, but the same (invertible) D, we find $b = c = 0$. If D runs through all permutation matrices and all rotation matrices, then we see that A is a diagonal matrix $\operatorname{diag}(x, \dots, x)$, where $x \in Z(C)$. By (3.4), $|a| = 1 = |x|$. Then

$$g = \begin{pmatrix} a & 0 \\ 0 & xI \end{pmatrix}.$$

Let $h = \begin{pmatrix} d & u \\ w^\top & B \end{pmatrix} \in U(\Psi_1, C)$. Then

$$\begin{pmatrix} ad & au \\ xw^\top & xB \end{pmatrix} = gh = hg = \begin{pmatrix} da & ux \\ w^\top a & Bx \end{pmatrix}.$$

Therefore $au = ux = xu$. For $u \neq 0$ it follows that $a = x$. Hence $g = a \operatorname{id}_E$ with $a \in Z^1(C)$. □

Proposition 3.2.17. *Let $g_1, g_2 \in U(\Psi_1, C)$. Then g_1 and g_2 induce the same isometry on B if and only if $g_1 h = g_2$ for some $h \in Z(\Psi_1, C)$.*

Proof. It suffices to show that exactly the elements in $Z(\Psi_1, C)$ induce id_B. Let $g \in U(\Psi_1, C)$ and suppose that

$$g = \begin{pmatrix} a & b \\ c^\top & A \end{pmatrix}$$

w. r. t. $\mathcal{B}(E)$, where $a \in C$, $b, c \in C^n$ and $A \in C^{n \times n}$. Then the induced isometry on B is

$$\widetilde{g}\colon \left\{ \begin{array}{lcl} B & \to & B \\ p & \mapsto & (a + pc^\top)^{-1}(b + pA). \end{array} \right.$$

Suppose that $\widetilde{g} = \operatorname{id}_B$. Then

$$0 = \widetilde{g}(0) = a^{-1} b,$$

which yields that $b = 0$ and, by (3.4), $|a| = 1$. Hence

$$g = \begin{pmatrix} a & 0 \\ c^\top & A \end{pmatrix}.$$

Now (3.4) shows that $0 = -a\overline{c}$, which implies that $c = 0$. Thus,

$$\widetilde{g}(p) = a^{-1} pA$$

for all $p \in B$. Suppose that $A = (a_{ij})_{i,j=1,\dots,n}$. Let $j \in \{1, \dots, n\}$ and consider $p = (p_i)_{i=1,\dots,n}$ with $p_j = \frac{1}{2}$ and $p_i = 0$ for $i \neq j$. Then $p \in B$ and

$$p = \widetilde{g}(p) = \tfrac{1}{2} a^{-1} (a_{j1}, \dots, a_{jn}).$$

Therefore $a_{ij} = 0$ for $i \neq j$ and $a_{jj} = a$. This shows that $A = aI$.

Now let $\zeta \in C$ with $|\zeta| < 1$. Then $(\zeta, 0) \in B$ and therefore

$$(\zeta, 0) = \widetilde{g}(\zeta, 0) = a^{-1}(\zeta, 0)a.$$

Hence $a\zeta = \zeta a$. By scaling, this equality holds for all $\zeta \in C$. Thus $a \in Z^1(C)$, which shows that $g \in Z(\Psi_1, C)$. Conversely, each element of $Z(\Psi_1, C)$ clearly induces id_B on B. ☐

Let

$$\mathrm{PU}(\Psi_1, C) := U(\Psi_1, C)/Z(\Psi_1, C)$$

and denote the coset of $g \in U(\Psi_1, C)$ by $[g]$. Recall that G is the full isometry group of B. As before we use the notation $\widetilde{g}$ for the isometry on B induced by $g \in U(\Psi_1, C)$.

Corollary 3.2.18. *The map*

$$j_{\Psi_1}\colon \begin{cases} \mathrm{PU}(\Psi_1, C) & \to \quad G \\ \quad [g] & \mapsto \quad \widetilde{g} \end{cases}$$

is a monomorphism of groups.

By Cor. 3.2.18, $\mathrm{PU}(\Psi_1, C)$ is isomorphic to a subgroup of G. For a characterization of this subgroup we will work with the space $\overline{P_C\big(E_-(\Psi_2)\big)}$.

Set of representatives for $\overline{P_C\big(E_-(\Psi_2)\big)}$

If $z = (\zeta, \eta, v) \in E_-(\Psi_2)$, then

$$0 > q_2\big((\zeta, \eta, v)\big) = -2\langle \zeta, \eta\rangle + |v|^2,$$

which shows that $\zeta \neq 0$. Therefore $[z] \in P_C\big(E_-(\Psi_2)\big)$ is represented by $(1, \zeta^{-1}\eta, \zeta^{-1}v)$ and this is the unique representative of the form $(1, *, *)$. We note that

$$0 > q_2\big((1, \zeta^{-1}\eta, \zeta^{-1}v)\big) = -2\langle 1, \zeta^{-1}\eta\rangle + |\zeta^{-1}v|^2 = -2\,\mathrm{Re}(\zeta^{-1}\eta) + |\zeta^{-1}v|^2,$$

and therefore

$$\mathrm{Re}(\zeta^{-1}\eta) > \tfrac{1}{2}|\zeta^{-1}v|^2.$$

Thus, if we define

$$H := \left\{(\tau, u) \in C \oplus V \;\middle|\; \mathrm{Re}(\tau) > \tfrac{1}{2}|u|^2\right\},$$

then

$$(\zeta^{-1}\eta, \zeta^{-1}v) \in H.$$

Conversely, if $(\tau, u) \in H$, then $[(1, \tau, u)] \in P_C(E_-(\Psi_2))$.

If $z = (\zeta, \eta, v) \in E_0(\Psi_2)$, then either $\zeta \neq 0$ or $z = (0, \eta, 0)$ with $\eta \neq 0$. Applying the previous argumentation we see that the set of elements in $P_C\big(E_0(\Psi_2)\big)$ which have a representative $(\zeta, \eta, v) \in E_0(\Psi_2)$ with $\zeta \neq 0$ is bijective to

$$\big\{(\tau, u) \in C \oplus V \bigm| \mathrm{Re}(\tau) = \tfrac{1}{2}|u|^2\big\}$$

via

$$[(\zeta, \eta, v)] \mapsto (\zeta^{-1}\eta, \zeta^{-1}v).$$

In the latter case, $[z]$ is represented by $(0, 1, 0)$. If

$$\overline{H}^g := \big\{(\tau, u) \in C \oplus V \bigm| \mathrm{Re}(\tau) \geq \tfrac{1}{2}|u|^2\big\} \cup \{\infty\}$$

denotes the closure of H in the one-point compactification $(C \oplus V) \cup \{\infty\}$ of $C \oplus V$, then the map $\tau_H \colon \overline{P_C\big(E_-(\Psi_2)\big)} \to \overline{H}^g$,

$$\tau_H\big([(\zeta, \eta, v)]\big) := \begin{cases} (\zeta^{-1}\eta, \zeta^{-1}v) & \text{if } \zeta \neq 0, \\ \infty & \text{if } \zeta = 0, \end{cases}$$

is a bijection with inverse map

$$\tau_H^{-1}(\infty) = [(0, 1, 0)] \qquad \tau_H^{-1}(\tau, u) = [(1, \tau, u)].$$

Since $\tau_H|_{P_C(E_-(\Psi_2))}$ is a restriction of the chart map φ_1 from Sec. 3.2.2, τ_H is a diffeomorphism between $P_C\big(E_-(\Psi_2)\big)$ and H.

Riemannian metric and induced isometries on H

Recall the model D from Sec. 1.4.1. The map

$$\beta\colon \left\{ \begin{array}{ccc} \overline{H}^g & \to & \overline{D}^g \\ \infty & \mapsto & \infty \\ (\zeta, v) & \mapsto & (\zeta, \sqrt{2}v) \end{array} \right.$$

with inverse

$$\beta^{-1}\colon \left\{ \begin{array}{ccc} \overline{D}^g & \to & \overline{H}^g \\ \infty & \mapsto & \infty \\ (\zeta, v) & \mapsto & (\zeta, \frac{1}{\sqrt{2}}v) \end{array} \right.$$

is clearly a diffeomorphism between the manifolds $\overline{H}^g$ and $\overline{D}^g$ with boundary, and hence a diffeomorphism between H and D. We endow H with a Riemannian metric by requiring that β be an isometry.

Let $\pi_H := \tau_H \circ \pi\colon E_-(\Psi_2) \to H$. As before, each $g \in U(\Psi_2, C)$ induces a (unique) map $\widetilde{g}$ on H which makes the diagram

$$\begin{tikzcd} E_-(\Psi_2) \arrow[r, "g"] \arrow[d, "\pi_H"'] & E_-(\Psi_2) \arrow[d, "\pi_H"] \\ H \arrow[r, dotted, "\widetilde{g}"] & H \end{tikzcd}$$

commutative.

Let $Z(\Psi_2, C)$ denote the center of $U(\Psi_2, C)$ and set

$$\mathrm{PU}(\Psi_2, C) := U(\Psi_2, C)/Z(\Psi_2, C).$$

In the following we will use the results from the previous subsections to show that $\mathrm{PU}(\Psi_2, C)$ is isomorphic to a subgroup of the full isometry group G of H.

We consider the map

$$T\colon \begin{cases} E & \to \quad E \\ (\zeta, \eta, v) & \mapsto \quad \left(\frac{1}{\sqrt{2}}(\zeta - \eta), \frac{1}{\sqrt{2}}(\zeta + \eta), v\right). \end{cases}$$

Then T is C-linear and invertible with inverse map $T^{-1}\colon E \to E$,

$$T^{-1}(\zeta, \eta, v) = \left(\tfrac{1}{\sqrt{2}}(\zeta + \eta), \tfrac{1}{\sqrt{2}}(-\zeta + \eta), v\right).$$

Lemma 3.2.19. *We have $\Psi_2 \circ (T \times T) = \Psi_1$. Further, $T\big(E_-(\Psi_1)\big) = E_-(\Psi_2)$ and $T\big(E_0(\Psi_1)\big) = E_0(\Psi_2)$.*

Proof. Let $(\zeta_1, \eta_1, v_1), (\zeta_2, \eta_2, v_2) \in E$. Then

$$\begin{aligned}
&\Psi_2\big(T(\zeta_1, \eta_1, v_1), T(\zeta_2, \eta_2, v_2)\big) = \\
&\quad = \Psi_2\left(\left(\tfrac{1}{\sqrt{2}}(\zeta_1 - \eta_1), \tfrac{1}{\sqrt{2}}(\zeta_1 + \eta_1), v_1\right), \left(\tfrac{1}{\sqrt{2}}(\zeta_2 - \eta_2), \tfrac{1}{\sqrt{2}}(\zeta_2 + \eta_2), v_2\right)\right) \\
&\quad = -\beta_1\left(\tfrac{1}{\sqrt{2}}(\zeta_1 - \eta_1), \tfrac{1}{\sqrt{2}}(\zeta_2 + \eta_2)\right) - \beta_1\left(\tfrac{1}{\sqrt{2}}(\zeta_1 + \eta_1), \tfrac{1}{\sqrt{2}}(\zeta_2 - \eta_2)\right) + \beta_2(v_1, v_2) \\
&\quad = -\tfrac{1}{2}\big[\beta_1(\zeta_1, \zeta_2) + \beta_1(\zeta_1, \eta_2) - \beta_1(\eta_1, \zeta_2) - \beta_1(\eta_1, \eta_2)\big] \\
&\qquad - \tfrac{1}{2}\big[\beta_1(\zeta_1, \zeta_2) - \beta_1(\zeta_1, \eta_2) + \beta_1(\eta_1, \zeta_2) - \beta_1(\eta_1, \eta_2)\big] + \beta_2(v_1, v_2) \\
&\quad = -\beta_1(\zeta_1, \zeta_2) + \beta_1(\eta_1, \eta_2) + \beta_2(v_1, v_2) \\
&\quad = \Psi_1\big((\zeta_1, \eta_1, v_1), (\zeta_2, \eta_2, v_2)\big).
\end{aligned}$$

This shows the claim. ☐

Proposition 3.2.20. *The map*

$$\lambda\colon \begin{cases} U(\Psi_1, C) & \to \quad U(\Psi_2, C) \\ g & \mapsto \quad T \circ g \circ T^{-1} \end{cases}$$

is an isomorphism with $\lambda\big(Z(\Psi_1, C)\big) = Z(\Psi_2, C)$.

Proof. Let $g \in U(\Psi_1, C)$. Since $T \in \mathrm{GL}_C(E)$, we have $\lambda(g) \in \mathrm{GL}_C(E)$. Lemma 3.2.19 shows that

$$\begin{aligned}
\Psi_2 \circ \big(TgT^{-1} \times TgT^{-1}\big) &= \Psi_2 \circ \big(T \times T\big) \circ \big(gT^{-1} \times gT^{-1}\big) \\
&= \Psi_1 \circ \big(g \times g\big) \circ \big(T^{-1} \times T^{-1}\big) = \Psi_1 \circ \big(T^{-1} \times T^{-1}\big) \\
&= \Psi_2.
\end{aligned}$$

Hence $\lambda(g) \in U(\Psi_2, C)$. Clearly, λ is an isomorphism of groups and hence $\lambda\big(Z(\Psi_1, C)\big) = Z(\Psi_2, C)$. ☐

For the proof of the following proposition recall the Cayley transform $\mathcal{C}\colon B \to D$ from Sec. 1.6. For each $\zeta \in C \smallsetminus \{1\}$ we have

$$\begin{aligned}(1-\zeta)^{-1}(1+\zeta) &= |1-\zeta|^{-2}(1-\overline{\zeta})(1+\zeta) = |1-\zeta|^{-2}(1-\overline{\zeta}+\zeta-|\zeta|^2)\\ &= |1-\zeta|^{-2}(1+2\operatorname{Im}\zeta-|\zeta|^2).\end{aligned}$$

Hence

$$\mathcal{C}(\zeta,v) = (1-\zeta)^{-1}(1+\zeta,2v).$$

Proposition 3.2.21. *Let $g \in U(\Psi_2,C)$. Then the induced map $\widetilde{g}$ on H is an isometry. Further, the map*

$$j_{\Psi_2}\colon \begin{cases} \mathrm{PU}(\Psi_2,C) & \to & G \\ [g] & \mapsto & \widetilde{g} \end{cases}$$

is well-defined and a monomorphism of groups.

Proof. The map T induces the map $\widehat{T}\colon P_C\big(E_-(\Psi_1)\big) \to P_C\big(E_-(\Psi_2)\big)$ given by

$$\widehat{T}\big([(\zeta,\eta,v)]\big) = \Big[\big(\tfrac{1}{\sqrt{2}}(\zeta-\eta), \tfrac{1}{\sqrt{2}}(\zeta+\eta), v\big)\Big]$$

and the map $\widetilde{T}\colon B \to H$ given by

$$\begin{aligned}\widetilde{T}\big((\eta,v)\big) &= \tau_H\big(\widehat{T}[(1,\eta,v)]\big) = \tau_H\Big(\big[\tfrac{1}{\sqrt{2}}(1-\eta), \tfrac{1}{\sqrt{2}}(1+\eta), v\big]\Big)\\ &= \tau_H\Big(\big[(1,(1-\eta)^{-1}(1+\eta),(1-\eta)^{-1}\sqrt{2}v)\big]\Big)\\ &= (1-\eta)^{-1}(1+\eta,\sqrt{2}v).\end{aligned}$$

Therefore, $\widetilde{T} = \beta^{-1}\circ\mathcal{C}$. From Prop. 1.4.4 and Prop. 3.2.12 it follows that the isometry group of B and that of H are identical. Then Prop. 3.2.15 and Prop. 3.2.20 imply that for each $g \in U(\Psi_2,C)$, the induced map $\widetilde{g}$ is an isometry on H. Moreover, Prop. 3.2.20 shows that λ factors to a map

$$\widetilde{\lambda}\colon \mathrm{PU}(\Psi_1,C) \to \mathrm{PU}(\Psi_2,C).$$

Recall the map j_{Ψ_1} from Cor. 3.2.18. Then the diagram

$$\begin{tikzcd} \mathrm{PU}(\Psi_1,C) \arrow[rr, "\widetilde{\lambda}"] \arrow[dr, "j_{\Psi_1}"'] & & \mathrm{PU}(\Psi_2,C) \arrow[dl, "j_{\Psi_2}"] \\ & G & \end{tikzcd}$$

commutes, which completes the proof. ∎

Lifted isometries

In this section we determine which isometries on H are induced from an element in $U(\Psi_2, C)$. To that end we need explicit formulas for the action of an element $g \in G$ on H, which are provided by the following remark.

Remark 3.2.22. Let $g \in G$. Sec. 1.6 provides the formulas for the action of g on D. Using the isometry $\beta\colon H \to D$, the action of g on H are given by

$$g^H := \beta^{-1} \circ g \circ \beta \colon H \to H.$$

Evaluating this formula, we find the following action laws. For the geodesic inversion σ we have

$$\sigma^H(\zeta, v) = \zeta^{-1}(1, -v).$$

For $a_s \in A$ we get

$$a_s^H(\zeta, v) = \big(s\zeta, s^{1/2}v\big).$$

For $n = (\xi, w) \in N$ it follows

$$n^H(\zeta, v) = \left(\zeta + \xi + \tfrac{1}{4}|w|^2 + \beta_2\big(v, \tfrac{1}{\sqrt{2}}w\big), \tfrac{1}{\sqrt{2}}w + v\right).$$

For $m = (\varphi, \psi) \in M$ we have

$$m^H(\zeta, v) = \big(\varphi(\zeta), \psi(v)\big).$$

Proposition 3.2.23. *A representative of σ^H in $U(\Psi_2, C)$ is*

$$g(\zeta, \eta, v) = (\eta, \zeta, -v).$$

Proposition 3.2.24. *Let $a_s \in A$. Then*

$$g(\zeta, \eta, v) = \left(s^{-1/2}\zeta, s^{1/2}\eta, v\right)$$

is a representative of a_s in $U(\Psi_2, C)$.

Proof. Obviously, g is C-linear and induces a_s on B. Further

$$\begin{aligned}
\Psi_2\big(g(\zeta_1, \eta_1, v_1), g(\zeta_2, \eta_2, v_2)\big) &= \Psi_2\big((s^{-1/2}\zeta_1, s^{1/2}\eta_1, v_1), (s^{-1/2}\zeta_2, s^{1/2}\eta_2, v_2)\big) \\
&= -s^{-1/2}\zeta_1\overline{\eta_2}s^{1/2} - s^{-1/2}\eta_1\overline{\zeta_2}s^{1/2} + \beta_2(v_1, v_2) \\
&= -\zeta_1\overline{\eta_2} - \eta_1\overline{\zeta}_2 + \beta_2(v_1, v_2) \\
&= \Psi_2\big((\zeta_1, \eta_1, v_1), (\zeta_2, \eta_2, v_2)\big).
\end{aligned}$$

Hence $g \in U(\Psi_2, C)$. □

Proposition 3.2.25. *Let $n = (\xi, w) \in N$. A representative of n in $U(\Psi_2, C)$ is*

$$g(\zeta, \eta, v) = \left(\zeta, \zeta\left(\xi + \tfrac{1}{4}|w|^2\right) + \eta + \tfrac{1}{\sqrt{2}}\beta_2(v, w), \tfrac{1}{\sqrt{2}}\zeta w + v\right).$$

Proof. The map g is clearly C-linear and induces n on B. To see that Ψ_2 is g-invariant, we calculate

$$\begin{aligned}
&\Psi_2\big(g(\zeta_1,\eta_1,v_1),g(\zeta_2,\eta_2,v_2)\big) = \\
&\qquad = \Psi_2\Big[\Big(\zeta_1,\zeta_1\left(\xi+\tfrac14|w|^2\right)+\eta_1+\tfrac{1}{\sqrt2}\beta_2(v_1,w),\tfrac{1}{\sqrt2}\zeta_1 w+v_1\Big),\\
&\qquad\qquad\Big(\zeta_2,\zeta_2\left(\xi+\tfrac14|w|^2\right)+\eta_2+\tfrac{1}{\sqrt2}\beta_2(v_2,w),\tfrac{1}{\sqrt2}\zeta_2 w+v_2\Big)\Big]\\
&\qquad = -\zeta_1\Big[\left(-\xi+\tfrac14|w|^2\right)\overline{\zeta}_2+\overline{\eta}_2+\tfrac{1}{\sqrt2}\beta_2(w,v_2)\Big]\\
&\qquad\quad - \Big[\zeta_1\left(\xi+\tfrac14|w|^2\right)+\eta_1+\tfrac{1}{\sqrt2}\beta_2(v_1,w)\Big]\overline{\zeta}_2+\beta_2\Big(\tfrac{1}{\sqrt2}\zeta_1 w+v_1,\tfrac{1}{\sqrt2}\zeta_2 w+v_2\Big)\\
&\qquad = \zeta_1\xi\overline{\zeta}_2-\tfrac14|w|^2\zeta_1\overline{\zeta}_2-\zeta_1\overline{\eta}_2-\tfrac{1}{\sqrt2}\zeta_1\beta_2(w,v_2)-\zeta_1\xi\overline{\zeta}_2-\tfrac14|w|^2\zeta_1\overline{\zeta}_2\\
&\qquad\quad -\eta_1\overline{\zeta}_2-\tfrac{1}{\sqrt2}\beta_2(v_1,w)\overline{\zeta}_2+\tfrac12|w|^2\zeta_1\overline{\zeta}_2+\tfrac{1}{\sqrt2}\zeta_1\beta_2(w,v_2)\\
&\qquad\quad +\tfrac{1}{\sqrt2}\beta_2(v_1,w)\overline{\zeta}_2+\beta_2(v_1,v_2)\\
&\qquad = \Psi_2\big((\zeta_1,\eta_1,v_1),(\zeta_2,\eta_2,v_2)\big).
\end{aligned}$$

This completes the proof. ◻

The remaining part of this section is devoted to the discussion which elements of M have a representative in $U(\Psi_2,C)$. The situation for M is much more involved than the proofs of Prop. 3.2.23-3.2.25. In particular, it will turn out that in general not each element of M can be lifted to $U(\Psi_2,C)$.

Lemma 3.2.26. *Let $m=(\varphi,\psi)\in M$ and suppose that φ is an inner automorphism of C. Then*

$$\left\{\beta_2\big(\psi(v_1),\psi(v_2)\big)\;\middle|\;v_1,v_2\in V\right\}=C.$$

Proof. Let $a\in C\smallsetminus\{0\}$ and suppose that $\varphi(\zeta)=a^{-1}\zeta a$ for all $\zeta\in C$. Choose $v\in V$ with $|v|=1$. For each $\zeta\in C$ we find

$$\begin{aligned}
\beta_2\big(\psi(a\zeta a^{-1}v),\psi(v)\big) &= \beta_2\big(\varphi(a\zeta a^{-1})\psi(v),\psi(v)\big)=\beta_2\big(\zeta\psi(v),\psi(v)\big)\\
&=\zeta\beta_2(\psi(v),\psi(v))=\zeta|\psi(v)|^2=\zeta|v|^2\\
&=\zeta.
\end{aligned}$$

Therefore

$$C\subseteq\left\{\beta_2\big(\psi(v_1),\psi(v_2)\big)\;\middle|\;v_1,v_2\in V\right\}.$$

The converse inclusion relation clearly holds by the range of β_2. ◻

Let $i_H\colon \overline{H}^g\to E_-(\Psi_2)\cup E_0(\Psi_2)$ be any section of

$$\pi_H=\tau_H\circ\pi\colon E_-(\Psi_2)\cup E_0(\Psi_2)\to\overline{H}^g.$$

Let $\widetilde{g}\in G$ and recall from Prop. 1.6.2 that g extends continuously to $\overline{H}^g$. If $g\in U(\Psi_2,C)$ is a representative of $\widetilde{g}$, then the diagram

$$\begin{tikzcd}
E_-(\Psi_2)\cup E_0(\Psi_2) \arrow[r, "g"] & E_-(\Psi_2)\cup E_0(\Psi_2) \arrow[d, "\pi_H"] \\
\overline{H}^g \arrow[u, "i_H"] \arrow[r, "\widetilde{g}"] & \overline{H}^g
\end{tikzcd}$$

commutes. We will make use of this fact in the proof of Prop. 3.2.27 below. For convenience we define i_H by

$$i_H(\infty) := (0,1,0) \quad \text{and} \quad i_H(\eta,v) := (1,\eta,v). \tag{3.5}$$

Proposition 3.2.27. *Let $m = (\varphi,\psi) \in M$. Then there is a representative of m in $U(\Psi_2, C)$ if and only if $\varphi = \mathrm{id}$. In this case,*

$$g(\zeta,\eta,v) = (\zeta,\eta,\psi(v))$$

is a representative of m in $U(\Psi_2, C)$.

Proof. Suppose first that $m = (\varphi,\psi)$ with $\varphi = \mathrm{id}$. We will show that

$$g\colon \left\{ \begin{array}{ccc} E & \to & E \\ (\zeta,\eta,v) & \mapsto & (\zeta,\eta,\psi(v)) \end{array} \right.$$

is an element of $U(\Psi_2, C)$. To that end let $(\zeta_1,\eta_1,v_1),(\zeta_2,\eta_2,v_2) \in E$ and $\zeta \in C$. Then

$$\begin{aligned} g\big(\zeta(\zeta_1,\eta_1,v_1)+(\zeta_2,\eta_2,v_2)\big) &= \big(\zeta\zeta_1+\zeta_2, \zeta\eta_1+\eta_2, \psi(\zeta v_1+v_2)\big) \\ &= \big(\zeta\zeta_1,\zeta\eta_1,\psi(\zeta v_1)\big) + \big(\zeta_2,\eta_2,\psi(v_2)\big) \\ &= \big(\zeta\zeta_1,\zeta\eta_1,\varphi(\zeta)\psi(v_1)\big) + g(\zeta_2,\eta_2,v_2) \\ &= \big(\zeta\zeta_1,\zeta\eta_1,\zeta\psi(v_1)\big) + g(\zeta_2,\eta_2,v_2) \\ &= \zeta g(\zeta_1,\eta_1,v_1) + g(\zeta_2,\eta_2,v_2). \end{aligned}$$

This shows that g is C-linear. The map g is obviously invertible, hence we have $g \in \mathrm{GL}_C(E)$. Clearly, g induces m.

Suppose now that $m = (\varphi,\psi) \in M$ and that there is a representative g of m in $U(\Psi_2, C)$. We have to show that $\varphi = \mathrm{id}$. Since $m(0) = 0$, it follows that

$$g\big(i_H(0)\big) = g(1,0,0) \in \pi_H^{-1}(0) = \big(C\smallsetminus\{0\}\big)\times\{0\}\times\{0\}.$$

Thus, there is $a \in C\smallsetminus\{0\}$ such that $g(1,0,0) = (a,0,0)$. Further $m(\infty) = \infty$. The same argument shows that there is $b \in C\smallsetminus\{0\}$ such that

$$g\big(i_H(\infty)\big) = g(0,1,0) = (0,b,0).$$

Then for each $\zeta \in C$ with $\operatorname{Re}\zeta \geq 0$ we have

$$\begin{aligned} (\varphi(\zeta),0) = m(\zeta,0) = \pi_H\big(g(i_H(\zeta,0))\big) = \pi_H\big(g(1,0,0)\big) = \pi_H\big((a,\zeta b,0)\big) \\ = (a^{-1}\zeta b, 0). \end{aligned}$$

Thus $\varphi(\zeta) = a^{-1}\zeta b$ for all $\zeta \in C$ with $\operatorname{Re}\zeta \geq 0$. Now

$$1 = \varphi(1) = a^{-1}b$$

and therefore $b = a$. Hence $\varphi(\zeta) = a^{-1}\zeta a$ for all $\zeta \in C$ with $\operatorname{Re}\zeta \geq 0$. For all $z = (\eta,v) \in H$ there exists $w = w(z) \in V$ such that

$$\begin{aligned} (\varphi(\eta),\psi(v)) = m(\eta,v) = \pi_H\big(g(i_H(z))\big) = \pi_H\big(g(1,\eta,v)\big) = \pi_H\big((a,\eta a,w)\big) \\ = \big(a^{-1}\eta a, a^{-1}w\big). \end{aligned}$$

Then $\psi(v) = a^{-1}w$ and therefore

$$g(1, \eta, v) = (a, \eta a, a\psi(v)).$$

Since g has to be C-linear, it follows that

$$g(\zeta, \eta, v) = (\zeta a, \eta a, a\psi(v))$$

for all $(\zeta, \eta, v) \in E$. We derive further properties of a. Let $v \in V \setminus \{0\}$. Then $g \in U(\Psi_2, C)$ yields that

$$|v|^2 = q_2\big((0,0,v)\big) = q_2\big(g(0,0,v)\big) = |a\psi(v)|^2 = |a|^2|\psi(v)|^2 = |a|^2|\psi(v)|^2,$$

hence $|a|^2 = 1$. Again using that $g \in U(\Psi_2, C)$ we find that for all $v_1, v_2 \in V$

$$\begin{aligned}
\beta_2\big(\psi(v_1), \psi(v_2)\big) &= \Psi_2\big((0,0,v_1),(0,0,v_2)\big) \\
&= \Psi_2\big(g(0,0,v_1), g(0,0,v_2)\big) = \Psi_2\big((0,0,a\psi(v_1)),(0,0,a\psi(v_2))\big) \\
&= \beta_2\big(a\psi(v_1), a\psi(v_2)\big) = a\beta_2\big(\psi(v_1),\psi(v_2)\big)\overline{a} \\
&= a\beta_2\big(\psi(v_1),\psi(v_2)\big)a^{-1}.
\end{aligned}$$

Before we can apply Lemma 3.2.26 we have to show that $\varphi(\zeta) = a^{-1}\zeta a$ for all $\zeta \in C$. Let $\zeta \in C$ with $\operatorname{Re}\zeta < 0$ and consider the decomposition $\zeta = \zeta_1 + \zeta_2$ with $\zeta_1 \in \mathbb{R}$ and $\zeta_2 \in C'$. Then $\mathbb{R}$-linearity of φ yields

$$\begin{aligned}
\varphi(\zeta) &= \varphi(\zeta_1 + \zeta_2) = \varphi(\zeta_1) + \varphi(\zeta_2) = -\varphi(-\zeta_1) + a^{-1}\zeta_2 a \\
&= -a^{-1}(-\zeta_1)a + a^{-1}\zeta_2 a = a^{-1}\zeta_1 a + a^{-1}\zeta_2 a = a^{-1}(\zeta_1+\zeta_2)a = a^{-1}\zeta a.
\end{aligned}$$

Then Lemma 3.2.26 implies that $a \in Z(C)$. Therefore $\varphi = \mathrm{id}$. ☐

We set

$$M^{\mathrm{res}} := \{(\varphi,\psi) \in M \mid \varphi = \mathrm{id}\} \quad \text{and} \quad G^{\mathrm{res}} := N\sigma M^{\mathrm{res}}AN.$$

Further we define a map $\varphi_H\colon G^{\mathrm{res}} \to \mathrm{PU}(\Psi_2, C)$ as follows: For $\widetilde{g} = \sigma$, $\widetilde{g} \in A$, $\widetilde{g} \in N$ or $\widetilde{g} \in M^{\mathrm{res}}$ we set $\varphi_H(\widetilde{g}) := [g]$, where g is the lift of $\widetilde{g}$ as in Prop. 3.2.23, 3.2.24, 3.2.25 or 3.2.27, resp. For $\widetilde{g} = n_2\sigma m a_s n_1 \in N\sigma M^{\mathrm{res}}AN$ we define

$$\varphi_H(\widetilde{g}) := \varphi_H(n_2)\varphi_H(\sigma)\varphi_H(m)\varphi_H(a_s)\varphi_H(n_1). \tag{3.6}$$

In other words, we extend φ_H to a group homomorphism. Since the Bruhat decomposition of an element $\widetilde{g} \in G^{\mathrm{res}}$ is unique and the Bruhat decomposition of G^{res} can be directly transfered to $\mathrm{PU}(\Psi_2, C)$, the map φ_H is indeed a group homomorphism, and by our previous considerations, even a group isomorphism.

The following remark shows that M^{res} is not necessarily all of M.

Remark 3.2.28. The well-known classification of normed division algebras over $\mathbb{R}$ (cf. [KR, Thm. 3.1]) of associative J^2C-module structures shows that C is either real or complex or quaternionic numbers. In the following we show that $M = M^{\mathrm{res}}$ for $C = \mathbb{R}$, but $M \neq M^{\mathrm{res}}$ for $C = \mathbb{C}$ or $C = \mathbb{H}$.

(i) Let $C = \mathbb{R}$ and suppose that $m = (\varphi, \psi) \in M$. We claim that $\varphi = \mathrm{id}$. Since $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a norm-preserving endomorphism of $\mathbb{R}$, the map φ is either id or $-\mathrm{id}$. Assume for contradiction that $\varphi = -\mathrm{id}$. Let $(\zeta, v) \in C \times V$ such that $\zeta \neq 0$ and $v \neq 0$. Then

$$\varphi(\zeta)\psi(v) = -\zeta\psi(v) = -\psi(\zeta v) \neq \psi(\zeta v).$$

Hence $m \notin M$. This is a contradiction and therefore $\varphi = \mathrm{id}$.

(ii) Let $C = \mathbb{C} = V$ and suppose that $\varphi = \psi$ are complex conjugation. For all $(\zeta, v) \in C \oplus V = \mathbb{C}^2$ we have $\varphi(\zeta)\psi(v) = \overline{\zeta}\,\overline{v} = \overline{\zeta v} = \psi(\zeta v)$. Clearly, φ, ψ are $\mathbb{R}$-linear endomorphism of the Euclidean vector space $\mathbb{C}$. Therefore $m = (\varphi, \psi) \in M$, but $m \notin M^{\mathrm{res}}$.

(iii) Let $C = \mathbb{H} = V$. Define $\varphi\colon C \to C$ and $\psi\colon V \to V$ by

$$\begin{aligned}\varphi(a + ib + jc + kd) &:= a - ib - jc + kd\\ \psi(a + ib + jc + kd) &:= -a + ib + jc - kd\end{aligned}$$

for $a + ib + jc + kd \in \mathbb{H}$. Clearly, φ, ψ are $\mathbb{R}$-linear endomorphisms of the Euclidean vector space $\mathbb{H}$. We claim that $J \circ (\varphi \times \psi) = \psi \circ J$. To that end let $\zeta = a_1 + ib_1 + jc_1 + kd_1 \in C$ and $v = a_2 + ib_2 + jc_2 + kd_2 \in V$. Then

$$\begin{aligned}\zeta v = a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 + i(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\\ + j(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2) + k(a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2).\end{aligned}$$

Therefore

$$\begin{aligned}\psi(\zeta v) = -a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 + i(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\\ + j(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2) + k(-a_1d_2 - b_1c_2 + c_1b_2 - d_1a_2).\end{aligned}$$

On the other side we get

$$\begin{aligned}\varphi(\zeta)\psi(v) &= (a_1 - ib_1 - jc_1 + kd_1)(-a_2 + ib_2 + jc_2 - kd_2)\\ &= -a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 + i(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\\ &\quad + j(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2) + k(-a_1d_2 - b_1c_2 + c_1b_2 - d_1a_2)\\ &= \psi(\zeta v).\end{aligned}$$

Therefore, $m = (\varphi, \psi) \in M$, but $m \notin M^{\mathrm{res}}$.

3.3. Isometric spheres via cocycles

For an element $g \in G^{\mathrm{res}} \smallsetminus G_\infty$, we give a characterization of the isometric sphere and its radius via a cocycle.

Using the isometry $\beta\colon H \to D$ we find for the height function on H the formula

$$\mathrm{ht}^H(\zeta, v) = \mathrm{Re}\,\zeta - \tfrac{1}{2}|v|^2,$$

and for the Cygan metric ($z_j = (\zeta_j, v_j)$)

$$\varrho^H(z_1,z_2) = \left|\tfrac{1}{2}|v_1|^2 + \tfrac{1}{2}|v_2|^2 + \left|\mathrm{ht}^H(z_1) - \mathrm{ht}^H(z_2)\right| + \mathrm{Im}\,\zeta_1 - \mathrm{Im}\,\zeta_2 - \beta_2(v_1,v_2)\right|^{1/2} \tag{3.7}$$

The basis point in H is $o^H = (1,0)$. The horospherical coordinates of $z = (\zeta, v) \in \overline{H}^g \smallsetminus \{\infty\}$ are

$$\left(\mathrm{ht}^H(z), \mathrm{Im}\,\zeta, \tfrac{1}{\sqrt{2}}v\right)_h.$$

In the following we derive a formula for isometric spheres which uses a cocycle. Let $Z^1(C)$ act diagonally on E and suppose that $\pi^{(1)}\colon E \to E/Z^1(C)$ is the canonical projection. Further set

$$\pi_H^{(1)} := \tau_H \circ \pi \circ \left(\pi^{(1)}\right)^{-1}\colon \left(E_-(\Psi_2) \cup E_0(\Psi_2)\right)/Z^1(C) \to \overline{H}^g.$$

Recall the section i_H of π_H from (3.5) and set

$$i_H^{(1)} := \pi^{(1)} \circ i_H\colon \overline{H}^g \to E/Z^1(C).$$

Further recall the isomorphism $\varphi_H\colon G^{\mathrm{res}} \to \mathrm{PU}(\Psi_2, C)$ from (3.6). For all $g \in G^{\mathrm{res}}$ the diagram

$$\begin{tikzcd} E/Z^1(C) \arrow[r, "\varphi_H(g)"] & E/Z^1(C) \arrow[d, "\pi_H^{(1)}"] \\ \overline{H}^g \arrow[u, "i_H^{(1)}"] \arrow[r, "g"] & \overline{H}^g \end{tikzcd}$$

commutes, but the diagram

$$\begin{tikzcd} E/Z^1(C) \arrow[r, "\varphi_H(g)"] & E/Z^1(C) \\ \overline{H}^g \arrow[u, "i_H^{(1)}"] \arrow[r, "g"] & \overline{H}^g \arrow[u, "i_H^{(1)}"'] \end{tikzcd}$$

in general not. The second diagram gives rise to the cocycle

$$j\colon G^{\mathrm{res}} \times \overline{H}^g \to C^\times/Z^1(C)$$

defined by

$$\varphi_H(g)\left(i_H^{(1)}(z)\right) = j(g,z)\, i_H^{(1)}(gz) \quad \forall\, g \in G^{\mathrm{res}}\ \forall\, z \in \overline{H}^g.$$

Lemma 3.3.1. *Let $g = n_2\sigma m a_t n_1 \in G^{res} \smallsetminus G_\infty$. Then*

$$j(g^{-1}, \infty) = t^{1/2} \mod Z^1(C).$$

Further, $R(g) = \left|j(g^{-1},\infty)\right|^{-1/2}$.

Proof. Suppose that $n_1 = (\xi_1, w_1)$, $n_2 = (\xi_2, w_2)$ and $m = (\mathrm{id}, \psi)$. Since $n_j^{-1} = (-\xi_j, -w_j)$ for $j = 1, 2$ and $g^{-1} = n_1^{-1}\sigma a_t m n_2^{-1}$, we have

$$g^{-1}\infty = n_1^{-1}\sigma\infty = n_1^{-1}0 = \left(-\xi_1 + \tfrac{1}{4}|w_1|^2, -\tfrac{1}{\sqrt{2}}w_1\right).$$

Further

$$\begin{aligned}
\varphi_H\big(g^{-1}\big)\big(i_H(\infty)\big) &= \varphi_H\big(g^{-1}\big)\big((0,1,0)\big)\\
&= \varphi_H\big(n_1^{-1}\sigma a_t m\big)\varphi_H\big(n_2^{-1}\big)\big((0,1,0)\big)\\
&= \varphi_H\big(n_1^{-1}\sigma a_t\big)\varphi_H\big(m\big)\big((0,1,0)\big)\\
&= \varphi_H\big(n_1^{-1}\sigma\big)\varphi_H\big(a_t\big)\big((0,1,0)\big)\\
&= \varphi_H\big(n_1^{-1}\big)\varphi_H\big(\sigma\big)\big((0,t^{1/2},0)\big)\\
&= \varphi_H\big(n_1^{-1}\big)\big((t^{1/2},0,0)\big)\\
&= \left(t^{1/2}, t^{1/2}\left(-\xi_1 + \tfrac{1}{4}|w_1|^2\right), -\tfrac{1}{\sqrt{2}}t^{1/2}w_1\right) \mod Z^1(C).
\end{aligned}$$

Hence $j(g^{-1},\infty) = t^{1/2} \mod Z^1(C)$. Then $R(g) = t^{-1/4} = \big|j(g^{-1},\infty)\big|^{-1/2}$. □

Proposition 3.3.2. *Let $z_1, z_2 \in \overline{H}^g \smallsetminus \{\infty\}$, $z_j = (\zeta_j, v_j)$. Then*

$$\varrho^H(z_1,z_2) = \left|\Psi_2\big(i_H(z_1), i_H(z_2)\big) + 2\min\left(\mathrm{ht}^H(z_1), \mathrm{ht}^H(z_2)\right)\right|^{1/2}.$$

Proof. For all $k_1, k_2 \in \mathbb{R}$ we have

$$k_1 + k_2 - |k_1 - k_2| = 2\min(k_1,k_2).$$

This and the definition of Ψ_2 show that

$$\begin{aligned}
&\left|\Psi_2\big(i_H(z_1), i_H(z_2)\big) + 2\min\left(\mathrm{ht}^H(z_1), \mathrm{ht}^H(z_2)\right)\right| =\\
&= \left|\Psi_2\big((1,\zeta_1,v_1),(1,\zeta_2,v_2)\big) + \mathrm{ht}^H(z_1) + \mathrm{ht}^H(z_2) - \left|\mathrm{ht}^H(z_1) - \mathrm{ht}^H(z_2)\right|\right|\\
&= \left| - \beta_1(1,\zeta_2) - \beta_1(\zeta_1,1) + \beta_2(v_1,v_2) + \mathrm{ht}^H(z_1) + \mathrm{ht}^H(z_2) - \left|\mathrm{ht}^H(z_1) - \mathrm{ht}^H(z_2)\right|\right|\\
&= \left|\zeta_1 - \mathrm{ht}^H(z_1) + \overline{\zeta}_2 - \mathrm{ht}^H(z_2) - \beta_2(v_1,v_2) + \left|\mathrm{ht}^H(z_1) - \mathrm{ht}^H(z_2)\right|\right|.
\end{aligned}$$

Since

$$\zeta_1 - \mathrm{ht}^H(z_1) = \zeta_1 - \operatorname{Re}\zeta_1 + \tfrac{1}{2}|v_1|^2 = \operatorname{Im}\zeta_1 + \tfrac{1}{2}|v_1|^2$$

and

$$\overline{\zeta}_2 - \mathrm{ht}^H(z_2) = \overline{\zeta}_2 - \operatorname{Re}\zeta_2 + \tfrac{1}{2}|v_2|^2 = -\operatorname{Im}\zeta_2 + \tfrac{1}{2}|v_2|^2,$$

it follows that

$$\begin{aligned}
&\left|\Psi_2\big(i_H(z_1), i_H(z_2)\big) + 2\min\left(\mathrm{ht}^H(z_1), \mathrm{ht}^H(z_2)\right)\right| =\\
&= \left|\tfrac{1}{2}|v_1|^2 + \operatorname{Im}\zeta_1 + \tfrac{1}{2}|v_2|^2 - \operatorname{Im}\zeta_2 - \beta_2(v_1,v_2) + \left|\mathrm{ht}^H(z_1) - \mathrm{ht}^H(z_2)\right|\right|\\
&= \varrho^H(z_1,z_2)^2.
\end{aligned}$$

Since $\varrho^H(z_1,z_2)$ is nonnegative, the claim follows. □

For the proof of the following proposition we note that the map

$$\Psi_2^{(1)}\colon \begin{cases} E/Z^1(C)\times E/Z^1(C) & \to & C/Z^1(C)\\ \big([z_1],[z_2]\big) & \mapsto & \big[\Psi_2(z_1,z_2)\big]\end{cases}$$

is well-defined. In particular, we have

$$\left|\Psi_2^{(1)}\big([z_1],[z_2]\big)\right| = \left|\Psi_2(z_1,z_2)\right|$$

for all $z_1, z_2 \in E$.

Proposition 3.3.3. *Let $g \in G^{res}\smallsetminus G_\infty$ and $z \in \overline{H}^g\smallsetminus\{\infty, g^{-1}\infty\}$. Then*

$$|j(g,z)|^{1/2} = |j(g^{-1},\infty)|^{1/2}\varrho^H(z,g^{-1}\infty) = \frac{\varrho^H(z,g^{-1}\infty)}{R(g)}.$$

Proof. The second equality is proven by Lemma 3.3.1. For the proof of the first equality recall from Prop. 3.3.2 that

$$\varrho^H(z,g^{-1}\infty) = \left|\Psi_2\big(i_H(z), i_H(g^{-1}\infty)\big) + 2\min\big(\mathrm{ht}^H(z), \mathrm{ht}^H(g^{-1}\infty)\big)\right|^{1/2}.$$

From $g^{-1}\infty \in \partial_g\smallsetminus\{\infty\}$ it follows that $\mathrm{ht}(g^{-1}\infty) = 0$. Hence

$$\varrho(z,g^{-1}\infty)^2 = \left|\Psi_2\big(i_H(z), i_H(g^{-1}\infty)\big)\right|.$$

Suppose that $gz = (\zeta, v)$. Then

$$\begin{aligned}\Psi_2\big(i_H(gz), i_H(\infty)\big) &= \Psi_2\big((1,gz),(0,1,0)\big)\\ &= -\beta_1(1,1) - \beta_1(\zeta,0) + \beta_2(v,0) = -1.\end{aligned}$$

It follows that

$$\begin{aligned}\varrho^H(z,g^{-1}\infty)^2 &= \left|\Psi_2\big(i_H(z), i_H(g^{-1}\infty)\big)\right| = \left|\Psi_2^{(1)}\big(i_H^{(1)}(z), i_H^{(1)}(g^{-1}\infty)\big)\right|\\ &= \left|\Psi_2^{(1)}\big(i_H^{(1)}(z), j(g^{-1},\infty)\varphi_H(g^{-1}) i_H^{(1)}(\infty)\big)\right|\\ &= |j(g^{-1},\infty)|^{-1}\left|\Psi_2^{(1)}\big(i_H^{(1)}(z), \varphi_H(g^{-1}) i_H^{(1)}(\infty)\big)\right|\\ &= |j(g^{-1},\infty)|^{-1}\left|\Psi_2^{(1)}\big(\varphi_H(g) i_H^{(1)}(z), i_H^{(1)}(\infty)\big)\right|\\ &= |j(g^{-1},\infty)|^{-1}|j(g,z)|\left|\Psi_2\big(i_H(gz), i_H(\infty)\big)\right|\\ &= |j(g^{-1},\infty)|^{-1}|j(g,z)|.\end{aligned}$$

This proves the claim. □

Proposition 3.3.4. *Let $g \in G^{res}\smallsetminus G_\infty$. Then*

$$\begin{aligned} I(g) &= \{z \in H \mid |j(g,z)| = 1\},\\ \mathrm{ext}\, I(g) &= \{z \in H \mid |j(g,z)| > 1\},\\ \mathrm{int}\, I(g) &= \{z \in H \mid |j(g,z)| < 1\}.\end{aligned}$$

Proof. This claim follows immediately from comparing Def. 2.2.8 and Prop. 3.3.3. □

3.4. A special instance of Theorem 2.3.4

The most frequent appearance of isometric fundamental regions in the literature is for properly discontinuous subgroups Γ of G^{res} for which ∞ is an ordinary point and the stabilizer Γ_∞ is trivial. We show that then Γ is of type (O) and $\Gamma\setminus\Gamma_\infty$ of type (F), which immediately allows to apply Theorem 2.3.4 to show the existence of an isometric fundamental domain for Γ.

Definition 3.4.1. Let U be a subset of $\overline{H}^g$ and Γ a subgroup of G. Then Γ is said to *act properly discontinuously* on U if for each compact subset K of U, the set $K \cap gK$ is nonempty for only finitely many g in Γ. The group Γ is said to be *properly discontinous* if Γ acts properly discontinuously on H.

Let
$$\partial_g H := \left\{(\zeta, v) \in C \oplus V \;\middle|\; \operatorname{Re}(\zeta) = \tfrac{1}{2}|v|^2\right\} \cup \{\infty\}$$
denote the boundary of H in $\overline{H}^g$.

Definition and Remark 3.4.2. Let Γ be a properly discontinuous subgroup of G and $z \in H$. The *limit set* $L(\Gamma)$ of Γ is the set of accumulation points of the orbit Γz. Since Γ is properly discontinuous, $L(\Gamma)$ is a subset of $\partial_g H$. The combination of Prop. 2.9 and Prop. 1.4 in [EO73] shows that $L(\Gamma)$ is independent of the choice of z. The *ordinary set* or *discontinuity set* $\Omega(\Gamma)$ of Γ is the complement of $L(\Gamma)$ in $\partial_g H$, hence $\Omega(\Gamma) := \partial_g H \setminus L(\Gamma)$.

Remark 3.4.3. Let $g, h \in G^{\mathrm{res}}$ and $z \in \overline{H}^g$. Since φ_H is a group isomorphism and $\varphi_H(g)$ is C-linear, it follows that

$$\begin{aligned}
\varphi_H(gh)\big(i_H(z)\big) &= \varphi_H(g)\left(\varphi_H(h)\big(i_H(z)\big)\right)\\
&= \varphi_H(g)\big(j(h,z)i_H(hz)\big)\\
&= j(h,z)\varphi_H(g)\big(i_H(hz)\big)\\
&= j(h,z)j(g,hz)i_H(hgz).
\end{aligned}$$

Further,
$$\varphi_H(gh)\big(i_H(z)\big) = j(gh,z)i_H(ghz).$$

Thus,
$$j(gh,z) = j(h,z)j(g,hz).$$

The proof of the following lemma proceeds along the lines of [For72, Sec. 17].

Lemma 3.4.4. *Let Γ be a properly discontinuous subgroup of G^{res} such that $\infty \in \Omega(\Gamma)$ and $\Gamma_\infty = \{\mathrm{id}\}$. Then*

(i) *The set of radii of isometric spheres is bounded from above.*

(ii) *The number of isometric spheres with radius exceeding a given positive quantity is finite.*

Proof. We start by proving some relations between radii and distances of centers of isometric spheres. Let $g, h \in \Gamma \smallsetminus \Gamma_\infty$ such that $g \neq h^{-1}$. The cocycle relation shows that

$$j\big((gh)^{-1}, \infty\big) = j(g^{-1}, \infty) j(h^{-1}, g^{-1}\infty).$$

Prop. 3.3.3 yields

$$\begin{aligned}\big|j\big((gh)^{-1}, \infty\big)\big|^{-1/2} &= |j(g^{-1}, \infty)|^{-1/2} |j(h^{-1}, g^{-1}\infty)|^{-1/2} \\ &= |j(g^{-1}, \infty)|^{-1/2} |j(h, \infty)|^{-1/2} \varrho^H(g^{-1}\infty, h\infty)^{-1}.\end{aligned}$$

Note that $gh \neq \mathrm{id}$ and therefore $gh \notin \Gamma_\infty$. Then Lemma 3.3.1 shows that

$$R(gh) = \frac{R(g)R(h^{-1})}{\varrho^H(g^{-1}\infty, h\infty)}. \tag{3.8}$$

Using the same arguments, we find

$$j(g^{-1}, \infty) = j\big(h(gh)^{-1}, \infty\big) = j\big((gh)^{-1}, \infty\big) j\big(h, (gh)^{-1}\infty\big)$$

and therefore

$$\begin{aligned}R(g) = |j(g^{-1}, \infty)|^{-1/2} &= \big|j\big((gh)^{-1}, \infty\big)\big|^{-1/2} \big|j\big(h, (gh)^{-1}\infty\big)\big|^{-1/2} \\ &= R(gh)\big|j\big(h, (gh)^{-1}\infty\big)\big|^{-1/2}.\end{aligned} \tag{3.9}$$

For $\big|j\big(h, (gh)^{-1}\infty\big)\big|^{1/2}$, Prop. 3.3.3 shows that identity

$$\big|j\big(h, (gh)^{-1}\infty\big)\big|^{1/2} = R(h)^{-1} \varrho^H\big((gh)^{-1}\infty, h^{-1}\infty\big). \tag{3.10}$$

Because $R(h) = R(h^{-1})$, it follows from (3.8)-(3.10) that

$$\begin{aligned}\varrho^H\big((gh)^{-1}\infty, h^{-1}\infty\big) = \big|j\big(h, (gh)^{-1}\infty\big)\big|^{1/2} R(h) &= \frac{R(gh)R(h)}{R(g)} \\ &= \frac{R(h)^2}{\varrho^H(g^{-1}\infty, h\infty)}.\end{aligned} \tag{3.11}$$

Since Γ is properly discontinuous and $\infty \in \Omega(\Gamma)$, [EO73, Prop. 8.5] shows[1] that there exists an open neighborhood U of ∞ in $\overline{H}^g$ such that $(\Gamma \smallsetminus \Gamma_\infty) \subseteq \overline{H}^g \smallsetminus U$. Since $\overline{H}^g \smallsetminus U$ is compact, we find $m > 0$ such that

$$\varrho^H(a\infty, b\infty) < m$$

for all $a, b \in \Gamma \smallsetminus \Gamma_\infty$. Then (3.11) shows that

$$R(h)^2 = \varrho^H\big((gh)^{-1}\infty, h^{-1}\infty\big) \varrho^H(g^{-1}\infty, h\infty) < m^2. \tag{3.12}$$

Thus, for each $a \in \Gamma \smallsetminus \Gamma_\infty$ we have $R(a) < m$, which proves (i).

[1] I would like to thank Martin Olbrich to point me to this article.

Now let $k > 0$ and suppose that $a, b \in \Gamma \smallsetminus \Gamma_\infty$ such that $I(a) \neq I(b^{-1})$ and $R(a), R(b) > k$. Since $a \neq b^{-1}$, (3.8) shows in combination with (3.12) that

$$\varrho^H(a^{-1}\infty, b\infty) = \frac{R(a)R(b)}{R(ab)} > \frac{k^2}{m}.$$

This means that the distance between the centers of isometric spheres whose radii exceed k is bounded from below by k^2/m. The centers of all these isometric spheres are contained in the compact set $\partial_g H \smallsetminus U$, which is bounded in $C \times V$ and hence also w. r. t. ϱ^H. It follows that there are only finitely many spheres with radius exceeding k. This proves (ii). □

Proposition 3.4.5. *Let Γ be a properly discontinuous subgroup of G^{res} such that $\infty \in \Omega(\Gamma)$ and $\Gamma_\infty = \{\mathrm{id}\}$. Then Γ is of type (O) and $\Gamma \smallsetminus \Gamma_\infty$ of type (F). Moreover,*

$$\mathcal{F} := \bigcap_{g \in \Gamma \smallsetminus \Gamma_\infty} \operatorname{ext} I(g)$$

is a fundamental domain for Γ in H.

Proof. For each $z \in H$, the map $\varrho^H(\cdot, z) \colon H \to \mathbb{R}$ is continuous. Therefore each ϱ^H-ball is open in H. Lemma 3.4.4(ii) implies that the set $\{\operatorname{int} I(g) \mid g \in \Gamma \smallsetminus \Gamma_\infty\}$ is locally finite. Then Remark 2.3.3 shows that Γ is of type (O). Note that the subgroup $\langle \Gamma \smallsetminus \Gamma_\infty \rangle$ of Γ which is generated by $\Gamma \smallsetminus \Gamma_\infty$ is exactly Γ. Let $z \in H$. Lemma 5 before Thm. 5.3.4 in [Rat06] states that Γz is a closed subset of H. Since $\infty \in \Omega(\Gamma)$, [EO73, Prop. 8.5] shows that we find an open neighborhood U of ∞ in $\overline{H}^g$ such that $\Gamma z \subseteq \overline{H}^g \smallsetminus U$. Now $\overline{H}^g \smallsetminus U$ is compact and therefore Γz is so. The height function is continuous, which shows that the maximum of $\{\operatorname{ht}(gz) \mid g \in \Gamma\}$ exists. Thus, $\Gamma \smallsetminus \Gamma_\infty$ is of type (F). By Theorem 2.3.4, $\mathcal{F}$ is a fundamental region for Γ in H. Obviously, $\mathcal{F}$ is connected and hence a fundamental domain for Γ in H. □

3.5. Isometric spheres and isometric fundamental regions in the literature

For real hyperbolic spaces, definitions of isometric spheres are given at several places, e. g., the original definition of Ford in [For72] for the plane, in [Kat92] for the upper half plane model and the ball model, in [Mar07] for three-dimensional space and in [Apa00] (or, earlier, in [Apa91]) for arbitrary dimensions. The definition for the upper half plane model is not equivalent to that in the ball model (see [Mar07]). The definition for the upper half plane model is the original definition by Ford and has been directly generalized to higher dimensions. Ford and Apanasov also treat the existence of isometric fundamental regions.

For complex hyperbolic spaces, the (to the knowledge of the author) only existing definitions are given by Parker in [Par94], Goldman in [Gol99], and Kamiya in [Kam03]. Kamiya also discusses the existence of isometric fundamental regions for special groups.

For quaternionic hyperbolic spaces, an investigation of isometric fundamental regions does not seem to exist. A definition of isometric spheres is provided by [KP03].

In this section we discuss the relation between the existing definitions of isometric spheres (for real hyperbolic space only exemplarily) and the existing statements on the existence of isometric fundamental regions with our definition of isometric sphere and with Theorem 2.3.4.

3.5.1. Real hyperbolic spaces

Ford [For72] provides proofs of isometric fundamental regions for group of isometries acting on real hyperbolic plane (see [For72, Thm. 15 and 22 in Ch. III]). His definition of fundamental region is substantially weaker than our definition. In particular, the translates of a fundamental region in sense of [For72] are not required to cover the whole space. For this reason, the hypothesis of [For72, Thm. 15, 22] are weaker than that of Theorem 2.3.4. However, the following discussion shows that the definition of isometric spheres in [For72] is contained in our definition.

In [Apa00], the model

$$D' = \{(t, Z) \in \mathbb{R} \times \mathfrak{z} \mid t > 0\}$$

of real hyperbolic space is used and the isometric sphere for an element $g \in G^{\mathrm{res}}$, acting on D', is defined as

$$I(g) := \{z \in D' \mid |g'(z)| = 1\}.$$

More precisely, Apanasov (as all other references) uses the coordinates in the order $\mathfrak{z} \times \mathbb{R}$. In particular, his model space for two-dimensional real hyperbolic space is the upper half plane, whereas D' is the right half plane. Clearly, this difference has no affect on his definition of isometric sphere. In Lemma 3.5.2 we will show that this definition of isometric spheres is subsumed by our definition. For the proof of this lemma we first derive explicit formulas for the action of G^{res} on D'.

The set D' is the space D from Sec. 1.4.1 constructed from the abelian H-type algebra $\mathfrak{n} = (\mathfrak{n}, \{0\}) = (\mathfrak{z}, \{0\})$. In Sec. 3-3.4 we had to work with the ordered decomposition $(\{0\}, \mathfrak{v}) = (\{0\}, \mathfrak{n})$. Then

$$D = \left\{(u, X) \in \mathbb{R} \times \mathfrak{v} \;\middle|\; u > \tfrac{1}{4}|X|^2\right\}.$$

Remark 3.5.1. According to [CDKR98], the isometry from D' to D is given by

$$\nu\colon \begin{cases} D' & \to & D \\ (t, Z) & \mapsto & (t^2 + |Z|^2, 2Z). \end{cases}$$

Its inverse is

$$\nu^{-1}\colon \begin{cases} D & \to & D' \\ (u, X) & \mapsto & \left(\sqrt{u - \tfrac{1}{4}|X|^2}, \tfrac{1}{2}X\right). \end{cases}$$

Let $(t, Z) \in D'$. Suppose that $n = (1, w) \in N$ (note that $C' = \{0\}$). Then

$$\begin{aligned}
\nu_G(n)(t, Z) &= \nu^{-1} \circ n \circ \nu(t, Z) \\
&= \nu^{-1} \circ n(t^2 + |Z|^2, 2Z) \\
&= \nu^{-1}\big(t^2 + |Z|^2 + \tfrac{1}{4}|w|^2 + \langle Z, w\rangle, w + 2Z\big) \\
&= \left(\sqrt{t^2 + |Z|^2 + \tfrac{1}{4}|w|^2 + \langle Z, w\rangle - \tfrac{1}{4}|w + 2Z|^2}, \tfrac{1}{2}w + Z\right) \\
&= \big(t, \tfrac{1}{2}w + Z\big).
\end{aligned}$$

Suppose that $m = (\mathrm{id}, \psi) \in M^{\mathrm{res}}$. Then

$$\begin{aligned}
\nu_G(m)(t, Z) &= \nu^{-1} \circ m \circ \nu(t, Z) \\
&= \nu^{-1} \circ m(t^2 + |Z|^2, 2Z) \\
&= \nu^{-1}(t^2 + |Z|^2, 2\psi(Z)) \\
&= \left(\sqrt{t^2 + |Z|^2 - \tfrac{1}{4}|2\psi(Z)|^2}, \psi(Z)\right) \\
&= (t, \psi(Z)).
\end{aligned}$$

Suppose that $a_s \in A$. Then

$$\begin{aligned}
\nu_G(a_s)(t, Z) &= \nu^{-1} \circ a_s \circ \nu(t, Z) \\
&= \nu^{-1} \circ a_s(t^2 + |Z|^2, 2Z) \\
&= \nu^{-1}\big(s(t^2 + |Z|^2), 2s^{1/2}Z\big) \\
&= \left(\sqrt{2(t^2 + |Z|^2) - \tfrac{1}{4}|2s^{1/2}Z|^2}, s^{1/2}Z\right) \\
&= \big(s^{1/2}t, s^{1/2}Z\big).
\end{aligned}$$

Finally,

$$\begin{aligned}
\nu_G(\sigma)(t, Z) &= \nu^{-1} \circ \sigma \circ \nu(t, Z) \\
&= \nu^{-1} \circ \sigma(t^2 + |Z|^2, 2Z) \\
&= \nu^{-1}\big((t^2 + |Z|^2)^{-1}, -2(t^2 + |Z|^2)^{-1}Z\big) \\
&= \left(\sqrt{(t^2 + |Z|^2)^{-1} - \tfrac{1}{4}|2(t^2 + |Z|^2)^{-1}Z|^2}, -(t^2 + |Z|^2)^{-1}Z\right) \\
&= (t^2 + |Z|^2)^{-1}(t, -Z).
\end{aligned}$$

Recall the isometry $\beta\colon H \to D$ from Sec. 3.2.3.

Lemma 3.5.2. *Let $z \in D'$ and $g \in G^{res}$. Then*

$$\big|j\big(g, \beta^{-1} \circ \nu(z)\big)\big|^{-1} = \big|\nu_G(g)'(z)\big|.$$

Proof. Let $z = (t, Z) \in D'$ and $g = n_2 \sigma m a_s n_1 \in G^{\mathrm{res}}$ with $n_j = (1, w_j)$, $j = 1, 2$, and $m = (\mathrm{id}, \psi)$. At first we calculate the value of $\big|j\big(g, \beta^{-1} \circ \nu(z)\big)\big|$. We have

$$\begin{aligned}
\varphi_H(g) i_H\big(\beta^{-1} \circ \nu(z)\big) &= \varphi_H(g)\big(1, t^2 + |Z|^2, \sqrt{2}Z\big) \\
&= \varphi_H(n_2 \sigma m a_s)\varphi_H(n_1)\big(1, t^2 + |Z|^2, \sqrt{2}Z\big) \\
&= \varphi_H(n_2 \sigma m)\varphi_H(a_s)\big(1, \tfrac{1}{4}|w_1|^2 + t^2 + |Z|^2 + \beta_2(Z, w_1), *\big) \\
&= \varphi_H(n_2 \sigma)\varphi_H(m)\big(s^{-1/2}, s^{1/2}\big(\tfrac{1}{4}|w_1|^2 + t^2 + |Z|^2 + \beta_2(Z, w_1)\big), *\big) \\
&= \varphi_H(n_2)\varphi_H(\sigma)\big(s^{-1/2}, s^{1/2}\big(\tfrac{1}{4}|w_1|^2 + t^2 + |Z|^2 + \beta_2(Z, w_1)\big), *\big) \\
&= \varphi_H(n_2)\big(s^{1/2}\big(\tfrac{1}{4}|w_1|^2 + t^2 + |Z|^2 + \beta_2(Z, w_1)\big), *, *\big) \\
&= \big(s^{1/2}\big(\tfrac{1}{4}|w_1|^2 + t^2 + |Z|^2 + \beta_2(Z, w_1)\big), *, *\big).
\end{aligned}$$

Thus

$$\big|j\big(g, \beta^{-1} \circ \nu(z)\big)\big| = s^{1/2}\big|\tfrac{1}{4}|w_1|^2 + t^2 + |Z|^2 + \beta_2(Z, w_1)\big|.$$

Since $\beta_2(Z, w_1) \in R$, we have $\beta_2(Z, w_1) = \langle Z, w_1 \rangle$ and therefore

$$\big|j\big(g, \beta^{-1} \circ \nu(z)\big)\big| = s^{1/2}\big|t^2 + \big|\tfrac{1}{2}w_1 + Z\big|^2\big| = \big|\big(t, \tfrac{1}{2}w_1 + Z\big)\big|^2.$$

Let $(u, W) \in \mathbb{R} \times \mathfrak{z}$. For the derivative $\nu_G(g)'(z)$ we find

$$\begin{aligned}
\nu_G(g)'(z)(u, W) &= \nu_G(n_2 \sigma m a_s)'\big(n_1(z)\big)\nu_G(n_1)'(z)(u, W) \\
&= \nu_G(n_2 \sigma m)'\big(\nu_G(a_s)(t, \tfrac{1}{2}w_1 + Z)\big)\nu_G(a_s)'(t, \tfrac{1}{2}w_1 + Z)(u, W) \\
&= \nu_G(n_2 \sigma m)'\big(s^{1/2}t, s^{1/2}(\tfrac{1}{2}w_1 + Z)\big)\big(s^{1/2}u, s^{1/2}W\big) \\
&= \nu_G(n_2 \sigma)'\big(s^{1/2}t, s^{1/2}\psi(\tfrac{1}{2}w_1 + Z)\big)\big)\big(s^{1/2}u, s^{1/2}\psi(W)\big) \\
&= \nu_G(n_2)(\dots)\Big(-s^{-1/2}\big(t, \psi(\tfrac{1}{2}w_1 + Z)\big)^{-1}\big(u, \psi(W)\big)\big(t, \psi(\tfrac{1}{2}w_1 + Z)\big)^{-1}\Big) \\
&= -s^{-1/2}\big(t, \psi(\tfrac{1}{2}w_1 + Z)\big)^{-1}\big(u, \psi(W)\big)\big(t, \psi(\tfrac{1}{2}w_1 + Z)\big)^{-1}.
\end{aligned}$$

Then

$$\begin{aligned}
\big|\nu_G(g)'(z)(u, W)\big| &= s^{-1/2}\big|\big(t, \psi\big(\tfrac{1}{2}w_1 + Z\big)\big)\big|^{-1}\big|\big(u, \psi(W)\big)\big|\big|\big(t, \psi\big(\tfrac{1}{2}w_1 + Z\big)\big)\big|^{-1} \\
&= s^{-1/2}\big|\big(t, \tfrac{1}{2}w_1 + Z\big)\big|^{-2}\big|(u, W)\big|.
\end{aligned}$$

Thus,

$$\big|\nu_G(g)'(z)\big| = s^{-1/2}\big|\big(t, \tfrac{1}{2}w_1 + Z\big)\big|^{-2} = \big|j\big(g, \beta^{-1} \circ \nu(z)\big)\big|^{-1}.$$

This completes the proof. □

Let $g \in G^{\mathrm{res}}$. From Lemma 3.5.2 and Prop. 3.3.4 it follows that

$$\operatorname{ext} I(g) = \big\{z \in D' \;\big|\; |\nu_G(g)'(z)| < 1\big\}.$$

Then [Apa00, Thm. 2.30] is a special case of Prop. 3.4.5. Lemma 2.31 in [Apa00] states an extension of Thm. 2.30 for subgroups Γ of G^{res} with $\Gamma_\infty \neq \{\mathrm{id}\}$. Unfortunately, the hypotheses of [Apa00, Lemma 2.31] are not completely stated, for which reason we cannot compare the lemma with Theorem 2.3.4.

3.5.2. Complex hyperbolic spaces

The isometric spheres for isometries of complex hyperbolic spaces in [Kam03] are identical to those in [Gol99]. Kamiya uses the model H and defines the Cygan metric by formula (3.7). Further, the isometric sphere of an element $f \in \mathrm{PU}(1,n;\mathbb{C}) = \mathrm{PU}(\Psi_2,\mathbb{C})$ with $f = (a_{ij})_{i,j=1,\dots,n+1}$ is defined as

$$I(f) = \left\{ z \in H \;\middle|\; \varrho(z, f^{-1}\infty) = R_f \right\}$$

where $R_f = |a_{12}|^{-1/2}$. The following lemma shows that this definition is covered by our definition of isometric spheres.

Lemma 3.5.3. *Let $f = (a_{ij})_{i,j=1,\dots,n+1} \in \mathrm{PU}(1,n;\mathbb{C})$. Then $R_f = R\big(\varphi_H^{-1}(f)\big)$.*

Proof. Set $g := \varphi_H^{-1}(f)$. We have

$$f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}a_{12}\\ \vdots \\ a_{n+1,2}\end{pmatrix}.$$

Therefore

$$|j(g,\infty)| = |a_{12}|.$$

Since $|j(g^{-1},\infty)| = |j(g,\infty)|$, Lemma 3.3.1 shows that $R(g) = |a_{12}|^{-1/2}$. □

[Kam03, Thm. 3.1] states the existence of isometric fundamental domains for discrete subgroups Γ of G^{res} for which, after possible conjugation of Γ, we have $\infty \in \Omega(\Gamma)$ and $\Gamma_\infty = \{\infty\}$. By [Rat06, Thm. 5.3.5], Γ is discrete if and only if Γ is properly discontinuous. Therefore, Kamiya's Theorem is a special case of Prop. 3.4.5.

In [Par94], Parker uses a section of the projection map from $E \smallsetminus \{0\}$ to horospherical coordinates which is reminiscent of the ball model. Therefore, we expect that, as in the real case, this definition is not equivalent to the definition from [Kam03].

3.5.3. Quaternionic hyperbolic spaces

In [KP03], Kim and Parker propose a definition of isometric spheres for isometries in G^{res} of quaternionic hyperbolic space. They use the model H and horospherical coordinates for the definition. With the bijection

$$\begin{cases} \mathfrak{z} \times \mathfrak{v} & \to \quad \mathfrak{z} \times \mathfrak{v} \\ (Z, X) & \mapsto \quad \big(Z, \tfrac{1}{2}X\big), \end{cases}$$

our Heisenberg group N and our Cygan metric is transfered into their one. After shuffling coordinates, they define the isometric sphere of an element $g = (a_{ij})_{i,j=1,\dots,n+1}$ in $\mathrm{PSp}(n,1)$ as, more precisely, they characterize it in their Prop. 4.3 as

$$I(g) = \left\{ z \in H \;\middle|\; \varrho(z, g^{-1}\infty) = \sqrt{2}\cdot |a_{12}|^{1/2} \right\}.$$

As in Sec. 3.5.2 we see that our definition gives $|a_{12}|^{1/2}$ as radius. The factor $\sqrt{2}$ is due to the factor $\frac{1}{2}$ in their choice of the section of the projection from $E_{-}(\Psi_2)$ to H. Note also that they use a slightly different indefinite form on E for the definition of the hyperbolic space. However, this difference does not affect the value of the radius of the isometric sphere.

Part II.

Cusp expansion

Let H be the hyperbolic plane and Γ a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of which ∞ is a cuspidal point. In this part we present the method of cusp expansion for the construction of a symbolic dynamics for the geodesic flow on the locally symmetric orbifold $\Gamma\backslash H$, where Γ has to satisfy some additional (mild) conditions.

The fundamental and starting object is the convex hyperbolic polyhedron

$$\mathcal{K} := \bigcap_{g\in\Gamma\smallsetminus\Gamma_\infty} \operatorname{ext} I(g),$$

which is the common part of the exteriors of the isometric spheres of Γ. Our conditions on Γ will imply that the boundary $\partial\mathcal{K}$ of $\mathcal{K}$ in H consists of non-trivial segments of isometric spheres. By a non-trivial segment we mean a connected subset which contains more than one element. An isometric sphere which effectively contributes to $\partial\mathcal{K}$ will be called *relevant*. We will require that the point of maximal height, the *summit*, of each relevant isometric sphere is contained in $\partial\mathcal{K}$. Let $\partial_g H$ denote the geodesic boundary of H. To each vertex v of $\mathcal{K}$ in H or $\partial_g H$ (other than ∞) we attach one (if $v \in H$) or two (if $v \in \partial_g H$) sets, which we call *precells in H*. If $v \in H$, the precell attached to v is the hyperbolic quadrilateral with vertices v, the two summits adjacent to v, and ∞. If $v \in \partial_g H$, then v might have one or two adjacent summits. In any case, one of the precells attached to v is the hyperbolic triangle with vertices v, one summit adjacent to v, and ∞. If there are two adjacent summits, then the other precell is of the same form but having the other summit as vertex. If there is only one adjacent summit, then the other precell is the vertical strip on H between v and the adjacent vertex of $\mathcal{K}$ in $\partial_g H$. The family of all precells in H is, up to boundary components, a decomposition of $\mathcal{K}$. Certain finite sets of precells in H will be called a *basal family of precells in H*. These sets are characterized by the property that the union of their elements is the closure of an isometric fundamental region for Γ in H. In particular, a basal family of precells in H is a set of representatives for the Γ_∞-orbits in the set of all precells in H.

The next step is to combine precells in H to so-called *cells in H*. The key idea behind this construction is that the family of cells in H should satisfy the following properties: Each cell in H shall be a union of precells in H such that the emerging set is a finite-sided n-gon with all vertices in $\partial_g H$. Further, each cell shall have two vertical sides (in other word, ∞ is a vertex of each cell) and each non-vertical side of a cell shall be a Γ-translate of a vertical side of some cell. Finally, the family of all cells in H shall provide a tesselation of H. Suppose that $\mathbb{A}$ is a basal family of precells in H. Then there is a side-pairing of the non-vertical sides of basal precells in H. Each precell which is attached to a vertex of $\mathcal{K}$ in H has two non-vertical sides. This fact allows to deduce from the side-paring a natural notion of *cycles* (cyclic sequences) in $\mathbb{A}\times\Gamma$, where a pair $(\mathcal{A}, g) \in \mathbb{A}\times\Gamma$ encodes that g maps one non-vertical side of $\mathcal{A}$ to a non-vertical side of some element in $\mathbb{A}$. This notion is easily extended to basal precells which are attached to vertices of $\mathcal{K}$ in $\partial_g H$. Moreover, there is a natural notion of equivalence of cycles. Each cell in H is the union of certain Γ-translates of the basal precells in some equivalence class of cycles. At this point the requirement that the summit of each relevant isometric sphere be in $\partial\mathcal{K}$ becomes crucial. It guarantees that each cell in H satisfies the requirements on its boundary structure mentioned above.

Let $\mathbb{B}$ denote the family of cells in H constructed from $\mathbb{A}$. Further suppose that $\pi\colon SH \to \Gamma\backslash SH$ denotes the canonical projection map of the unit tangent bundle of H onto the unit tangent bundle of the orbifold $\Gamma\backslash H$. Let $\widetilde{\mathrm{BS}}$ denote the set of boundary points of the elements in $\mathbb{B}$ and suppose that $\widetilde{\mathrm{CS}}$ denotes the set of unit tangent vectors based on $\widetilde{\mathrm{BS}}$ which are not tangent to $\widetilde{\mathrm{BS}}$. We will show that $\widehat{\mathrm{CS}} := \pi\big(\widetilde{\mathrm{CS}}\big)$ is a cross section for the geodesic flow with respect to certain measures μ and we will also characterize these measures. To that end we extend the notions of precells and cells in H to SH. Each precell in H induces a precell in SH in a geometric way. There is even a canonical bijection between precells in H and precells in SH. As before, let $\mathbb{A}$ be a basal family of precells in H. For each equivalence class of cycles in $\mathbb{A} \times \Gamma$ we fix a so-called generator. The set of chosen generators will be denoted by $\mathbb{S}$. Relative to $\mathbb{S}$ we perform a construction of cells in SH from basal precells in SH similar to the construction in H. The union of all cells in SH will be seen to be a fundamental set for Γ in SH. It is exactly this property of cells in SH which will allow to characterize the geodesics on $\Gamma\backslash H$ which intersect $\widehat{\mathrm{CS}}$ infinitely often in past and future, and which in turn allows to characterize the measures μ. It will turn out that exactly those geodesics do not intersect $\widehat{\mathrm{CS}}$ infinitely often in future or past which have one endpoint in the geodesic boundary of the orbifold.

The switch from H to SH brings an additional degree of freedom to the construction without destroying any features. One is allowed to shift each cell in SH (independently from each other) by any element of Γ_∞. The map which fixes for each cell $\widetilde{\mathcal{B}}$ in SH an element of Γ_∞ by which $\widetilde{\mathcal{B}}$ is shifted will be denoted by $\mathbb{T}$. The boundary structure of cells in H and the choices of $\mathbb{S}$ and $\mathbb{T}$ will be seen to induce a natural labeling of $\widehat{\mathrm{CS}}$. In this way, we have geometrically constructed a cross section and a symbolic dynamics to which we refer as geometric cross section and geometric symbolic dynamics. The geometric cross section does not depend on the choice of $\mathbb{A}$, $\mathbb{S}$ or $\mathbb{T}$; its labeling, however, does.

Suppose that R denotes the first return map of the cross section. Then $(\widehat{\mathrm{CS}}, R)$ is the to $\widehat{\mathrm{CS}}$ associated discrete dynamical system. In general, $(\widehat{\mathrm{CS}}, R)$ is not conjugate to a discrete dynamical system $(\widetilde{\mathrm{DS}}, \widetilde{F})$ for some $\widetilde{\mathrm{DS}} \subseteq \mathbb{R}\times\mathbb{R}$. But $\widehat{\mathrm{CS}}$ contains a subset $\widehat{\mathrm{CS}}_{\mathrm{red}}$ for which $(\widehat{\mathrm{CS}}_{\mathrm{red}}, R)$ is naturally conjugate to a discrete dynamical system in some subset of $\mathbb{R}\times\mathbb{R}$. The set $\widehat{\mathrm{CS}}_{\mathrm{red}}$ is itself a cross section (with respect to the same measures as $\widehat{\mathrm{CS}}$) and can be constructed effectively in a geometric way from $\widehat{\mathrm{CS}}$. The labeling of $\widehat{\mathrm{CS}}$ induces a labeling of $\widehat{\mathrm{CS}}_{\mathrm{red}}$ and the conjugate discrete dynamical system $(\widetilde{\mathrm{DS}}, \widetilde{F})$ is of continued fraction type. In contrast to $\widehat{\mathrm{CS}}$, the set $\widehat{\mathrm{CS}}_{\mathrm{red}}$ depends on the choice of $\mathbb{A}$, $\mathbb{S}$ and $\mathbb{T}$.

The definition of precells in H and the construction of cells in H from precells in based on ideas in [Vul99]. Our construction differs from Vulakh's in three aspects: We define three kinds of precells in H of which only the non-cuspidal ones are precells in sense of Vulakh. In turn, cells arising from cuspidal or strip precells are not cells in sense of Vulakh. Finally, we extend the considerations to precells and cells in unit tangent bundle.

The necessary properties of the setting in which we will work are stated in Sec. 4. Sec. 5 introduces the notions of symbolic dynamics. In Sec. 6 we present the cusp expansion method and perform it for some examples. The transfer operators of the discrete dynamical systems arising from cusp expansion are briefly treated in Sec. 7. Finally, in Sec. 8, we discuss the relation of the cross section constructed using cusp expansion for the modular group $\mathrm{PSL}(2,\mathbb{Z})$ to the cross section constructed by Series [Ser85].

4. Preliminaries

We take the upper half plane

$$H := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$$

with the Riemannian metric given by the line element $ds^2 = y^{-2}(dx^2 + dy^2)$ as a model for two-dimensional real hyperbolic space. The associated Riemannian metric will be denoted by d_H. The space H is the model D' from Sec. 3.5.1 with $\mathfrak{z} = \mathbb{R}$ and the coordinates in the order $\mathfrak{z} \times \mathbb{R}$. We identify the group of orientation-preserving isometries with $\mathrm{PSL}(2, \mathbb{R})$ via the left action

$$\begin{cases} \mathrm{PSL}(2,\mathbb{R}) \times H & \to & H \\ \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\right) & \mapsto & \frac{az+b}{cz+d}. \end{cases}$$

One easily checks that the center of $\mathrm{SL}(2, \mathbb{R})$ is $\{\pm \mathrm{id}\}$. Therefore $\mathrm{PSL}(2, \mathbb{R})$ is the quotient group

$$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}.$$

We denote an element of $\mathrm{PSL}(2, \mathbb{R})$ by any of its representatives in $\mathrm{SL}(2, \mathbb{R})$. The one-point compactification of the closure of H in $\mathbb{C}$ will be denoted by $\overline{H}^g$, hence

$$\overline{H}^g = \{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\} \cup \{\infty\}.$$

It is homeomorphic to the geodesic compactification of H. The action of $\mathrm{PSL}(2, \mathbb{R})$ extends continuously to the boundary $\partial_g H = \mathbb{R} \cup \{\infty\}$ of H in $\overline{H}^g$.

The geodesics on H are the semicircles centered on the real line and the vertical lines. All geodesics shall be oriented and parametrized by arc length. For each element v of the unit tangent bundle SH there exists a unique geodesic γ_v on H such that $\gamma_v'(0) = v$. We call γ_v the *geodesic determined by* $v \in SH$. The (unit speed) *geodesic flow* on H is the dynamical system

$$\Phi\colon \begin{cases} \mathbb{R} \times SH & \to & SH \\ (t, v) & \mapsto & \gamma_v'(t). \end{cases}$$

Let Γ be a discrete (or, equivalently, a properly discontinuous) subgroup of $\mathrm{PSL}(2, \mathbb{R})$. The orbit space

$$Y := \Gamma\backslash H$$

is naturally equipped with the structure of a good Riemannian orbifold. Since H is a symmetric space of rank one, we call Y a *locally symmetric good orbifold of rank one*. This notion is a natural extension of the notion of locally symmetric spaces of rank one. The orbifold Y inherits all geometric properties of H that are Γ-invariant. Vice versa, several geometric entities of Y can be understood as the

Γ-equivalence class of the corresponding geometric entity on H. In particular, the geodesics on Y correspond to Γ-equivalence classes of geodesics on H, and the unit tangent bundle SY of Y is the orbit space of the induced Γ-action on the unit tangent bundle SH. Let $\pi\colon H \to Y$ and $\pi\colon SH \to SY$ denote the canonical projection maps. Then the *geodesic flow* on Y is given by

$$\widehat{\Phi} := \pi \circ \Phi \circ (\mathrm{id} \times \pi^{-1})\colon \mathbb{R} \times SY \to SY.$$

Here, π^{-1} is an arbitrary section of π. One easily sees that $\widehat{\Phi}$ does not depend on the choice of π^{-1}.

Since Γ is discrete, there exists a fundamental domain $\mathcal{F}$ for Γ. The set $\mathcal{F}$ might touch $\partial_g H$ at some points. By touching a point $z \in \partial_g H$ we mean that there is a neighborhood U of z in the topology of $\overline{H}^g$ such that the intersection of the closures in $\overline{H}^g$ of all boundary components of $\mathcal{F}$ that are intersected by U is z. In some cases one can characterize these points as fixed points of parabolic elements of Γ (see [Rat06, Thm. 12.3.7]). Those points will play a special role in the cusp expansion.

An element $g \in \Gamma$ is called *parabolic* if $g \neq \mathrm{id}$ and $|\operatorname{tr}(g)| = 2$, or equivalently, if g fixes exactly one point in $\partial_g H$. An element z in $\partial_g H$ is a *cuspidal point of* Γ if Γ contains a parabolic element that stabilizes z. A *cusp* of Γ is a Γ-equivalence class of a cuspidal point of Γ. If $\mathcal{F}$ is a fundamental domain for Γ such that $\overline{\mathcal{F}}$ is a convex fundamental polyhedron, then $\mathcal{F}$ touches $\partial_g H$ in at least one representative of each cusp (see [Rat06, Thm. 12.3.7, Cor. 2 of Thm. 12.3.5]).

Because a convex fundamental polyhedron for Γ can (up to Γ-equivalent boundary points) be identified with Y, the following definition is natural. Let z be a cuspidal point of Γ and extend the projection $\pi\colon H \to Y$ to $\pi\colon \overline{H}^g \to \Gamma\backslash\overline{H}^g$. Then $\pi(z)$ is called a *cusp of* Y or also a *cusp of* Γ.

We conclude this section with some conventions.

Let I be an interval in $\mathbb{R}$. A *geodesic arc* is a curve $\alpha\colon I \to H$ that can be extended to a geodesic. In particular, each geodesic is a geodesic arc. A *geodesic segment* is the image of a geodesic arc. If α is a geodesic, then $\alpha(\mathbb{R})$ is called a *complete geodesic segment*. A geodesic segment is called *non-trivial* if it contains more than one element.

We let $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ denote the two-point compactification of $\mathbb{R}$ and extend the ordering of $\mathbb{R}$ to $\overline{\mathbb{R}}$ by the definition $-\infty < a < \infty$ for each $a \in \mathbb{R}$.

If $\alpha\colon I \to H$ is a geodesic arc and $a < b$ are the boundary points of I in $\overline{\mathbb{R}}$, then the points

$$\alpha(a) := \lim_{t \to a} \alpha(t) \in \overline{H}^g \qquad \text{and} \qquad \alpha(b) := \lim_{t \to b} \alpha(t) \in \overline{H}^g$$

are called the *endpoints* of α and of the associated geodesic segment $\alpha(I)$.

The geodesic segment $\alpha(I)$ is often denoted as

$$\alpha(I) = \begin{cases} [\alpha(a), \alpha(b)] & \text{if } a, b \in I, \\ [\alpha(a), \alpha(b)) & \text{if } a \in I,\, b \notin I, \\ (\alpha(a), \alpha(b)] & \text{if } a \notin I,\, b \in I, \\ (\alpha(a), \alpha(b)) & \text{if } a, b \notin I. \end{cases}$$

If $\alpha(a), \alpha(b) \in \partial_g H$, it will always be made clear whether we refer to a geodesic segment or an interval in $\mathbb{R}$.

Let U be a subset of H. Recall that the closure of U in H is denoted by $\overline{U}$ or $\mathrm{cl}(U)$, its boundary is denoted by ∂U. Its interior is denoted by U°. To increase clarity, we denote the closure of a subset V of $\overline{H}^g$ in $\overline{H}^g$ by $\overline{V}^g$ or $\mathrm{cl}_{\overline{H}^g} V$. Moreover, we set $\partial_g V := \overline{V}^g \cap \partial_g H$, which can be understood as the geodesic boundary of V. For a subset $W \subseteq \mathbb{R}$ let $\mathrm{int}_{\mathbb{R}}(W)$ denote the interior of W in $\mathbb{R}$ and $\partial_{\mathbb{R}} W$ the boundary of W in $\mathbb{R}$. If X is a subset of $\partial_g H$, then $\mathrm{int}_g(X)$ denotes the interior of X in $\partial_g H$. If $X \subseteq \mathbb{R}$, then $\mathrm{int}_g(X) = \mathrm{int}_{\mathbb{R}}(X)$.

Recall that for two sets A, B, the complement of B in A is denoted by $A \smallsetminus B$. In contrast, if Γ acts on A, the space of left cosets is written as $\Gamma \backslash A$. For example, $\Gamma_\infty \backslash (\Gamma \smallsetminus \Gamma_\infty)$ is the set of orbits of the Γ_∞-action on the set $\Gamma \smallsetminus \Gamma_\infty$.

5. Symbolic dynamics

Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and set $Y := \Gamma\backslash H$. Let $\widehat{\mathrm{CS}}$ be a subset of SY. Suppose that $\widehat{\gamma}$ is a geodesic on Y. If $\widehat{\gamma}'(t) \in \widehat{\mathrm{CS}}$, then we say that $\widehat{\gamma}$ *intersects* $\widehat{\mathrm{CS}}$ *in* t. Further, $\widehat{\gamma}$ is said to *intersect* $\widehat{\mathrm{CS}}$ *infinitely often in future* if there is a sequence $(t_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = \infty$ and $\hat{\gamma}'(t_n) \in \widehat{\mathrm{CS}}$ for all $n \in \mathbb{N}$. Analogously, $\hat{\gamma}$ is said to *intersect* $\widehat{\mathrm{CS}}$ *infinitely often in past* if we find a sequence $(t_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = -\infty$ and $\hat{\gamma}'(t_n) \in \widehat{\mathrm{CS}}$ for all $n \in \mathbb{N}$. Let μ be a measure on the space of geodesics on Y. A *cross section* $\widehat{\mathrm{CS}}$ (w. r. t. μ) for the geodesic flow $\widehat{\Phi}$ is a subset of SY such that

(C1) μ-almost every geodesic $\hat{\gamma}$ on Y intersects $\widehat{\mathrm{CS}}$ infinitely often in past and future,

(C2) each intersection of $\hat{\gamma}$ and $\widehat{\mathrm{CS}}$ is *discrete in time:* if $\hat{\gamma}'(t) \in \widehat{\mathrm{CS}}$, then there is $\varepsilon > 0$ such that $\hat{\gamma}'((t-\varepsilon, t+\varepsilon)) \cap \widehat{\mathrm{CS}} = \{\hat{\gamma}'(t)\}$.

We call a subset $\widehat{U}$ of Y a *totally geodesic suborbifold of* Y if $\pi^{-1}(\widehat{U})$ is a totally geodesic submanifold of H. Let $\mathrm{pr}\colon SY \to Y$ denote the canonical projection on base points. If $\mathrm{pr}(\widehat{\mathrm{CS}})$ is a totally geodesic suborbifold of Y and $\widehat{\mathrm{CS}}$ does not contain elements tangent to $\mathrm{pr}(\widehat{\mathrm{CS}})$, then $\widehat{\mathrm{CS}}$ automatically satisfies (C2).

Suppose that $\widehat{\mathrm{CS}}$ is a cross section for $\widehat{\Phi}$. If $\widehat{\mathrm{CS}}$ in addition satisfies the property that *each* geodesic intersecting $\widehat{\mathrm{CS}}$ at all intersects it infinitely often in past and future, then $\widehat{\mathrm{CS}}$ will be called a *strong cross section*, otherwise a *weak cross section*. Clearly, every weak cross section contains a strong cross section.

The *first return map* of $\widehat{\Phi}$ w. r. t. the strong cross section $\widehat{\mathrm{CS}}$ is the map

$$R\colon \begin{cases} \widehat{\mathrm{CS}} & \to & \widehat{\mathrm{CS}} \\ \hat{v} & \mapsto & \widehat{\gamma_v}'(t_0) \end{cases}$$

where $\pi(v) = \hat{v}$, $\pi(\gamma_v) = \widehat{\gamma_v}$ and

$$t_0 := \min\left\{ t > 0 \,\middle|\, \widehat{\gamma}_v'(t) \in \widehat{\mathrm{CS}} \right\}.$$

Recall that γ_v denotes the geodesic on H determined by v. Further, t_0 is called the *first return time* of $\hat{v}$ resp. of $\widehat{\gamma_v}$. This definition requires that $t_0 = t_0(\widehat{v})$ exists for each $\widehat{v} \in \widehat{\mathrm{CS}}$, which will indeed be the case in our situation. For a weak cross section $\widehat{\mathrm{CS}}$, the first return map can only be defined on a subset of $\widehat{\mathrm{CS}}$. In general, this subset is larger than the maximal strong cross section contained in $\widehat{\mathrm{CS}}$.

Suppose that $\widehat{\mathrm{CS}}$ is a strong cross section and let Σ be an at most countable set. Decompose $\widehat{\mathrm{CS}}$ into a disjoint union $\bigcup_{\alpha\in\Sigma}\widehat{\mathrm{CS}}_\alpha$. To each $\hat{v}\in\widehat{\mathrm{CS}}$ we assign the (two-sided infinite) *coding sequence* $(a_n)_{n\in\mathbb{Z}}\in\Sigma^{\mathbb{Z}}$ defined by

$$a_n := \alpha \quad \text{iff } R^n(\hat{v})\in\widehat{\mathrm{CS}}_\alpha.$$

Note that R is invertible and let Λ be the set of all sequences that arise in this way. Then Λ is invariant under the left shift $\sigma\colon \Sigma^{\mathbb{Z}}\to\Sigma^{\mathbb{Z}}$,

$$\big(\sigma\big((a_n)_{n\in\mathbb{Z}}\big)\big)_k := a_{k+1}.$$

Suppose that the map $\widehat{\mathrm{CS}}\to\Lambda$ is also injective, which it will be in our case. Then we have the natural map $\mathrm{Cod}\colon\Lambda\to\widehat{\mathrm{CS}}$ which maps a coding sequence to the element in $\widehat{\mathrm{CS}}$ it was assigned to. Obviously, the diagram

$$\begin{tikzcd}
\widehat{\mathrm{CS}} \arrow[r, "R"] & \widehat{\mathrm{CS}} \\
\Lambda \arrow[u, "\mathrm{Cod}"] \arrow[r, "\sigma"] & \Lambda \arrow[u, "\mathrm{Cod}"']
\end{tikzcd}$$

commutes. The pair (Λ,σ) is called a *symbolic dynamics* for $\widehat{\Phi}$. If $\widehat{\mathrm{CS}}$ is only a weak cross section and hence R is only partially defined, then Λ also contains one- or two-sided finite coding sequences.

Let CS' be a set of representatives for the cross section $\widehat{\mathrm{CS}}$, that is, CS' is a subset of SH such that $\pi|_{\mathrm{CS}'}$ is a bijection $\mathrm{CS}'\to\widehat{\mathrm{CS}}$. Define the map $\tau\colon\widehat{\mathrm{CS}}\to\partial_g H\times\partial_g H$ by $\tau(\hat{v}) := (\gamma_v(\infty),\gamma_v(-\infty))$ where $v=(\pi|_{\mathrm{CS}'})^{-1}(\hat{v})$. For some cross sections $\widehat{\mathrm{CS}}$ it is possible to choose CS' in a such way that τ is a bijection between $\widehat{\mathrm{CS}}$ and some subset $\widetilde{\mathrm{DS}}$ of $\mathbb{R}\times\mathbb{R}$. In this case the dynamical system $(\widehat{\mathrm{CS}},R)$ is conjugate to $(\widetilde{\mathrm{DS}},\widetilde{F})$ by τ, where $\widetilde{F}:=\tau\circ R\circ\tau^{-1}$ is the induced selfmap on $\widetilde{\mathrm{DS}}$ (partially defined if $\widehat{\mathrm{CS}}$ is only a weak cross section). Moreover, to construct a symbolic dynamics for $\widehat{\Phi}$, one can start with a decomposition of $\widetilde{\mathrm{DS}}$ into pairwise disjoint subsets $\widetilde{D}_\alpha$, $\alpha\in\Sigma$.

Finally, let (Λ,σ) be a symbolic dynamics with alphabet Σ. Suppose that we have a map $i\colon\Lambda\to\mathrm{DS}$ for some $\mathrm{DS}\subseteq\mathbb{R}$ such that $i\big((a_n)_{n\in\mathbb{Z}}\big)$ depends only on $(a_n)_{n\in\mathbb{N}_0}$, a (partial) selfmap $F\colon\mathrm{DS}\to\mathrm{DS}$, and a decomposition of DS into a disjoint union $\bigcup_{\alpha\in\Sigma}D_\alpha$ such that

$$F\big(i((a_n)_{n\in\mathbb{Z}})\big)\in D_\alpha \quad\Leftrightarrow\quad a_1=\alpha$$

for all $(a_n)_{n\in\mathbb{Z}}\in\Lambda$. Then F, more precisely the triple $\big(F,i,(D_\alpha)_{\alpha\in\Sigma}\big)$, is called a *generating function for the future part* of (Λ,σ). If such a generating function exists, then the future part of a coding sequence is independent of the past part.

6. Cusp expansion

Let Γ be a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. On the way of the development of cusp expansion we will gradually impose the requirements on Γ that it be discrete, has ∞ as a cuspidal point and satisfies the conditions (A1) and (A2) which are defined below. The cusp $\pi(\infty)$ plays a special role. All definitions and constructions will be made with respect to this cusp.

At the beginning of each (sub-)section we state the properties of Γ which we assume throughout that (sub-)section.

6.1. Isometric fundamental domains

In Theorem 2.3.4 we proved the existence of isometric fundamental regions for certain subgroups Γ of $\mathrm{Isom}(D)$. The hypotheses of this theorem are that Γ be of type (O), that $\Gamma\setminus\Gamma_\infty$ be of type (F) and that a certain condition on the boundary of a fundamental region for Γ_∞ be satisfied. In this section we specialize to D being the upper half plane and the group Γ being a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. In Sec. 6.1.1 we recall the definition of isometric spheres. From Sec. 6.1.2 on we require that Γ be discrete and that ∞ be a cuspidal point of Γ. Under these conditions, the set of interiors of all isometric spheres is locally finite. This immediately implies that Γ is of type (O). In Sec. 6.1.3 we suppose that Γ satisfies in addition a condition which is weaker than to require that $\Gamma\setminus\Gamma_\infty$ be of type (F). It will turn out that in presence of the other properties of Γ, this condition is equivalent to $\Gamma\setminus\Gamma_\infty$ being of type (F). The purpose of Sec. 6.1.4 is to bring together the results of the previous sections to prove the existence of isometric fundamental domains $\mathcal{F}(r)$, $r\in\mathbb{R}$, for Γ if the fundamental domain $\mathcal{F}_\infty(r)$ for Γ_∞ in H is chosen to be a vertical strip in H. To that end we investigate the structure of the set $\mathcal{K} := \bigcap_{g\in\Gamma\setminus\Gamma_\infty} \operatorname{ext} I(g)$. Also here, the fact that the set of interiors of all isometric spheres is locally finite plays a crucial role. In Sec. 6.1.5 we study the fine structure of the boundary of $\mathcal{K}$ by investigating the isometric spheres that contribute to $\partial\mathcal{K}$ and their relation to each other. For that we introduce the notion of a relevant isometric sphere and its relevant part. The results of this section are of rather technical nature, but they are essential for the construction of cross sections. Moreover, in Sec. 6.1.6 we use these insights on $\partial\mathcal{K}$ to show that for certain parameters $r\in\mathbb{R}$, the closure of the isometric fundamental domain $\mathcal{F}(r)$ is a geometrically finite, exact, convex fundamental polyhedron for Γ in H. This, in turn, will show that Γ is a geometrically finite group and will allow to characterize the cuspidal points of Γ via $\mathcal{F}(r)$. Finally, in Sec. 6.1.7, we show that, conversely, each geometrically finite group of which ∞ is a cuspidal point satisfies all requirements we imposed so far on Γ.

6.1.1. Isometric spheres

Let Γ be a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and suppose that $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \smallsetminus \Gamma_\infty$. For each $z \in H$ we have
$$g'(z) = \frac{1}{(cz+d)^2}.$$
From Lemma 3.5.2 and Prop. 3.3.4 it follows that the isometric sphere of g is
$$I(g) = \{z \in H \mid |cz+d| = 1\},$$
that
$$\operatorname{ext} I(g) = \{z \in H \mid |cz+d| > 1\}$$
is its exterior, and that
$$\operatorname{int} I(g) = \{z \in H \mid |cz+d| < 1\}$$
is its interior.

Remark 6.1.1. Let $g \in \mathrm{PSL}(2,\mathbb{R}) \smallsetminus \mathrm{PSL}(2,\mathbb{R})_\infty$ and suppose that the representative $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ of g is chosen such that $c > 0$. Then the isometric sphere
$$I(g) = \left\{ z \in H \,\middle|\, \left| z + \tfrac{d}{c} \right| = \tfrac{1}{c} \right\}$$
of g is the complete geodesic segment with endpoints $-\frac{d}{c} - \frac{1}{c}$ and $-\frac{d}{c} + \frac{1}{c}$. Let $z_0 = x_0 + iy_0$ be an element of $I(g)$. Then the geodesic segment (z_0, ∞) is contained in $\operatorname{ext} I(g)$, and the geodesic segment (x_0, z_0) belongs to $\operatorname{int} I(g)$. Moreover, $H = \operatorname{ext} I(g) \cup I(g) \cup \operatorname{int} I(g)$ is a partition of H into convex subsets such that $\partial \operatorname{ext} I(g) = I(g) = \partial \operatorname{int} I(g)$.

6.1.2. Type (O)

Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and suppose that ∞ is a cuspidal point of Γ. Then (see [Bor97, 3.6]) there is a unique generator $t_\lambda := \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ with $\lambda > 0$ of Γ_∞.

Recall that a family $\{S_j \mid j \in J\}$ of subsets of H is called *locally finite* if for each $z \in H$ there exists a neighborhood U of z in H such that the set $\{j \in J \mid U \cap S_j \neq \emptyset\}$ is finite.

Our next goal is to show that the set of the interiors of all isometric spheres is locally finite, or in other words, that the family of the interiors of all isometric spheres is locally finite if it is indexed by the set of all isometric spheres. To this end we will characterize the set of all isometric spheres as the set of classes of a certain equivalence relation of elements in $\Gamma \smallsetminus \Gamma_\infty$. The equivalence relation on $\Gamma \smallsetminus \Gamma_\infty$ is given by considering two elements as equivalent when they generate the same isometric sphere. It will turn out that this equivalence relation is very easily expressed via a group action, namely it is the left action of Γ_∞ on $\Gamma \smallsetminus \Gamma_\infty$. This characterization of isometric spheres allows to apply a result in [Bor97] which directly translates to a statement on the radii of isometric spheres. We start by investigating when two elements generate the same isometric sphere.

Lemma 6.1.2. *Let $g_1, g_2 \in \Gamma \smallsetminus \Gamma_\infty$. Then the isometric spheres $I(g_1)$ and $I(g_2)$ are equal if and only if $g_1 g_2^{-1} \in \Gamma_\infty$.*

Proof. Let $g_j := \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$ for $j = 1, 2$. Then $I(g_1) = I(g_2)$ if and only if $c_1 = c_2 =: c$ and $d_1 = d_2 =: d$. Suppose that $I(g_1) = I(g_2)$. Then

$$g_1 g_2^{-1} = \begin{pmatrix} a_1 d - b_1 c & -a_1 b_2 + b_1 a_2 \\ 0 & -cb_2 + da_2 \end{pmatrix} = \begin{pmatrix} 1 & -a_1 b_2 + b_1 a_2 \\ 0 & 1 \end{pmatrix},$$

where we used that $\det(g_1) = 1 = \det(g_2)$. Hence $g_1 g_2^{-1} \in \Gamma_\infty$.
Now suppose that $g_1 g_2^{-1} \in \Gamma_\infty$. Then $g_1 g_2^{-1} = \left(\begin{smallmatrix} 1 & m\lambda \\ 0 & 1 \end{smallmatrix}\right)$ for some $m \in \mathbb{Z}$. Hence

$$g_1 = \begin{pmatrix} 1 & m\lambda \\ 0 & 1 \end{pmatrix} g_2 = \begin{pmatrix} a_2 + m\lambda c_2 & b_2 + m\lambda d_2 \\ c_2 & d_2 \end{pmatrix}.$$

Thus, $c_1 = c_2$ and $d_1 = d_2$. ☐

Lemma 6.1.2 shows that the generator g of the isometric sphere $I(g)$ is uniquely determined up to left multiplication with elements in Γ_∞. Let

$$\mathrm{IS} := \{ I(g) \mid g \in \Gamma \smallsetminus \Gamma_\infty \}$$

denote the set of all isometric spheres. Then the map

$$\Upsilon \colon \left\{ \begin{array}{ccc} \Gamma_\infty \backslash (\Gamma \smallsetminus \Gamma_\infty) & \to & \mathrm{IS} \\ {[g]} & \mapsto & I(g) \end{array} \right.$$

is a bijection.

Contrary to left multiplication, right multiplication of g with t_λ^m induces a shift by $-m\lambda$. This fact will be needed to show that of the interiors of isometric spheres which are generated by the elements of a double coset $\Gamma_\infty g \Gamma_\infty$ only finitely many intersect small neighborhoods of a given point in H.

Lemma 6.1.3. *Let $g \in \Gamma \smallsetminus \Gamma_\infty$ and $m \in \mathbb{Z}$. Then $I(g t_\lambda^m) = I(g) - m\lambda$.*

Proof. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c > 0$ be a representative of g. Then

$$\begin{aligned} I(g t_\lambda^m) &= \{ z \in H \mid |cz + cm\lambda + d| = 1 \} \\ &= \{ z \in H \mid |c(z + m\lambda) + d| = 1 \} \\ &= \{ w - m\lambda \in H \mid |cw + d| = 1 \} \\ &= I(g) - m\lambda. \end{aligned}$$

This shows the claim. ☐

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we set $c(g) := |c|$. The map $c \colon \Gamma \to \mathbb{R}$ is well-defined. Moreover, for each $m, n \in \mathbb{Z}$ it holds

$$t_\lambda^m g t_\lambda^n = \begin{pmatrix} a + m\lambda c & n\lambda a + mn\lambda^2 c + b + m\lambda d \\ c & n\lambda c + d \end{pmatrix}.$$

Hence, c is constant on the double coset $\Gamma_\infty g \Gamma_\infty$ of g in Γ. In particular, c induces the map

$$\overline{c}\colon \begin{cases} \Gamma_\infty\backslash(\Gamma\smallsetminus\Gamma_\infty) & \to \mathbb{R}^+ \\ [g] & \mapsto c(g). \end{cases}$$

Using the bijection $\Upsilon\colon \Gamma_\infty\backslash(\Gamma\smallsetminus\Gamma_\infty) \to \mathrm{IS}$ we define the map $\tilde{c}\colon \mathrm{IS} \to \mathbb{R}^+$, $\tilde{c} := \overline{c}\circ\Upsilon^{-1}$. Note that $1/\tilde{c}(I)$ is the radius of the isometric sphere $I \in \mathrm{IS}$.

The following lemma is one of the key points for the proof of Prop. 6.1.5 below. It uses the characterization of isometric spheres via Υ and Lemma 6.1.3.

Lemma 6.1.4. *Let $a,b\in\mathbb{R}$, $a<b$, and let $U := (a,b)+i\mathbb{R}^+$ be the vertical strip in H spanned by a and b. For each $k\in\mathbb{R}^+$, the set*

$$\{\operatorname{int} I \mid I\in\mathrm{IS},\ \tilde{c}(I)=k,\ \operatorname{int} I\cap U\neq\emptyset\}$$

is finite.

Proof. Let $g\in\Gamma\smallsetminus\Gamma_\infty$ such that $c(g)=k$. At first we will show that the set $g\Gamma_\infty$ contains only finitely many elements h such that $\operatorname{int} I(h)\cap U\neq\emptyset$. If for all elements h in $g\Gamma_\infty$, the interior of $I(h)$ does not intersect U, we are done. Suppose that this is not the case and fix some $h\in g\Gamma_\infty$ such that $\operatorname{int} I(h)\cap U\neq\emptyset$. We may assume w.l.o.g. that $h=g$. Recall that $1/c(g)$ is the radius of $I(g)$. If $w\in\operatorname{int} I(g)$, then $\operatorname{int} I(g)$ is contained in the vertical strip

$$\big(\operatorname{Re} w-\tfrac{2}{k}, \operatorname{Re} w+\tfrac{2}{k}\big)+i\mathbb{R}^+.$$

Since $\operatorname{int} I(g)\cap U\neq\emptyset$, we find that

$$\operatorname{int} I(g)\subseteq \big(a-\tfrac{2}{k}, b+\tfrac{2}{k}\big)+i\mathbb{R}^+ =: P.$$

Let $t_\lambda^m\in\Gamma_\infty$. Lemma 6.1.3 implies that

$$\operatorname{int} I(gt_\lambda^m)\subseteq P-m\lambda = \big(a-\tfrac{2}{k}-m\lambda, b+\tfrac{2}{k}-m\lambda\big)+i\mathbb{R}^+.$$

For $m\leq \frac{1}{\lambda}\big(-b+a-\frac{4}{k}\big)$ one easily calculates that

$$b+\tfrac{2}{k}\leq a-\tfrac{2}{k}-m\lambda,$$

and for $m\geq\frac{1}{\lambda}\big(b-a+\frac{4}{k}\big)$ one has

$$a-\tfrac{2}{k}\geq b+\tfrac{2}{k}-m\lambda.$$

Thus, for $|m|\geq\frac{1}{\lambda}\big(b-a+\frac{4}{k}\big)$, the interior of $I(gt_\lambda^m)$ does not intersect P. Note that $U\subseteq P$. Hence there are only finitely many elements h in $g\Gamma_\infty$ such that $\operatorname{int} I(h)\cap U\neq\emptyset$.

By Lemma 6.1.2, $\operatorname{int} I(g)=\operatorname{int} I(t_\lambda^m g)$ for all $m\in\mathbb{Z}$. Therefore, the set

$$\{\operatorname{int} I(h)\mid h\in\Gamma_\infty g\Gamma_\infty,\ \operatorname{int} I(h)\cap U\neq\emptyset\}$$

is finite. [Bor97, Lemma 3.7] states that there are only finitely many double cosets $\Gamma_\infty g\Gamma_\infty$ in Γ such that $c(g)=k$ for some (and hence any) representative g of $\Gamma_\infty g\Gamma_\infty$. This completes the proof. ∎

Recall that H equals the model D' from Sec. 3.5.1 of two-dimensional real hyperbolic space with changed order of coordinates. Therefore, for the height of an element $z \in H$ we have $\operatorname{ht}(z) = \operatorname{Im} z$. Then the maximal height of an element of an isometric sphere I is $1/\tilde{c}(I)$. This in turn means that $\operatorname{int} I$ is contained in the horizontal strip $\{z \in H \mid \operatorname{ht}(z) < 1/\tilde{c}(I)\}$.

For each $z \in H$ and $r > 0$, the set $B_r(z)$ denotes the open Euclidean ball with radius r and center z.

Proposition 6.1.5. *The set* $\{\operatorname{int} I \mid I \in \mathrm{IS}\}$ *of all interiors of isometric spheres is locally finite.*

Proof. Let $z \in H$. Fix $\varepsilon \in (0, \operatorname{ht}(z))$ and set $U := B_\varepsilon(z)$. We will show that $U \cap \operatorname{int} I \neq \emptyset$ for some $I \in \mathrm{IS}$ implies that $\tilde{c}(I)$ belongs to a finite set. Since U is contained in the vertical strip $(\operatorname{Re} z - \varepsilon, \operatorname{Re} z + \varepsilon) + i\mathbb{R}^+$, Lemma 6.1.4 then implies the claim. To that end set $m := \operatorname{ht}(z) - \varepsilon$. Suppose that $I \in \mathrm{IS}$ with $\tilde{c}(I) \geq \frac{1}{m}$. For each $w \in \operatorname{int} I$ we have

$$\operatorname{ht}(w) < \frac{1}{\tilde{c}(I)} \leq m = \operatorname{ht}(z) - \varepsilon.$$

Therefore $w \notin U$ and hence $U \cap \operatorname{int} I = \emptyset$. This means that if $U \cap \operatorname{int} I \neq \emptyset$, then $\tilde{c}(I) < \frac{1}{m}$. Now [Bor97, Lemma 3.7] implies that the map c assumes only finitely many values less than $\frac{1}{m}$. Hence also the map $\tilde{c}$ does so. The previous argument shows that the proof is complete. □

Now Remark 2.3.3 implies that Γ is of type (O).

Proposition 6.1.6. *We have*

$$\overline{\bigcup_{I\in\mathrm{IS}} \operatorname{int} I} = \bigcup_{I\in\mathrm{IS}} \overline{\operatorname{int} I} = H \smallsetminus \bigcap_{I\in\mathrm{IS}} \operatorname{ext} I.$$

Remark 6.1.7. If Λ is a subgroup of Γ, then the set of all interiors of isometric spheres of Λ is a subset of the set of all interiors of isometric spheres of Γ. Hence, if ∞ is a cuspidal point of Λ, then Λ is of type (O).

6.1.3. Type (F)

Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of which ∞ is a cuspidal point. Suppose that Γ satisfies the following condition:

For each $z \in H$, the set $\mathcal{H}_z := \left\{ \operatorname{ht}(gz) \,\middle|\, g \in \langle \Gamma \smallsetminus \Gamma_\infty \rangle \right\}$ is bounded from above. (A1)

The condition (A1) is clearly weaker and easier to check than the requirement that $\Gamma \smallsetminus \Gamma_\infty$ be of type (F). Since the height is invariant under Γ_∞, we see that $\mathcal{H}_z = \{\operatorname{ht}(gz) \mid g \in \Gamma\}$, and hence (A1) is equivalent to

For each $z \in H$, the set $\mathcal{H}_z := \{\operatorname{ht}(gz) \mid g \in \Gamma\}$ is bounded from above. (A1')

Prop. 6.1.9 below shows that the properties of Γ already implies that $\Gamma\diagdown\Gamma_\infty$ is of type (F). For its proof we need the following lemma.

Lemma 6.1.8. *Let $\alpha, \beta \in \mathbb{R}^+$, $\alpha > \beta$, and suppose that we have $w \in H$ with $\operatorname{Im} w \in [\beta, \alpha]$. Further let $\delta \in (0, \alpha)$. Then there exists $\eta > 0$ such that for each $g \in \mathrm{PSL}(2, \mathbb{R})$ for which $\operatorname{Im}(gw) \in [\beta, \alpha]$ we have $B_\eta(gw) \subseteq gB_\delta(w)$.*

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2, \mathbb{R})$ with $\operatorname{Im}(gw) \in [\beta, \alpha]$ and pick $z \in \partial B_\delta(w)$. Then (see [Kat92, Thm. 1.2.6])

$$\cosh d_H(z, w) = 1 + \frac{|z - w|^2}{2 \operatorname{Im} z \cdot \operatorname{Im} w} = 1 + \frac{\delta^2}{2 \operatorname{Im} z \cdot \operatorname{Im} w}$$

and

$$\cosh d_H(gz, gw) = 1 + \frac{|gz - gw|^2}{2 \operatorname{Im}(gz) \cdot \operatorname{Im}(gw)}.$$

Since $d_H(z, w) = d_H(gz, gw)$, we have

$$|gz - gw|^2 = \frac{\operatorname{Im}(gz) \cdot \operatorname{Im}(gw)}{\operatorname{Im} z \cdot \operatorname{Im} w} \cdot \delta^2.$$

From $\operatorname{Im}(gz) = \operatorname{Im}(z)/|cz + d|^2$ it follows that

$$|gz - gw|^2 = \frac{\delta^2}{|cz + d|^2 |cw + d|^2}.$$

We will now show that there is a universal upper bound (that is, it does not depend on g, z or w) for $|cz + d|^2 |cw + d|^2$. By assumption,

$$\beta \le \operatorname{Im}(gw) = \frac{\operatorname{Im} w}{|cw + d|^2} \le \alpha,$$

hence

$$\beta |cw + d|^2 \le \operatorname{Im} w \le \alpha |cw + d|^2.$$

Then $\beta \le \operatorname{Im} w \le \alpha$ implies that

$$\beta |cw + d|^2 \le \alpha \quad \text{and} \quad \beta \le \alpha |cw + d|^2.$$

Therefore

$$\frac{\beta}{\alpha} \le |cw + d|^2 \le \frac{\alpha}{\beta}. \tag{6.1}$$

Moreover,

$$\sqrt{\frac{\alpha}{\beta}} \ge |cw + d| \ge |\operatorname{Im}(cw + d)| = |c| \operatorname{Im} w \ge |c| \beta.$$

Thus

$$|c| \le \sqrt{\frac{\alpha}{\beta^3}}. \tag{6.2}$$

Finally, (6.1) and (6.2) give

$$\begin{aligned}|cz+d| &= |c(z-w)+cw+d| \\ &\leq |c||z-w|+|cw+d| \\ &\leq \sqrt{\frac{\alpha}{\beta^3}}\cdot\delta+\sqrt{\frac{\alpha}{\beta}}.\end{aligned}$$

Hence, for all $w\in H$ with $\operatorname{Im} w\in[\beta,\alpha]$, for all $z\in\partial B_\delta(w)$ and all $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ in $\mathrm{PSL}(2,\mathbb{R})$ with $\operatorname{Im}(gw)\in[\beta,\alpha]$ we have

$$|cz+d|^2|cw+d|^2\leq\left(\sqrt{\frac{\alpha}{\beta^3}}\cdot\delta+\sqrt{\frac{\alpha}{\beta}}\right)^2\cdot\frac{\alpha}{\beta}=:\frac{1}{k}.$$

Therefore

$$|gz-gw|\geq\sqrt{k}\delta=:\eta.$$

Since $gB_\delta(w)$ is connected, it follows that $B_\eta(gw)\subseteq gB_\delta(w)$. $\square$

Let $t_\lambda=\left(\begin{smallmatrix}1&\lambda\\0&1\end{smallmatrix}\right)$ be the unique generator of Γ_∞ such that $\lambda>0$. For each $r\in\mathbb{R}$, the set

$$\mathcal{F}_\infty(r):=(r,r+\lambda)+i\mathbb{R}^+$$

is a fundamental domain for Γ_∞ in H. The proof of the following proposition extensively uses that Γ satisfies (A1) and that each point $w\in H$ is Γ_∞-equivalent to a point in $\overline{\mathcal{F}_\infty(r)}$ of same height.

Proposition 6.1.9. *The set $\Gamma\smallsetminus\Gamma_\infty$ is of type (F), i. e., for each $z\in H$, the maximum of $\mathcal{H}_z$ exists.*

Proof. Let $z\in H$ and set $\alpha:=\sup\mathcal{H}_z$. Note that α is finite by (A1) (resp. by (A1')). Assume for contradiction that the maximum of $\mathcal{H}_z$ does not exist. Let $\varepsilon>0$ and set $K:=[0,\lambda]+i[\alpha-\varepsilon,\alpha]$. We claim that the set

$$T:=\{h\in\Gamma\mid hz\in K\}$$

is infinite. To see this, let $n\in\mathbb{N}$. By our assumption that $\max\mathcal{H}_z$ does not exist, there are infinitely many $g\in\Gamma$ with $\operatorname{ht}(gz)>\alpha-\frac{1}{n}$. Since $\operatorname{ht}(t_\lambda^m w)=\operatorname{ht}(w)$ for each $w\in H$ and $m\in\mathbb{Z}$, there is at least one $g\in\Gamma$ such that $\operatorname{ht}(gz)>\alpha-\frac{1}{n}$ and $gz\in\overline{\mathcal{F}_\infty(0)}$, thus $gz\in K$. By varying n, we see that T is infinite.

Fix some $g\in T$ and set $w:=gz$. Since Γ is discrete, it acts properly discontinuously on H (see [Kat92, Thm. 2.2.6]). Thus we find $\delta>0$ such that

$$\Lambda:=\{k\in\Gamma\mid kB_\delta(w)\cap B_\delta(w)\neq\emptyset\}$$

is finite. We will show that this contradicts to T being infinite. By Lemma 6.1.8 we find $\eta>0$ such that $B_\eta(hg^{-1}w)\subseteq hg^{-1}B_\delta(w)$ for all $h\in T$. For each $h\in T$ let S_h denote the subset of T such that $B_\eta(hg^{-1}w)$ intersects each $B_\eta(kg^{-1}w)$, $k\in S_h$. We claim that at least one S_h is infinite. Assume for contradiction that each S_h is finite. We construct a sequence $(h_n)_{n\in\mathbb{N}}$ in T as follows: Pick any

$h_1 \in T$ and choose $h_2 \in T \smallsetminus S_{h_1}$. Suppose that we have already chosen $h_1, \ldots, h_j$ such that $h_k \in T \smallsetminus \bigcup_{l=1}^{k-1} S_l$, $k = 2, \ldots, j$. Since T is infinite, the set $T \smallsetminus \bigcup_{l=1}^{j} S_l$ is non-empty. Pick any $h_{j+1} \in T \smallsetminus \bigcup_{l=1}^{j} S_l$. The axiom of choice shows that we get an infinite sequence $(h_n)_{n\in\mathbb{N}}$ in T. By construction, for $n_1, n_2 \in \mathbb{N}$, $n_1 \neq n_2$, the balls $B_\eta(h_{n_1} g^{-1} w)$, $B_\eta(h_{n_2} g^{-1} w)$ are disjoint.

Let vol denote the Euclidean volume and note that the η-neighborhood $B_\eta(K)$ of K is bounded. Then

$$\infty > \operatorname{vol}(B_\eta(K)) \geq \operatorname{vol}\left(\bigcup_{h\in T} B_\eta(hg^{-1}w)\right) \geq \operatorname{vol}\left(\bigcup_{j=1}^{\infty} B_\eta(h_j g^{-1} w)\right) =$$

$$= \sum_{j=1}^{\infty} \operatorname{vol}\left(B_\eta(h_j g^{-1} w)\right) = 2\pi\eta \sum_{j=1}^{\infty} 1 = \infty,$$

which is a contradiction.

Hence there exists $h \in T$ such that S_h is infinite. But then, for each $k \in S_h$, we have

$$\emptyset \neq B_\eta(hg^{-1}w) \cap B_\eta(kg^{-1}w) \subseteq hg^{-1} B_\delta(w) \cap kg^{-1} B_\delta(w),$$

and therefore

$$\emptyset \neq B_\delta(w) \cap gh^{-1}kg^{-1} B_\delta(w).$$

This contradicts to Λ being finite. In turn, the maximum of $\mathcal{H}_z$ exists. □

Remark 6.1.10. Prop. 6.1.9 implies that whenever Λ is a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ such that $\Lambda \smallsetminus \Lambda_\infty$ is of type (F), then for each discrete subgroup Γ of Λ one has that $\Gamma \smallsetminus \Gamma_\infty$ is of type (F) as long as ∞ is a cusp point of Γ.

6.1.4. The set $\mathcal{K}$ and isometric fundamental domains

Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ which has ∞ as cuspidal point and satisfies (A1). Let

$$\mathcal{K} := \bigcap_{I \in \mathrm{IS}} \operatorname{ext} I$$

be the common part of the exteriors of all isometric spheres. Here we will prove the existence of isometric fundamental domains for Γ. To that end we will show that the boundary of $\mathcal{K}$ is contained in a locally finite union of isometric spheres. The first lemma implies that the set of all isometric spheres is locally finite.

Lemma 6.1.11. *The families* $\left\{\overline{\operatorname{int} I} \;\middle|\; I \in \mathrm{IS}\right\}$ *and* IS *are locally finite.*

Proof. By Prop. 6.1.5 the family $\{\operatorname{int} I \mid I \in \mathrm{IS}\}$ is locally finite. Then [vQ79, Hilfssatz 7.14] states that the family $\{\overline{\operatorname{int} I} \mid I \in \mathrm{IS}\}$ is locally finite as well. For each $I \in \mathrm{IS}$, the isometric sphere I is a subset of $\overline{\operatorname{int} I}$. Hence IS is locally finite. □

Proposition 6.1.12. *The boundary $\partial\mathcal{K}$ of $\mathcal{K}$ is contained in a locally finite union of isometric spheres.*

Proof. By Prop. 6.1.6, $\mathcal{K} = \bigcap_{I\in\mathrm{IS}} \operatorname{ext} I$ is open. Then Prop. 6.1.6 (for the third and the last equality) and 2.2.11 (for the last equality) imply that

$$\begin{aligned}\partial\mathcal{K} &= \overline{\bigcap_{I\in\mathrm{IS}} \operatorname{ext} I} \setminus \bigcap_{I\in\mathrm{IS}} \operatorname{ext} I = \overline{\bigcap_{I\in\mathrm{IS}} \operatorname{ext} I} \cap \Big(H \setminus \bigcap_{I\in\mathrm{IS}} \operatorname{ext} I \Big) \\ &= \overline{\bigcap_{I\in\mathrm{IS}} \operatorname{ext} I} \cap \overline{\bigcup_{I\in\mathrm{IS}} \operatorname{int} I} = \bigcap_{I\in\mathrm{IS}} \overline{\operatorname{ext} I} \cap \bigcup_{I\in\mathrm{IS}} \overline{\operatorname{int} I}.\end{aligned}$$

Therefore $z \in \partial\mathcal{K}$ if and only if

$$\forall\, I \in \mathrm{IS}\colon z \in \overline{\operatorname{ext} I} \quad \text{and} \quad \exists\, J \in \mathrm{IS}\colon z \in \overline{\operatorname{int} J}.$$

Since $\overline{\operatorname{ext} J} \cap \overline{\operatorname{int} J} = J$ for all $J \in \mathrm{IS}$, we see that $z \in \partial\mathcal{K}$ if and only if

$$\forall\, I \in \mathrm{IS}\colon z \in \overline{\operatorname{ext} I} \quad \text{and} \quad \exists\, J \in \mathrm{IS}\colon z \in J.$$

Thus, $\partial\mathcal{K} \subseteq \bigcup_{I\in\mathrm{IS}} I$. The set IS is locally finite by Lemma 6.1.11. □

Remark 6.1.13. Prop. 6.1.12 does not show that the family of connected components of $\partial\mathcal{K}$ is locally finite. That this is indeed the case will be proven in Sec. 6.1.5.

Lemma 6.1.14. *The set $\mathcal{K}$ is convex. Moreover, if $z \in \overline{\mathcal{K}}$, then the geodesic segment (z,∞) is contained in $\mathcal{K}$.*

Proof. Recall from Remark 6.1.1 that $\operatorname{ext} I$ is a convex set for each $I \in \mathrm{IS}$. Thus, $\mathcal{K} = \bigcap_{I\in\mathrm{IS}} \operatorname{ext} I$ is so. Let $z \in \overline{\mathcal{K}}$. Prop. 2.2.11 shows that

$$z \in \overline{\bigcap_{I\in\mathrm{IS}} \operatorname{ext} I} = \bigcap_{I\in\mathrm{IS}} \overline{\operatorname{ext} I}.$$

Hence, for each $I \in \mathrm{IS}$ we have $z \in \overline{\operatorname{ext} I}$. By Remark 6.1.1, for each $I \in \mathrm{IS}$, the geodesic segment (z,∞) is contained in $\operatorname{ext} I$. Therefore (z,∞) is contained in $\bigcap_{I\in\mathrm{IS}} \operatorname{ext} I = \mathcal{K}$. □

Recall that we suppose that Γ is discrete with cuspidal point ∞ and fulfills (A1). As before, let $t_\lambda := \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ be the unique generator of Γ_∞ with $\lambda > 0$. For each $r \in \mathbb{R}$ set $\mathcal{F}_\infty(r) := (r, r+\lambda) + i\mathbb{R}^+$, which is a fundamental domain for Γ_∞ in H. Let

$$\mathcal{F}(r) := \mathcal{F}_\infty(r) \cap \mathcal{K}.$$

Theorem 6.1.15. *For each $r \in \mathbb{R}$, the set $\mathcal{F}(r)$ is a convex fundamental domain for Γ in H. Moreover,*

$$\partial\mathcal{F}(r) = \big(\partial\mathcal{F}_\infty(r) \cap \overline{\mathcal{K}}\big) \cup \big(\overline{\mathcal{F}_\infty(r)} \cap \partial\mathcal{K}\big).$$

Proof. Let $r \in \mathbb{R}$. Then $\mathcal{F}_\infty(r)$ is obviously a convex domain. Lemma 6.1.14 states that $\mathcal{K}$ is convex, and Prop. 6.1.6 implies that $\mathcal{K}$ is open. Therefore, $\mathcal{F}_\infty(r) \cap \mathcal{K}$ is a convex domain.

Now we show that $\overline{\mathcal{F}_\infty(r)} \cap \overline{\mathcal{K}} = \overline{\mathcal{F}_\infty(r) \cap \mathcal{K}}$. Clearly, we have $\overline{\mathcal{F}_\infty(r) \cap \mathcal{K}} \subseteq \overline{\mathcal{F}_\infty(r)} \cap \overline{\mathcal{K}}$. To prove the opposite inclusion relation let $z \in \overline{\mathcal{F}_\infty(r)} \cap \overline{\mathcal{K}}$, hence we are in one of the following four cases:

Case 1: Suppose $z \in \mathcal{F}_\infty(r) \cap \mathcal{K}$. Obviously, $z \in \overline{\mathcal{F}_\infty(r) \cap \mathcal{K}}$.

Case 2: Suppose $z \in \partial\mathcal{F}_\infty(r) \cap \mathcal{K}$. Fix a neighborhood V of z such that $V \subseteq \mathcal{K}$. For each neighborhood U of z with $U \subseteq V$ we have $U \cap \mathcal{F}_\infty(r) \neq \emptyset$. Hence $U \cap \mathcal{F}_\infty(r) \cap \mathcal{K} \neq \emptyset$. Therefore $z \in \overline{\mathcal{F}_\infty(r) \cap \mathcal{K}}$.

Case 3: Suppose $z \in \mathcal{F}_\infty(r) \cap \partial\mathcal{K}$. Analogously to Case 2 we find $z \in \overline{\mathcal{F}_\infty(r) \cap \mathcal{K}}$.

Case 4: Suppose $z \in \partial\mathcal{F}_\infty(r) \cap \partial\mathcal{K}$. Fix a neighborhood V of z such that $\partial\mathcal{K} \cap V$ is contained in the finite union of the isometric spheres $I_1, \ldots, I_n$. Then $\partial\mathcal{F}_\infty(r)$ intersects each I_j transversely (if at all). Therefore, for each neighborhood U of z with $U \subseteq V$ we find $U \cap \mathcal{F}_\infty(r) \cap \mathcal{K} \neq \emptyset$. Hence $z \in \overline{\mathcal{F}_\infty(r) \cap \mathcal{K}}$.

Thus, $\overline{\mathcal{F}_\infty(r)} \cap \overline{\mathcal{K}} = \overline{\mathcal{F}_\infty(r) \cap \mathcal{K}}$. By Prop. 6.1.6, the group Γ is of type (O), and Prop. 6.1.9 shows that $\Gamma\backslash\Gamma_\infty$ is of type (F). Thus all hypotheses of Theorem 2.3.4 are satisfied, which states that $\mathcal{F}(r)$ is a fundamental region for Γ in H.

Finally,

$$
\begin{aligned}
\partial\mathcal{F}(r) &= \overline{\mathcal{F}(r)} \setminus \mathcal{F}(r) \\
&= \left(\overline{\mathcal{F}_\infty(r) \cap \mathcal{K}}\right) \cap \complement\left(\mathcal{F}_\infty(r) \cap \mathcal{K}\right) \\
&= \left(\overline{\mathcal{F}_\infty(r)} \cap \overline{\mathcal{K}}\right) \cap \left(\complement\mathcal{F}_\infty(r) \cup \complement\mathcal{K}\right) \\
&= \left(\overline{\mathcal{F}_\infty(r)} \cap \complement\mathcal{F}_\infty(r) \cap \overline{\mathcal{K}}\right) \cup \left(\overline{\mathcal{F}_\infty(r)} \cap \overline{\mathcal{K}} \cap \complement\mathcal{K}\right) \\
&= \left(\partial\mathcal{F}_\infty(r) \cap \overline{\mathcal{K}}\right) \cup \left(\overline{\mathcal{F}_\infty(r)} \cap \partial\mathcal{K}\right).
\end{aligned}
$$

This completes the proof. $\square$

Example 6.1.16. For $n \in \mathbb{N}$, $n \geq 3$, let $\lambda_n := 2\cos\frac{\pi}{n}$. The subgroup of $\mathrm{PSL}(2,\mathbb{R})$ which is generated by

$$
S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T_n := \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix}
$$

is called the *Hecke triangle group* G_n. Using Poincaré's Theorem (see [Mas71]) one sees that

$$
\mathcal{F}_n := \left\{ z \in H \;\middle|\; |z| > 1,\ |\operatorname{Re} z| < \tfrac{\lambda_n}{2} \right\}
$$

is a fundamental domain for G_n in H (see Fig. 6.1). The group G_n has ∞ as cuspidal point. The stabilizer of ∞ is

$$
(G_n)_\infty = \left\{ \left(\begin{smallmatrix} 1 & m\lambda_n \\ 0 & 1 \end{smallmatrix}\right) \;\middle|\; m \in \mathbb{Z} \right\}.
$$

Hence each vertical strip $\mathcal{F}_\infty(r) := (r, r+\lambda_n) + i\mathbb{R}^+$ of width λ_n is a fundamental domain for $(G_n)_\infty$ in H. The complete geodesic segment

$$
(-1, 1) = \{ z \in H \mid |z| = 1 \}
$$

is the isometric sphere $I(S)$ of S. Its subsegment $(-\overline{\varrho}_n, \varrho_n)$ with

$$
\varrho_n := \frac{\lambda_n}{2} + i\sqrt{1 - \frac{\lambda_n^2}{4}} = \cos\tfrac{\pi}{n} + i\sin\tfrac{\pi}{n}
$$

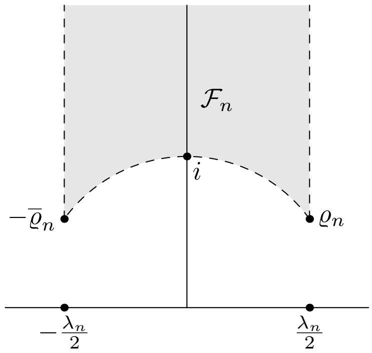


Figure 6.1: The fundamental domain $\mathcal{F}_n$ for G_n in H.

is the non-vertical side of $\mathcal{F}_n$. Therefore

$$\mathcal{F} = \mathcal{F}_\infty\big(-\tfrac{\lambda_n}{2}\big) \cap \operatorname{ext} I(S).$$

This in turn implies that

$$\mathcal{K} = \bigcap_{g \in G_n \smallsetminus (G_n)_\infty} \operatorname{ext} I(g) = \bigcap_{m \in \mathbb{Z}} \operatorname{ext} I(ST_n^m).$$

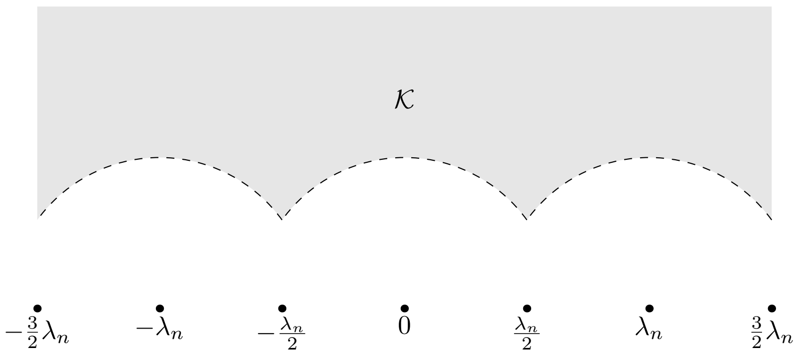


Figure 6.2: The set $\mathcal{K}$ for G_n.

Example 6.1.17. We consider the group

$$\mathrm{P}\Gamma_0(5) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{Z}) \;\middle|\; c \equiv 0 \mod 5 \right\}.$$

This group has ∞ as cuspidal point. The stabilizer of ∞ is given by

$$\mathrm{P}\Gamma_0(5)_\infty = \left\{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \;\middle|\; b \in \mathbb{Z} \right\}.$$

Therefore, each vertical strip $\mathcal{F}_\infty(r) := (r, r+1) + i\mathbb{R}^+$ of width 1 is a fundamental domain for $\mathrm{P}\Gamma_0(5)_\infty$. The isometric spheres of $\mathrm{P}\Gamma_0(5)$ are the sets

$$I_{c,d} = \{ z \in H \mid |5cz + d| = 1 \} = \left\{ z \in H \;\middle|\; \left| z + \tfrac{d}{5c} \right| = \tfrac{1}{5c} \right\}$$

where $c \in \mathbb{N}$ and $d \in \mathbb{Z}$. This clearly shows that the set of all interiors of isometric spheres is locally finite, which implies that $\mathrm{P}\Gamma_0(5)$ is of type (O). One easily shows that $\mathrm{P}\Gamma_0(5)$ is of type (F). The set $\mathcal{K}$ is given by

$$\mathcal{K} = \bigcap_{d\in\mathbb{Z}} \left\{ z \in H \;\middle|\; \left|z + \tfrac{d}{5}\right| > \tfrac{1}{5} \right\},$$

see Fig. 6.3.

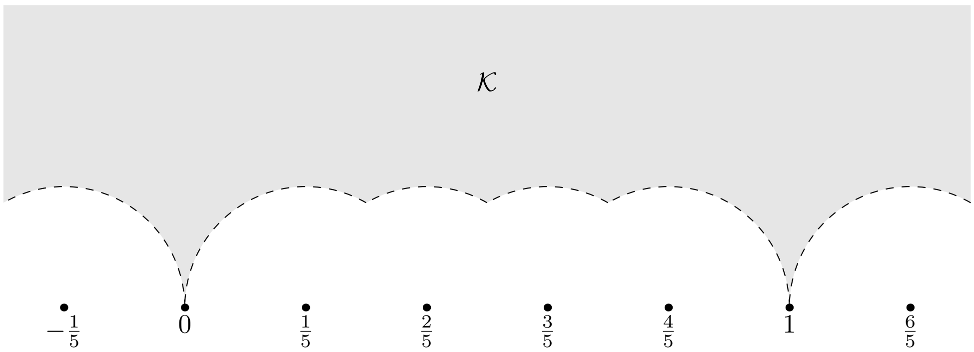


Figure 6.3: The set $\mathcal{K}$ for G_n.

Then the set

$$\mathcal{F} = \mathcal{F}_\infty(0) \cap \mathcal{K} = \mathcal{F}_\infty(0) \cap \bigcap_{d=1}^{4} \left\{ z \in H \;\middle|\; \left|z - \tfrac{d}{5}\right| > \tfrac{1}{5} \right\}$$

is an isometric fundamental domain for $\mathrm{P}\Gamma_0(5)$ in H.

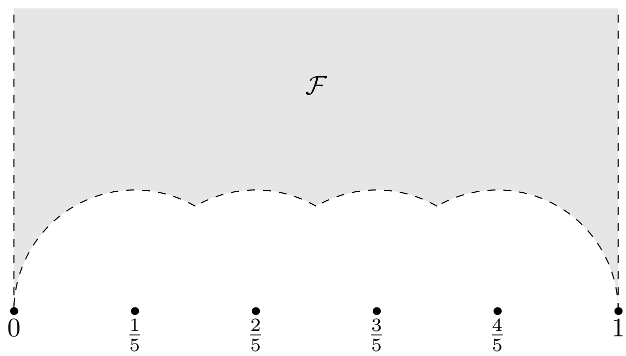


Figure 6.4: The fundamental domain $\mathcal{F}$ for $\mathrm{P}\Gamma_0(5)$ in H.

6.1.5. Relevant sets and the boundary structure of $\mathcal{K}$

Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of which ∞ is a cuspidal point and suppose that Γ satisfies (A1). Recall that

$$\mathcal{K} = \bigcap_{I\in \mathrm{IS}} \operatorname{ext} I.$$

Definition 6.1.18. An isometric sphere I is called *relevant* if $I \cap \partial\mathcal{K}$ contains a submanifold of H of codimension one. If the isometric sphere I is relevant, then $I \cap \partial\mathcal{K}$ is called its *relevant part*.

Example 6.1.19. Recall the Hecke triangle group G_n from Example 6.1.16. The relevant isometric spheres for G_n are $I(ST_n^m)$ with $m \in Z$. The relevant part of $I(ST_n^m)$ is the geodesic segment $[-\overline{\varrho}_n, \varrho_n] - m\lambda_n$.

From now on, to simplify notation and exposition, we use the following convention. Let $\alpha\colon I \to H$ be a geodesic arc where I is an interval of the form $[a,\infty)$ or (a,∞) with $a \in \mathbb{R}$. Then $\alpha(\infty) \in \partial_g H$. In contrast to the definition on p. 86, we denote the associated geodesic segment by $[a, \alpha(\infty)]$ resp. $(a, \alpha(\infty)]$, even though we do not consider $\alpha(\infty)$ as belonging to $\alpha(I)$. Further, we use the obvious analogous convention if I is of the form $(-\infty, a]$ or $(-\infty, a)$ or $(-\infty,\infty)$.

There are only two situations in which we do consider the endpoints in $\partial_g H$ to belong to the geodesic segment. If $s_1 = [a,b]$, $s_2 = [b,c]$ are two geodesic segments with $b \in \partial_g H$, then we say that b is an intersection point of s_1 and s_2. Further, if A is a subset of $\overline{H}^g$ or if A is a subset of H but considered as a subset of $\overline{H}^g$, then $s_1 \subseteq A$ means that indeed also the points a and b belong to A.

On the other hand, if A is some subset of H, then $s_1 \subseteq A$ means that $s_1 \cap H$ is a subset of A. The context will always clarify which interpretation of $[a,b]$ is used.

Lemma 6.1.20.

(i) *The relevant part of a relevant isometric sphere is a geodesic segment. Suppose that a is an endpoint of the relevant part s of the isometric sphere I. If $a \in H$, then $a \in s$.*

(ii) *Suppose that I and J are two different relevant isometric spheres and let $s_I := [a,b]$ resp. $s_J := [c,d]$ be their relevant parts with* $\operatorname{Re} a < \operatorname{Re} b$ *and* $\operatorname{Re} c < \operatorname{Re} d$. *Then s_I and s_J intersect in at most one point. Moreover, if I intersects s_J, then s_I intersects s_J. In this case, the intersection point is either $a = d$ or $b = c$.*

(iii) *Let I be a relevant isometric sphere and $s_I := [a,b]$ its relevant part. If $a \in H$, then there is a relevant isometric sphere J different from I with relevant part $s_J := [c,a]$. Moreover, we have either* $\operatorname{Re} c < \operatorname{Re} a < \operatorname{Re} b$ *or* $\operatorname{Re} b < \operatorname{Re} a < \operatorname{Re} c$.

(iv) *If $c \in \partial\mathcal{K}$, then there is a relevant isometric sphere which contains c.*

Proof.

(i) Let I be a relevant isometric sphere and let $s := I \cap \partial\mathcal{K}$ denote its relevant part. Suppose that $a, b \in s$ and let c be an element of the geodesic segment (a, b). We will show that $c \in s$. Note that $c \in H$. Since $\mathcal{K}$ is convex, $c \in \overline{\mathcal{K}}$. Moreover, (a, b) is a subset of the complete geodesic segment I, thus $c \in I$. Therefore $c \in I \cap \overline{\mathcal{K}}$. Because

$$I \cap \mathcal{K} = I \cap \bigcap_{J \in \mathrm{IS}} \operatorname{ext} J \subseteq I \cap \operatorname{ext} I = \emptyset, \tag{6.3}$$

and $\mathcal{K}$ is open (see Prop. 6.1.6) we get that

$$I \cap \overline{\mathcal{K}} = I \cap \partial\mathcal{K}. \tag{6.4}$$

Therefore, $c \in I \cap \partial\mathcal{K} = s$. This shows that s is a geodesic segment.

Finally, since I and $\partial\mathcal{K}$ are closed subsets of H, the set s is closed as well. Since s is a geodesic segment, this means that it contains all its endpoints that are in H.

(ii) In this part we consider all geodesic segments and in particular the isometric spheres as subsets of $\overline{H}^g$. Since I and J are different geodesic segments, they intersect (in $\overline{H}^g$) in at most one point. In particular, their relevant parts do so. Suppose that I intersects s_J in z. Note that possibly $z \in \partial_g H$. Since $\mathcal{K}$ is convex, $z \in \mathrm{cl}_{\overline{H}^g}(\mathcal{K})$. Because $\left(\mathrm{cl}_{\overline{H}^g}(\mathcal{K})\right)^\circ = \mathcal{K}$, we find analogously to (6.4) that $I \cap \mathrm{cl}_{\overline{H}^g}(\mathcal{K}) = I \cap \partial_g\mathcal{K}$. Then $z \in I \cap \partial_g\mathcal{K} = s_I$.

We will now show that z is either a or b. If $z \in \partial_g H$, then this is clear. Suppose that $z \in H$. Assume for contradiction that $z \in (a, b)$. The intersection of I and J in z is transversal. Then either $[a, z)$ or $(z, b]$ is contained in $\operatorname{int} J$. But then, since $\complement\overline{\mathcal{K}} = \bigcup_{I' \in \mathrm{IS}} \operatorname{int} I'$, the geodesic segment $[a, z)$ resp. $(z, b]$ is disjoint to $\partial\mathcal{K}$ and therefore disjoint to the relevant part s_I of I. This is a contradiction. Therefore $z \in \{a, b\}$. Analogously, we find $z \in \{c, d\}$.

W.l.o.g. we assume that $z = a$. Then $a = c$ or $a = d$. We will prove that $a = d$. Assume for contradiction that $a = c$. Then, since we have that $\operatorname{Re} a = \operatorname{Re} c < \operatorname{Re} b, \operatorname{Re} d$, we find $b' \in (a, b]$ and $d' \in (c, d]$ such that $\operatorname{Re} b' = \operatorname{Re} d'$. If $\operatorname{Im} d' = \operatorname{Im} b'$, then the non-trivial geodesic segments $[a, b']$ and $[c, d']$ would coincide, which would imply that $I = J$. Thus, we may assume that $\operatorname{Im} d' < \operatorname{Im} b'$. Then $d' \in \operatorname{int} I$, which means that d' is not contained in the relevant part of J. This is a contradiction. Hence $a = d$.

(iii) Our first goal is to show that there is an isometric sphere J different to I with $a \in J$. By Lemma 6.1.11 we find an open connected neighborhood U of a in H which intersects $\overline{\operatorname{int} J}$ for only finitely many $J \in \mathrm{IS}$, say

$$\left\{J \in \mathrm{IS} \;\middle|\; U \cap \overline{\operatorname{int} J} \neq \emptyset\right\} = \{J_1, \ldots, J_n\} =: \mathcal{J}$$

and suppose that $J_1 = I$. If there is $J \in \mathrm{IS}$ such that $J \neq I$ and $a \in J$, then $J \in \{J_2, \ldots, J_n\}$. Assume for contradiction that $a \notin J_j$ for $j = 2, \ldots, n$.

By choice of U, for all $J \in \mathrm{IS} \smallsetminus \mathcal{J}$ we have $U \subseteq \operatorname{ext} J$. We claim that by shrinking U we find an open connected neighborhood V of a such that $V \subseteq \bigcap\{\operatorname{ext} J \mid J \in \mathrm{IS} \smallsetminus \{I\}\}$. Let $j \in \{2, \ldots, n\}$. Since

$$a \in \partial\mathcal{K} \subseteq \overline{\mathcal{K}} = \overline{\bigcap_{J \in \mathrm{IS}} \operatorname{ext} J} = \bigcap_{J \in \mathrm{IS}} \overline{\operatorname{ext} J}$$

(see Prop 2.2.11), it follows that $a \in \operatorname{ext} J_j$. Because $\operatorname{ext} J_j$ is open, there is an open connected neighborhood U_j of a such that $U_j \subseteq \operatorname{ext} J_j$. Set $V := U \cap \bigcap_{j=2}^{n} U_j$, which is an open connected neighborhood of a. Then $V \cap I = (z, w)$ with $a \in (z, w)$. Employing (6.4) we find

$$\begin{aligned} I \cap \partial\mathcal{K} = I \cap \overline{\mathcal{K}} \supseteq I \cap V \cap \partial\mathcal{K} &= I \cap \Big(V \cap \bigcap_{J \in \mathrm{IS}} \overline{\operatorname{ext} J}\Big) \\ &= I \cap \big(V \cap \overline{\operatorname{ext} I}\big) = I \cap V \\ &= (z, w). \end{aligned}$$

Therefore $a \in (z, w) \subseteq s_I$, in contradiction to a being an endpoint of s_I. Hence, there is an isometric sphere J with $J \neq I$ and $a \in J$. Note that we have already shown that there are only finitely many of these.

We now prove the existence of a relevant isometric sphere $J \neq I$ with $a \in J$. Let x, y be the endpoint of I. By assumption, the non-trivial geodesic segment $[x, a)$ is disjoint to $s_I = I \cap \partial\mathcal{K}$. Since $[x, a) \subseteq I$ and $I \cap \mathcal{K} = \emptyset$, it follows that $\emptyset = [x, a) \cap \big(\mathcal{K} \cap \partial\mathcal{K}\big)$. Hence

$$[x, a) \subseteq \complement\overline{\mathcal{K}} = \bigcup_{J \in \mathrm{IS}} \operatorname{int} J.$$

Let J be an isometric sphere with $J \neq I$ and $a \in J$. Since J intersects I only once, it follows that $[x, a) \subseteq \operatorname{int} J$.

Now consider all isometric spheres I_j such that $I_j \neq I$ and $a \in I_j$. As already seen, there are only finitely many, say $I_1, \ldots, I_k$. Suppose that for $j = 1, \ldots, k$, the isometric sphere I_j is the complete geodesic segment $[x_j, y_j]$ with $x_j < y_j$ and suppose further that $y_1 < \ldots < y_k$. For the endpoints x, y of I suppose that $x < y$. W.l.o.g. assume that $\operatorname{Re} a < \operatorname{Re} b$. Then

$$x_1 < \ldots < x_k < x < \operatorname{Re} a < y_1 < \ldots y_k < y.$$

This implies that $\operatorname{int} I_1$ contains the geodesic segment $[x_j, a)$ for $j = 2, \ldots, k$, and that

$$[x_1, a) \subseteq \bigcap_{j=2}^{k} \operatorname{ext} I_j \cap \operatorname{ext} I.$$

We claim that I_1 is relevant. To that end let U be chosen as above. Then $I, I_1, \ldots, I_k \in \mathcal{J}$. For each $J \in \mathcal{J}$ with $a \notin J$ choose an open connected neighborhood V_J of a such that $V_J \subseteq \operatorname{ext} J$. Set $V := U \cap \bigcap_{J \in \mathcal{J}, a \notin J} V_J$.

Since V is an open connected neighborhood of a, there exists $e \neq a$ such that $(e,a) = V \cap [x_1, a)$. Then

$$\begin{aligned} I_1 \cap \partial\mathcal{K} = I_1 \cap \overline{\mathcal{K}} &\supseteq I_1 \cap V \cap \bigcap_{J \in \mathrm{IS}} \overline{\operatorname{ext} J} \\ &= I_1 \cap V \cap \bigcap_{j=1}^{k} \overline{\operatorname{ext} I_j} \cap \overline{\operatorname{ext} I} \\ &\supseteq V \cap [x_1, a) = (e,a). \end{aligned}$$

Therefore I_1 is relevant. The remaining claims now follow from (ii).

(iv) By Lemma 6.1.11 we find an open connected neighborhood U of c in H which intersects only finitely many isometric spheres. Say

$$\mathcal{I} := \{I \in \mathrm{IS} \mid I \cap U \neq \emptyset\}.$$

Each isometric sphere which contains c is an element of $\mathcal{I}$. Prop. 6.1.12 shows that at least one element of $\mathcal{I}$ does contain c. Let $\mathcal{J} := \{J_1, \ldots, J_k\}$ be the subset of $\mathcal{I}$ of isometric spheres which contain c. Suppose that for $j = 1, \ldots, k$, the isometric sphere J_j is the complete geodesic segment $[x_j, y_j]$ with $x_j < y_j$ and suppose further that $y_1 < \ldots < y_k$. As in (iii) one concludes that J_1 is relevant. □

The following example shows that Lemma 6.1.20(iii) does not have an analogous statement for $a \in \partial_g H$, nor Lemma 6.1.20(iv) for $c \in \partial_g \mathcal{K}$.

Example 6.1.21. Let $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$, and denote by Γ the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ which is generated by S and T. One easily sees that

$$\mathcal{F} := \{z \in H \mid |\operatorname{Re} z| < 2,\ |z| > 1\}$$

is a fundamental domain for Γ in H, either by using Poincaré's Theorem (see

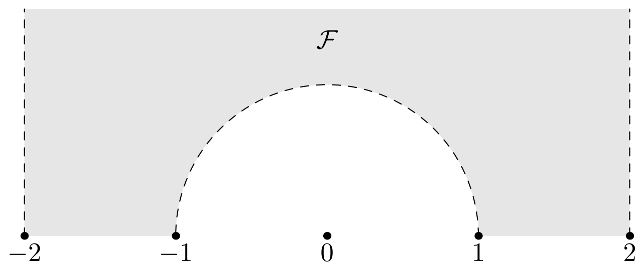


Figure 6.5: The fundamental domain $\mathcal{F}$.

[Mas71]). If we set $\mathcal{F}_\infty := \{z \in H \mid |\operatorname{Re} z| < 2\}$, then it follows that

$$\mathcal{F} = \mathcal{F}_\infty \cap \operatorname{ext} I(S).$$

Hence, $\mathcal{F}$ is an isometric fundamental domain. The relevant isometric spheres are $I(S) = \{z \in H \mid |z| = 1\}$ and its translates by T^m, $m \in \mathbb{Z}$. Moreover, $I(S)$ is the relevant part of $I(S)$, and there is no relevant isometric sphere with relevant part $[1, c]$ for some $c \in \overline{H}^g \smallsetminus \{-1\}$. Further there is no relevant isometric sphere with endpoint $3/2 \in \partial_g \mathcal{K}$.

Let $\mathrm{pr}_\infty \colon \overline{H}^g \smallsetminus \{\infty\} \to \mathbb{R}$ denote the *geodesic projection from* ∞ to $\partial_g H$, i. e.,

$$\mathrm{pr}_\infty(z) := z - \mathrm{ht}(z) = \mathrm{Re}\, z.$$

For $a, b \in \mathbb{R}$ set

$$\langle a, b\rangle := \begin{cases} [a, b] & \text{if } a \le b, \\ [b, a] & \text{otherwise.} \end{cases}$$

Let Rel be the set of all relevant isometric spheres.

Definition 6.1.22. Let $\mathrm{Rel} \neq \emptyset$. A *vertex of* $\mathcal{K}$ is an endpoint of the relevant part of a relevant isometric sphere. Suppose that v is a vertex of $\mathcal{K}$. If $v \in H$, then v is said to be an *inner vertex*, otherwise v is an *infinite vertex*.

If v is an infinite vertex and there are two different relevant isometric spheres I_1, I_2 with relevant parts $[a, v]$ resp. $[v, b]$, then v is called a *two-sided infinite vertex*, otherwise v is said to be a *one-sided infinite vertex*.

Example 6.1.23. For each of our sample groups we consider the set $\mathcal{K}$ and its vertices.

(i) Recall the Hecke triangle group G_n from Example 6.1.16. The set $\mathcal{K}$ has only inner vertices, namely ϱ_n and its $(G_n)_\infty$-translates.

(ii) Recall the congruence group $\mathrm{P}\Gamma_0(5)$ from Example 6.1.17. For this group, the set $\mathcal{K}$ has inner as well as infinite vertices. All infinite vertices are two-sided.

(iii) For the group Γ from Example 6.1.21 we have

$$\mathcal{K} = \bigcap_{m \in \mathbb{Z}} \mathrm{ext}\, I(ST^m) = \{z \in H \mid \forall\, m \in \mathbb{Z} \colon |z + 4m| > 1\},$$

see Fig. 6.6. Each vertex of $\mathcal{K}$ is one-sided infinite.

Prop. 6.1.26 below justifies the notions in Def. 6.1.22. For a precise statement, we need the following two definitions.

Definition 6.1.24. A *side* of a subset A of H is a non-empty maximal convex subset of ∂A. A side S is called *vertical* if $\mathrm{pr}_\infty(S)$ is a singleton, otherwise it is called *non-vertical*.

Definition 6.1.25. Let $\{A_j \mid j \in J\}$ be a family of (possibly bounded) real submanifolds of H or $\overline{H}^g$, and let $n := \max\{\dim A_j \mid j \in J\}$. The union $\bigcup_{j \in J} A_j$ is said to be *essentially disjoint* if for each $i, j \in J$, $i \neq j$, the intersection $A_i \cap A_j$ is contained in a (possibly bounded) real submanifold of dimension $n - 1$.

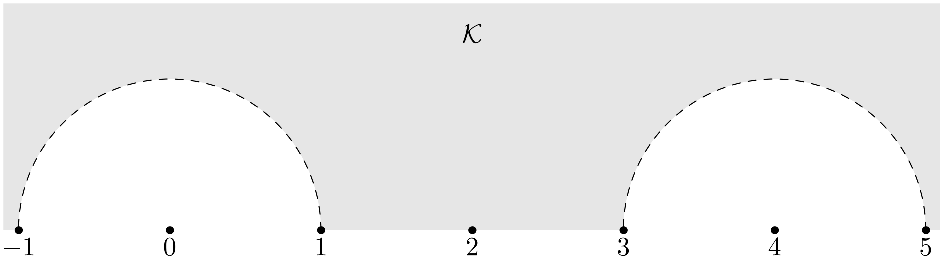


Figure 6.6: The set $\mathcal{K}$ for the group Γ from Example 6.1.21.

The following proposition gives a first insight in the boundary structure of $\mathcal{K}$. It is an immediate consequence of Lemma 6.1.20 and Prop. 6.1.12.

Proposition 6.1.26. *Suppose that* $\mathrm{Rel} \neq \emptyset$. *The set* $\partial\mathcal{K}$ *is the essentially disjoint union of the relevant parts of all relevant isometric spheres. The sides of* $\mathcal{K}$ *are precisely these relevant parts. Each side of* $\mathcal{K}$ *is non-vertical. The family of sides of* $\mathcal{K}$ *is locally finite.*

Remark 6.1.27. If $\mathrm{Rel} = \emptyset$, then Prop. 6.1.26 is essentially void. In this case, $\mathcal{K} = H$, hence $\partial\mathcal{K} = \emptyset$ and $\partial_g\mathcal{K} = \partial_g H$.

Prop. 6.1.29 below provides a deeper insight in the structure of $\partial\mathcal{K}$ by showing that the isometric sphere $I(g)$ is relevant if and only if $I(g^{-1})$ is relevant and that even the relevant parts are mapped to each other by g resp. g^{-1}. For its proof we need the following lemma.

Lemma 6.1.28. *Let* $g_1, g_2 \in \Gamma \smallsetminus \Gamma_\infty$ *such that* $I(g_1) \cap \operatorname{int} I(g_2) \neq \emptyset$. *Then* $g_2 g_1^{-1} \in \Gamma \smallsetminus \Gamma_\infty$ *and*

$$g_1\big(I(g_1) \cap \operatorname{int} I(g_2)\big) = I\big(g_1^{-1}\big) \cap \operatorname{int} I\big(g_2 g_1^{-1}\big).$$

Proof. For $j = 1, 2$ let $g_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$. Fix $z \in I(g_1) \cap \operatorname{int} I(g_2)$ and set $w := g_1 z$. By Lemma 2.2.12, $w \in I(g_1^{-1})$. Hence it remains to prove that $w \in \operatorname{int} I(g_2 g_1^{-1})$. We have

$$\begin{aligned} 1 > |c_2 z + d_2| = |c_2 g_1^{-1} w + d_2| &= \left| c_2 \frac{d_1 w - b_1}{-c_1 w + a_1} + d_2 \right| \\ &= \frac{|(d_1 c_2 - c_1 d_2) w + a_1 d_2 - b_1 c_2|}{|-c_1 w + a_1|} \\ &= |(c_2 d_1 - c_1 d_2) w + a_1 d_2 - b_1 c_2|, \end{aligned}$$

where the last equality holds because $w \in I(g_1^{-1})$. Now

$$g_2 g_1^{-1} = \begin{pmatrix} d_1 a_2 - c_1 b_2 & -b_1 a_2 + a_1 b_2 \\ d_1 c_2 - c_1 d_2 & -b_1 c_2 + a_1 d_2 \end{pmatrix} \in \Gamma.$$

If $d_1 c_2 - c_1 d_2$ would vanish, then $g_2 g_1^{-1}$ would be of the form $\left(\begin{smallmatrix} * & * \\ 0 & c_3 \end{smallmatrix}\right) \in \Gamma_\infty$ with $|c_3| = |a_1 d_2 - b_1 c_2| < 1$. But, since ∞ is a cuspidal point of Γ, each element of Γ_∞

is of the form $\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)$. This gives a contradiction. Therefore, $d_1c_2 - c_1d_2 \neq 0$ and $g_2g_1^{-1} \in \Gamma\smallsetminus\Gamma_\infty$. The calculation from above shows that $w \in \operatorname{int} I(g_2g_1^{-1})$. $\square$

Proposition 6.1.29. *Let $I(g)$ be a relevant isometric sphere with relevant part $[a,b]$. Then the isometric sphere $I(g^{-1})$ is relevant and its relevant part is $g[a,b] = [ga, gb]$.*

Proof. Let $z \in [a,b] \cap H$. We show that $gz \in \partial\mathcal{K}$. Assume for contradiction that $gz \notin \partial\mathcal{K}$. Then either $gz \in H\smallsetminus\overline{\mathcal{K}} = \bigcup_{I\in\mathrm{IS}} \operatorname{int} I$ or $gz \in \mathcal{K} = \bigcap_{I\in\mathrm{IS}} \operatorname{ext} I$. Suppose that $gz \in \bigcup_{I\in\mathrm{IS}} \operatorname{int} I$ and pick $h \in \Gamma\smallsetminus\Gamma_\infty$ such that $gz \in \operatorname{int} I(h)$. Lemma 2.2.12 shows that $gz \in I(g^{-1}) \cap \operatorname{int} I(h)$. But then Lemma 6.1.28 states that $z \in \operatorname{int} I(hg)$, which contradicts to $z \in \partial\mathcal{K}$. Thus, $gz \in \overline{\mathcal{K}}$. From $gz \in I(g^{-1})$ it follows that $gz \notin \mathcal{K} \subseteq \operatorname{ext} I(g^{-1})$. If we suppose that $gz \in \mathcal{K}$, then the previous argument gives a contradiction. Hence, $gz \in \partial\mathcal{K}$.

This shows that the submanifold $g[a,b] = [ga, gb]$ of H of codimension one (and possibly with boundary) is contained in $I(g^{-1}) \cap \partial\mathcal{K}$. Thus, $I(g^{-1})$ is relevant. Suppose that $[c,d]$ is the relevant part of $I(g^{-1})$. The previous argument shows that $g^{-1}[c,d]$ is contained in the relevant part of $I(g)$. Hence

$$[a,b] = g^{-1}g[a,b] \subseteq g^{-1}[c,d] \subseteq [a,b],$$

and therefore $[c,d] = g[a,b]$. $\square$

Remark 6.1.30. Prop. 6.1.29 clearly implies that inner vertices of $\mathcal{K}$ are mapped to inner vertices, and infinite vertices to infinite ones. But it does not show whether two-sided infinite vertices are mapped to two-sided infinite vertices, and one-sided infinite vertices to one-sided ones. We do not know whether this is true for all discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ of which ∞ is a cuspidal point and which satisfy (A1).

6.1.6. The structure of the isometric fundamental domains

Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of which ∞ is a cuspidal point and which satisfies (A1). Let $t_\lambda = \left(\begin{smallmatrix}1&\lambda\\0&1\end{smallmatrix}\right)$ be the generator of Γ_∞ with $\lambda > 0$ and recall that for each $r \in \mathbb{R}$, the set $\mathcal{F}_\infty(r) := (r, r+\lambda) + i\mathbb{R}^+$ is a fundamental domain for Γ_∞ in H. As before set $\mathcal{K} := \bigcap_{I\in\mathrm{IS}} \operatorname{ext} I$ and define

$$\mathcal{F}(r) := \mathcal{F}_\infty(r) \cap \mathcal{K}$$

for $r \in \mathbb{R}$. In this section we will show that, for some choices of $r \in \mathbb{R}$, the fundamental domain $\mathcal{F}(r)$ is a geometrically finite, exact, convex fundamental polyhedron for Γ in H. This in turn will show that Γ is a geometrically finite group and will allow to characterize the cuspidal points of Γ.

Definition 6.1.31. Let Λ be a subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

(i) A *convex polyhedron* in H is a non-empty, closed, convex subset of H such that the family of its sides is locally finite.

(ii) A fundamental region R for Λ is *locally finite* if $\{g\overline{R} \mid g \in \Lambda\}$ is a locally finite family of subsets of H. If R is a fundamental domain and a locally finite fundamental region for Λ in H, then R is called a *locally finite fundamental domain* for Λ in H.

(iii) A *convex fundamental polyhedron* for Λ in H is a convex polyhedron in H whose interior is a locally finite fundamental domain for Λ in H.

(iv) A convex fundamental polyhedron P for Λ in H is *exact* if for each side S of P there is an element $g \in \Lambda$ such that $S = P \cap gP$.

(v) A convex polyhedron P in H is *geometrically finite* if for each point x in $\partial_g P$ there is an open neighborhood N of x in $\overline{H}^g$ that meets just the sides of P with endpoint x.

Prop. 6.1.34 below discusses the boundary structure of $\mathcal{F}(r)$. This result is a major input for the proof that Γ is geometrically finite and, even more, allows to introduce the notion of boundary intervals in Sec. 6.2 which in turn determines a type of precells and cells in H and finally takes part in the proof that the base manifold of the cross sections is totally geodesic (see Sec. 6.5). The non-vertical sides of $\mathcal{F}(r)$ are contained in relevant isometric spheres. Isometric spheres which coincide with $\partial\mathcal{F}(r)$ in a single point or not at all are not interesting for the structure of $\mathcal{F}(r)$.

Definition 6.1.32. Let $r \in \mathbb{R}$. We say that the isometric sphere I *contributes* to $\partial\mathcal{F}(r)$ if $I \cap \partial\mathcal{F}(r)$ contains more than one point.

Note that an isometric sphere which contributes to $\partial\mathcal{F}(r)$ is necessarily relevant.

Lemma 6.1.33. *There exists $M \geq 0$ such that the set*

$$H_M^\circ := \{z \in H \mid \mathrm{ht}(z) > M\}$$

is contained in $\mathcal{K}$. Moreover, $H_M^\circ \cap \partial K = \emptyset$.

Proof. Recall the map $\tilde{c}\colon \mathrm{IS} \to \mathbb{R}^+$ from p. 94. [Bor97, Lemma 3.7] implies that $\tilde{c}$ assumes its minimum. Necessarily, $\min \tilde{c}(\mathrm{IS}) > 0$. Choose $M > 1/\min\tilde{c}(\mathrm{IS})$. Pick $z \in H_M^\circ$ and let $I \in \mathrm{IS}$. Since the radius of I is $1/\tilde{c}(I)$, this isometric sphere is height-bounded from above by M. Remark 6.1.1 implies that $z \in \operatorname{ext} I$. Therefore, $z \in \bigcap_{I\in\mathrm{IS}} \operatorname{ext} I = \mathcal{K}$. Now $\mathcal{K}$ is open by Prop. 6.1.6, thus $H_M^\circ \cap \partial\mathcal{K} = \emptyset$. □

Proposition 6.1.34. *Let $r \in \mathbb{R}$. The fundamental domain $\mathcal{F}(r)$ has two vertical sides. These are the connected components of $\partial\mathcal{F}_\infty(r) \cap \overline{\mathcal{K}}$. The set of non-vertical sides of $\mathcal{F}(r)$ is given by*

$$\big\{I \cap \partial\mathcal{F}(r) \;\big|\; I \text{ contributes to } \partial\mathcal{F}(r)\big\}.$$

In particular, each relevant isometric sphere induces at most one side of $\mathcal{F}(r)$. Moreover, the family of sides of $\mathcal{F}(r)$ is locally finite.

Proof. Recall the boundary structure of $\mathcal{F}(r)$ from Theorem 6.1.15. We start by showing that there are two connected components of $\partial\mathcal{F}_\infty(r)\cap\mathcal{K}$ and that these are vertical sides of $\mathcal{K}$. The set $\partial\mathcal{F}_\infty(r)$ consists of two connected components given by the geodesic segments (r,∞) and $(r+\lambda,\infty)$. Consider (a,∞) where $a\in\{r,r+\lambda\}$. Lemma 6.1.33 shows that $(a,\infty)\cap\overline{\mathcal{K}}\neq\emptyset$. Let z be any element of $(a,\infty)\cap\overline{\mathcal{K}}$. Then Lemma 6.1.14 shows that the geodesic segment $[z,\infty)$ is contained in $\overline{\mathcal{K}}$. Clearly, it is contained in $\partial\mathcal{F}_\infty(r)$. Thus, each connected component of $\partial\mathcal{F}_\infty(r)\cap\overline{\mathcal{K}}$ is non-empty and a vertical side of $\mathcal{F}(r)$. Moreover, this shows that the non-vertical sides of $\mathcal{F}(r)$ are contained in $\overline{\mathcal{F}_\infty(r)}\cap\partial\mathcal{K}$. We will show that each side of $\mathcal{F}(r)$ which intersects $\mathcal{F}_\infty(r)\cap\partial\mathcal{K}$ is non-vertical. Then each vertical side of $\mathcal{F}(r)$ which intersects $\overline{\mathcal{F}_\infty(r)}\cap\partial\mathcal{K}$ is necessarily one of the two above, which shows that there are only these two vertical sides.

Let S be a side of $\mathcal{F}(r)$ which intersects $\mathcal{F}_\infty(r)\cap\partial\mathcal{K}$. Hence there exists $c\in\partial\mathcal{K}$ such that $c\in S$. Lemma 6.1.20 shows that there exists a relevant isometric sphere I with relevant part s_I such that $c\in s_I$. Since $\mathcal{F}_\infty(r)$ is convex and open, the intersection

$$s_I\cap\mathcal{F}_\infty(r)=I\cap\partial\mathcal{K}\cap\mathcal{F}_\infty(r)$$

is a non-trivial geodesic segment. Therefore, $s_I\cap\mathcal{F}_\infty(r)\subseteq S$ and I contributes to $\partial\mathcal{F}(r)$. Since $s_I\cap\mathcal{F}_\infty(r)$ is non-vertical, S is so. By definition, S is a geodesic segment. Since I is a complete geodesic segment which intersects S non-trivially, $S\subseteq I$. Hence, since $S\subseteq\overline{\mathcal{F}_\infty(r)}\cap\partial\mathcal{K}$,

$$S=I\cap\partial\mathcal{K}\cap\overline{\mathcal{F}_\infty(r)}=I\cap\partial\mathcal{F}(r).$$

This shows that each side of $\mathcal{F}(r)$ which intersects $\mathcal{F}_\infty(r)\cap\partial\mathcal{K}$ is non-vertical and of the form $I\cap\partial\mathcal{F}(r)$ for some isometric sphere I which contributes to $\partial\mathcal{F}(r)$.

Suppose now that I is an isometric sphere contributing to $\partial\mathcal{F}(r)$. Since I is non-vertical, $I\cap\partial\mathcal{F}(r)$ is contained in some non-vertical side S of $\mathcal{F}(r)$. Using that I is a complete geodesic segment, we get that $S=I\cap\partial\mathcal{F}(r)$. Hence, the set of non-vertical sides of $\mathcal{F}(r)$ is precisely

$$\{I\cap\partial\mathcal{F}(r)\mid I\text{ contributes to }\partial\mathcal{F}(r)\}$$

and each relevant isometric sphere generates at most one side of $\mathcal{F}(r)$. Now Lemma 6.1.11, or alternatively Prop. 6.1.26, shows that the family of non-vertical sides is locally finite. Since there are only two vertical sides, the family of all sides is locally finite. $\square$

Lemma 6.1.35. *Let I be a relevant isometric sphere with relevant part s. If t is a subset of s (in H), then $\mathrm{pr}_\infty^{-1}(\mathrm{pr}_\infty(t))\cap\partial\mathcal{K}=t$.*

Proof. Let $V:=\mathrm{pr}_\infty^{-1}(\mathrm{pr}_\infty(s))$ and pick $z\in V\cap\partial\mathcal{K}$. By Lemma 6.1.14, the geodesic segment $[z,\infty)$ is contained in $\overline{\mathcal{K}}$ with $(z,\infty)\subseteq\mathcal{K}$. By Prop. 6.1.12 there is an isometric sphere J such that $z\in J$. Remark 6.1.1 shows that $(\mathrm{pr}_\infty(z),z)\subseteq\operatorname{int}J$ and therefore $(\mathrm{pr}_\infty(z),z)\cap\overline{\mathcal{K}}=\emptyset$. Hence $(\mathrm{pr}_\infty(z),\infty)\cap\partial\mathcal{K}=$

$\{z\}$. Let $w \in s$. Then $w \cap V \cap \partial\mathcal{K}$ and $\mathrm{pr}_\infty^{-1}(\mathrm{pr}_\infty(w)) \cap \partial\mathcal{K} = \{w\}$. This proves the claim. □

Recall that Rel denotes the set of all relevant isometric spheres.

Proposition 6.1.36. *Let $r \in \mathbb{R}$. Then the set $\overline{\mathcal{F}(r)}$ is a geometrically finite convex polyhedron. In particular, $\mathcal{F}(r)$ is finite-sided.*

Proof. If $\mathrm{Rel} = \emptyset$, then $\mathcal{F}(r) = \mathcal{F}_\infty(r)$ and the statements are obviously true. Suppose that $\mathrm{Rel} \neq \emptyset$. Theorem 6.1.15 shows that $\overline{\mathcal{F}(r)}$ is convex and Prop. 6.1.34 states that the family of sides of $\mathcal{F}(r)$, which is the same as that of $\overline{\mathcal{F}(r)}$, is locally finite. Therefore, $\overline{\mathcal{F}(r)}$ is a convex polyhedron.

We will now show that $\overline{\mathcal{F}(r)}$ is geometrically finite. Let $x \in \partial_g\overline{\mathcal{F}(r)}$. Because $\overline{\mathcal{F}(r)}$ is a convex polyhedron in a two-dimensional space, there are at most two sides of $\overline{\mathcal{F}(r)}$ with endpoint x.

Suppose that $x = \infty$. There are two sides of $\overline{\mathcal{F}(r)}$ with endpoint ∞, namely the vertical ones. Lemma 6.1.33 shows that we find $M \geq 0$ such that the set $H_M^\circ := \{z \in H \mid \mathrm{ht}(z) > M\}$ is contained in $\mathcal{K}$ and $H_M^\circ \cap \partial\mathcal{K} = \emptyset$. Let $\varepsilon > 0$. Then

$$U := \Big(H_M^\circ \cap \big\{z \in \overline{H}^g \smallsetminus \{\infty\} \bigm| \mathrm{Re}\, z \notin [r-\varepsilon, r+\lambda+\varepsilon]\big\}\Big) \cup \{\infty\}$$

is a neighborhood of ∞ in $\overline{H}^g$. Using Thm. 6.1.15 we find

$$\begin{aligned} U \cap \partial\mathcal{F}(r) &= \Big(U \cap \big(\partial\mathcal{F}_\infty(r) \cap \overline{\mathcal{K}}\big)\Big) \cup \Big(U \cap \big(\overline{\mathcal{F}_\infty(r)} \cap \partial\mathcal{K}\big)\Big) \\ &= \Big(\big(U \cap \overline{\mathcal{K}}\big) \cap \partial\mathcal{F}_\infty(r)\Big) \cup \Big(\big(U \cap \partial\mathcal{K}\big) \cap \overline{\mathcal{F}_\infty(r)}\Big) \\ &= U \cap \partial\mathcal{F}_\infty(r). \end{aligned}$$

Hence U intersects only the two vertical sides of $\overline{\mathcal{F}(r)}$.

Suppose now that $x \in \mathbb{R}$. In the following we construct vertical strips in H for all possible intersection situations at x. Afterwards these vertical strips are combined to a neighborhood of x.

If x is the endpoint of the relevant part $s := [a, x]$ of some relevant isometric sphere I with $\mathrm{Re}\, a < x$ such that $s \cap \partial\mathcal{F}(r)$ is not empty or a singleton, then choose $\varepsilon > 0$ such that $\max\{r, \mathrm{Re}\, a\} < x - \varepsilon$ and consider the vertical strip $V := (x-\varepsilon, x] + i\mathbb{R}^+$ where $(x-\varepsilon, x]$ denotes an interval in $\mathbb{R}$. Clearly, $V \subseteq \mathcal{F}_\infty(r)$. By Lemma 6.1.35, $V \cap \partial\mathcal{K} \subseteq s$. By Thm. 6.1.15 we find

$$\begin{aligned} V \cap \partial\mathcal{F}(r) &= \Big(V \cap \big(\partial\mathcal{F}_\infty(r) \cap \overline{\mathcal{K}}\big)\Big) \cup \Big(V \cap \big(\overline{\mathcal{F}_\infty(r)} \cap \partial\mathcal{K}\big)\Big) \\ &= \Big(\big(V \cap \partial\mathcal{F}_\infty(r)\big) \cap \overline{\mathcal{K}}\Big) \cup \Big(\big(V \cap \partial K\big) \cap \overline{\mathcal{F}_\infty(r)}\Big) \\ &= s \cap \overline{\mathcal{F}_\infty(r)}. \end{aligned}$$

Hence V intersects only the side of $\overline{\mathcal{F}(r)}$ which is contained in s.

If x is the endpoint of a vertical side s of $\overline{\mathcal{F}(r)}$, then $x = r$ or $x = r + \lambda$ and the vertical strip $V := (-\infty, r] + i\mathbb{R}^+$ resp. $V := [r + \lambda, \infty) + i\mathbb{R}^+$ intersects only s.

If x is the endpoint of at most one side of $\overline{\mathcal{F}(r)}$, then there exists an interval I of the form $[x, y)$ or $(y, x]$ in $\partial_g \mathcal{F}(r)$. More precisely, if $x = r$, then I is of the form $[x, y)$. If $x = r + \lambda$, then I is of the form $(y, x]$. If x is the endpoint of the relevant part $[a, x]$ of some relevant sphere and if $\operatorname{Re} a < x$, then $I = [x, y)$. If $\operatorname{Re} a > x$, then $I = (y, x]$. If x is not the endpoint of any side, then there exists an interval of each kind. In all cases, Lemma 6.1.14 implies that the vertical strip $V := I + i\mathbb{R}^+$ is contained in $\mathcal{K}$ and therefore in $\overline{\mathcal{F}(r)}$. If $x = r$, then V intersects the vertical side (r, ∞). If $x = r + \lambda$, then V intersects the vertical side $(r + \lambda, \infty)$. In all other cases, V does not intersect any side of $\mathcal{F}(r)$.

Combining these results with Lemma 6.1.20 we see that in each situation there is an open interval I which contains x and for which the vertical strip $V := I + i\mathbb{R}^+$ intersects only the sides of $\mathcal{F}(r)$ with endpoint x. Now note that the neighborhood $\overline{V}^g \setminus \{\infty\}$ of x in $\overline{H}^g$ intersects exactly those sides which are intersected by V. Thus, $\overline{\mathcal{F}(r)}$ is geometrically finite. By [Rat06, Cor. 2 of Thm. 12.4.1], $\overline{\mathcal{F}(r)}$ is finite-sided. $\square$

Let v be an inner vertex of $\mathcal{K}$. Then there are two relevant isometric spheres I_1, I_2 with relevant parts $[a, v]$ resp. $[v, b]$. Let $\alpha(v)$ denote the angle at v inside $\mathcal{K}$ between $[a, v]$ and $[v, b]$.

Corollary 6.1.37. *There exists $k > 0$ such that for each inner vertex v of $\mathcal{K}$ we have $\alpha(v) \geq k$.*

Proof. Let v be an inner vertex of $\mathcal{K}$. Then v is the intersection point of two isometric spheres, or more generally, of two complete geodesic segments, say s_1 and s_2. Let γ_{11}, γ_{12} resp. γ_{21}, γ_{22} be the geodesics such that $\gamma_{11}(\mathbb{R}) = s_1 = \gamma_{12}(\mathbb{R})$ resp. $\gamma_{21}(\mathbb{R}) = s_2 = \gamma_{22}(\mathbb{R})$. Let $w_{11}, w_{12}, w_{21}, w_{22}$ be the unit tangent vector to $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}$, resp., at v. Since $v \in H$, each of the sets $\{w_{11}, w_{21}\}$, $\{w_{11}, w_{22}\}$, $\{w_{12}, w_{21}\}$ and $\{w_{12}, w_{22}\}$ contains two elements. Now $\alpha(v)$ is one of the angles between the elements of one of these sets. Therefore $\alpha(v) > 0$.

Let $r \in \mathbb{R}$ and consider the set $\mathcal{V}_{\text{in}}$ of all inner vertices of $\mathcal{K}$ that are contained in $\partial \mathcal{F}(r)$. Each element of $\mathcal{V}_{\text{in}}$ is the endpoint of a side of $\mathcal{F}(r)$. Prop. 6.1.36 shows that $\mathcal{F}(r)$ is finite-sided, hence $\mathcal{V}_{\text{in}}$ is finite. Thus, in turn, there exists $k > 0$ such that for all $v \in \mathcal{V}_{\text{in}}$, $\alpha(v) \geq k$. Each inner vertex of $\mathcal{K}$ is G_∞-equivalent to some element of $\mathcal{V}_{\text{in}}$. Since the angle is invariant under G_∞, the statement is proved. $\square$

Recall the geodesic projection $\operatorname{pr}_\infty \colon \overline{H}^g \setminus \{\infty\} \to \mathbb{R}$ from p. 107.

Theorem 6.1.38. *If $\operatorname{Rel} \neq \emptyset$, then let v be a vertex of $\mathcal{K}$ and set $r := \operatorname{pr}_\infty(v) = \operatorname{Re}(v)$. If $\operatorname{Rel} = \emptyset$, then pick any $r \in \mathbb{R}$. The set $\overline{\mathcal{F}(r)}$ is a geometrically finite, exact, convex fundamental polyhedron for Γ in H.*

Proof. Theorem 6.1.15 shows that $\mathcal{F}(r)$ is open and Prop. 6.1.36 states that $\overline{\mathcal{F}(r)}$ is a convex polyhedron. Therefore, $\overline{\mathcal{F}(r)}^\circ = \mathcal{F}(r)$. By Theorem 6.1.15

and Prop. 6.1.36, it remains to show that $\mathcal{F}(r)$ is locally finite and that $\overline{\mathcal{F}(r)}$ is exact. If $\mathrm{Rel} = \emptyset$, then $\mathcal{F}(r) = (r, r+\lambda) + i\mathbb{R}^+$. Obviously, $\mathcal{F}(r)$ is locally finite and $\overline{\mathcal{F}(r)}$ is exact.

Suppose that $\mathrm{Rel} \neq \emptyset$. We start by determining the exact boundary structure of $\mathcal{F}(r)$. Let s be the relevant part of some relevant isometric sphere and suppose that $\mathrm{pr}_\infty(s) \cap (r, r+\lambda) \neq \emptyset$. We claim that $\mathrm{pr}_\infty(s) \subseteq [r, r+\lambda]$. Let I_1 be a relevant isometric sphere such that its relevant part s_1 has v as an endpoint. Suppose that $r \in \mathrm{pr}_\infty(s)$ and note that $r \in \mathrm{pr}_\infty(s_1)$. Lemma 6.1.35 implies that

$$r \in \mathrm{pr}_\infty^{-1}(\mathrm{pr}_\infty(r)) \cap \overline{\mathcal{K}} \subseteq s \cap s_1.$$

Thus s and s_1 intersect. By Lemma 6.1.20, either $s = s_1$ or $s \cap s_1 = \{v\}$. In both cases, v is an endpoint of s. Therefore, since $\mathrm{pr}_\infty(s) \cap (r, r+\lambda) \neq \emptyset$, $\mathrm{pr}_\infty(s) \subseteq [r, \infty)$. If $r \notin \mathrm{pr}_\infty(s)$, then clearly $\mathrm{pr}_\infty(s) \subseteq (r, \infty)$. Cor. 2.2.15 shows that $\mathcal{K}$ is Γ_∞-invariant, and so is $\partial\mathcal{K}$. Therefore $v+\lambda$ is a vertex of $\mathcal{K}$. A parallel argumentation shows that $\mathrm{pr}_\infty(s) \subseteq (-\infty, r+\lambda]$. Hence $\mathrm{pr}_\infty(s) \subseteq [r, r+\lambda]$.

Because $\mathcal{F}_\infty(r)$ is connected, we find that $s \subseteq \partial\mathcal{F}(r)$. Hence, if the relevant part of some relevant isometric sphere contributes non-trivially to $\partial\mathcal{F}(r)$, then this relevant part is completely contained in $\partial\mathcal{F}(r)$. In combination with Prop. 6.1.34 we see that $\partial\mathcal{F}(r)$ consists of two vertical sides and a (finite) number of relevant parts of relevant isometric spheres.

Suppose first that S is a vertical side. Then $t_\lambda^\varepsilon\overline{\mathcal{F}(r)} \cap \overline{\mathcal{F}(r)} = S$ for either $\varepsilon = 1$ or $\varepsilon = -1$. Suppose now that S is a non-vertical side, and let I be the relevant isometric sphere with relevant part S. Suppose that $g \in \Gamma\smallsetminus\Gamma_\infty$ is a generator of I. Prop. 6.1.29 shows that gS is the relevant part of $I(g^{-1})$. Then there is some $m \in \mathbb{Z}$ such that $t_\lambda^m gS$ intersects non-trivially a non-vertical side of $\overline{\mathcal{F}(r)}$. Since $t_\lambda^m gS$ is the relevant part of $I((t_\lambda^m g)^{-1})$, it is a side of $\overline{\mathcal{F}(r)}$. Now, $I(t_\lambda^m g) = I(g)$, and therefore $(t_\lambda^m g)^{-1}\overline{\mathcal{F}(r)} \cap \overline{\mathcal{F}(r)} = S$. Thus, $\overline{\mathcal{F}(r)}$ is exact if $\mathcal{F}(r)$ is locally finite.

Now let $z \in H$. If z is Γ-equivalent to some point in $\mathcal{F}(r)$ or is contained in a side of $\mathcal{F}(r)$ but not an endpoint of it, then the (argument for the) exactness of $\mathcal{F}(r)$ shows that there is a neighborhood of U which intersects only finitely many Γ-translates of $\overline{\mathcal{F}(r)}$. Suppose that z is an endpoint in H of some side of $\mathcal{F}(r)$. Then z is an inner vertex of $\mathcal{K}$. Suppose that there is $v \in \overline{\mathcal{F}(r)}$ and $g \in \Gamma$ such that $gv = z$. Since $\mathcal{F}(r)$ is a fundamental domain, $v \in \partial\mathcal{F}(r)$. Prop. 6.1.29 implies that v is an inner vertex of $\mathcal{K}$ as well. Then Cor. 6.1.37 implies that there are only finitely many pairs $(v, g) \in \overline{\mathcal{F}(r)} \times \Gamma$ such that $gv = z$. Hence there is a neighborhood U of z intersecting only finitely many Γ-translates of $\overline{\mathcal{F}(r)}$. $\square$

A subgroup Λ of $\mathrm{PSL}(2,\mathbb{R})$ is called *geometrically finite* if there is a geometrically finite, exact, convex fundamental polyhedron for Λ in H.

Corollary 6.1.39. *The group Γ is geometrically finite.*

We end this section with a discussion of the nature of the points in $\partial_g\mathcal{F}(r)$.

Recall from Def. 3.4.2 that the limit set of Γ is the set $L(\Gamma)$ of all accumulation points of $\Gamma \cdot z$ for some $z \in H$. Further recall that $L(\Gamma)$ is a subset of $\partial_g H$, moreover, that for each pair z_1, z_2, the accumulation points of $\Gamma \cdot z_1$ and $\Gamma \cdot z_2$ are identical.

Theorem 6.1.40. *Let r be as in Theorem 6.1.38. Then $\partial_g \mathcal{F}(r) \cap L(\Gamma)$ is finite and consists of cuspidal points of Γ. Moreover, each cusp of Γ has a representative in $\partial_g \mathcal{F}(r) \cap L(\Gamma)$.*

Proof. The first statement is an application of [Rat06, Cor. 3 of Thm. 12.4.4] to Theorem 6.1.38. The second statement follows immediately from the combination of Thm. 12.3.6, Cor. 2 of Thm. 12.3.5, Thm. 12.3.7 and 12.1.1 in [Rat06]. □

Corollary 6.1.41. *If Γ is cofinite, then each infinite vertex of $\mathcal{K}$ is two-sided and a cuspidal point of Γ.*

Proof. In [Kat92, Thm 4.5.1, Thm 4.5.2] it is shown that Γ is cofinite if and only if $L(\Gamma) = \partial H$. Then the statement follows from Theorem 6.1.40. □

6.1.7. A characterization of the group Γ

Let Λ be a geometrically finite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of which ∞ is a cuspidal point. By [Rat06, Thm. 6.6.3] the group Λ is discrete. In this section we show that Λ satisfies (A1) (and (A1')), i. e., for each $z \in H$, the set

$$\mathcal{H}_z = \{\mathrm{ht}(gz) \mid g \in \Lambda\}$$

is bounded from above. This shows that the conditions on Γ from the previous sections are equivalent to require that Γ be geometrically finite and has ∞ as cuspidal point. The strategy of the proof is as follows.

Let $t_\mu := \left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)$ be the generator of Λ_∞ with $\mu > 0$ and recall that for each $a \in \mathbb{R}$, the set

$$\mathcal{F}_\infty(a) := \mathrm{pr}_\infty^{-1}\left((a, a+\mu)\right) \cap H$$

is a fundamental domain for Λ_∞. We show that there is a geometrically finite, exact, convex fundamental polyhedron P for Λ in H of which ∞ is an infinite vertex. Then there exist $a \in \mathbb{R}$, $r > 0$ and a finite number $P_1, \ldots, P_l$ of Γ-translates of P such that

$$N(a) := \{z \in H \mid \mathrm{ht}(z) \geq r\} \cap \overline{\mathcal{F}_\infty(a)} = \{z \in H \mid \mathrm{ht}(z) \geq r\} \cap \bigcup_{j=1}^{l} P_j.$$

Now, for each $z \in H$, the set $N(a) \cap \Gamma \cdot z$ is finite, which implies that $\mathcal{H}_z$ is bounded from above.

The following definition is consistent with Def. 6.1.22.

Definition 6.1.42. Let P be a convex polyhedron in H. We call $x \in \partial_g P$ an *infinite vertex* of P if x is the endpoint of a side of P.

Lemma 6.1.43. *There exists a geometrically finite, exact, convex fundamental polyhedron P for Λ which has ∞ as an infinite vertex.*

Proof. Let $D(p)$ be a Dirichlet domain for Λ with center p. Then $\overline{D(p)}$ is a geometrically finite, exact, convex fundamental polyhedron for Λ. By [Rat06, Thm. 12.3.6] there exists $g \in \Lambda$ such that $\infty \in g\partial_g D(p)$. Now, $gD(p) = D(gp)$, hence $\infty \in \partial_g D(gp)$. Then [Rat06, Cor. 2 of Thm. 12.3.5, Thm. 12.3.7] shows that ∞ is an infinite vertex of $\overline{D(gp)}$. Set $P := \overline{D(gp)}$. $\square$

Let P be a geometrically finite, exact, convex fundamental polyhedron for Λ of which ∞ is an infinite vertex. For each side S of P let $g_S \in \Lambda$ be the unique element such that $S = P \cap g_S^{-1}(P)$. The existence and uniqueness of g_S is given by [Rat06, Thm. 6.7.5]. The set $g_S(S)$ is a side of P. The set

$$Q := \{g_S \mid S \text{ is a side of } P\}$$

is called the *side-pairing of* P. Each infinite vertex of P is an endpoint of exactly one or two sides of P. We assign to each infinite vertex of x of P one or two finite sequences $\big((x_j, g_j)\big)$ by the following algorithm:

(step 1) Set $x_1 := x$ and let S_1 be a side of P with endpoint x_1. Set $g_j := g_{S_j}$ and let $x_2 := g_1(x_1)$. Set $j := 2$.

(step j) If x_j is an endpoint of exactly one side, then the algorithm terminates. In this case, x_j does not belong to the sequence. If x_j is an endpoint of the two sides $g_{j-1}(S_{j-1})$ and S_j of P, then set $g_j := g_{S_j}$. If $x_j = x_1$ and $S_j = S_1$, the algorithm terminates. If $x_j \neq x_1$ or $S_j \neq S_1$, set $x_{j+1} := g_j(x_j)$ and continue with (step $j+1$).

Since P is finite-sided (see [Rat06, Cor. 2 of Thm. 12.4.1]), the previous algorithm terminates for each $x \in \partial_g P$.

Lemma 6.1.44. *Let S be a side of P with endpoint ∞. Then $g_S(\infty)$ is an endpoint of two sides of P.*

Proof. For contradiction assume that $a := g_S(\infty)$ is an endpoint of only one side of P. Then there exists $b \in \mathbb{R}$ such that the interval $\langle a, b\rangle$ is contained in $\partial_g P$. Then there is $r > 0$ such that $K := \langle a, b\rangle + i(0, r]$ is contained in P. Note that $g_S^{-1}(b) \in \mathbb{R}$. The set $g_S^{-1}(K^\circ)$, and therefore $g_S^{-1}(P^\circ)$, contains one of the open half-spaces $\{z \in H \mid \operatorname{Re} z > g_S^{-1}(b)\}$ and $\{z \in H \mid \operatorname{Re} z < g_S^{-1}(b)\}$. Both of which contain points that are equivalent under t_μ, which is a contradiction to $g_S^{-1}(P^\circ)$ being a fundamental domain for Λ. Thus, the claim is proved. $\square$

Proposition 6.1.45. *Let $\big((x_j, g_j)\big)_{j=1,\ldots,k}$ be one of the sequences assigned to $x = \infty$. Then there exists $a \in \mathbb{R}$, $m \in \mathbb{N}_0$ and $r > 0$ such that*

$$\{z \in H \mid \mathrm{ht}(z) \geq r\} \cap \bigcup_{j=1}^{k} g_1^{-1} \cdots g_j^{-1} P = \{z \in H \mid \mathrm{ht}(z) \geq r\} \cap \bigcup_{j=0}^{m} t_\mu^j \overline{\mathcal{F}_\infty(a)}.$$

Further, there exists $l \in \{1, \ldots, k\}$ such that

$$\{z \in H \mid \mathrm{ht}(z) \geq r\} \cap \bigcup_{j=1}^{l} g_1^{-1} \cdots g_j^{-1} P = \{z \in H \mid \mathrm{ht}(z) \geq r\} \cap \overline{\mathcal{F}_\infty(a)}.$$

Proof. Let $T_1 = [\alpha_1, \infty]$, $T_2 = [\alpha_2, \infty]$ be the two sides of P of which ∞ is an endpoint. Suppose that $\operatorname{Re}\alpha_1 < \operatorname{Re}\alpha_2$ and suppose further for simplicity that $S_1 = T_2$. The argumentation for $S_1 = T_1$ is analogous. For $j = 1, \ldots, k$ let $S_j = [a_j, x_j]$ be the side of P such that $g_j = g_{S_j}$, and set $h_{j+1} := g_j g_{j-1} \cdots g_2 g_1$ and $h_1 := \mathrm{id}$. By construction, $x_j = h_j \infty$ and

$$h_{j+1}^{-1} P \cap h_j^{-1} P = h_j^{-1}(g_j^{-1} P \cap P) = h_j^{-1} S_j.$$

for $j = 1, \ldots, k$. Set $r := \max\big\{ \mathrm{ht}(h_j^{-1} a_j) \bigm| j = 1, \ldots, k \big\}$. Further define $H_r := \{z \in H \mid \mathrm{ht}(z) \geq r\}$. Then

$$\mathrm{pr}_\infty^{-1} \big([\operatorname{Re} a_1, \operatorname{Re} h_2^{-1} a_2] \big) \cap H_r = h_2^{-1} P \cap H_r$$

and S_1, $h_2^{-1} S_2$ are precisely the (vertical) sides of $h_2^{-1} P$ with ∞ as an endpoint. Inductively one sees that, for $j = 2, \ldots, k+1$,

$$\mathrm{pr}_\infty^{-1} \big([\operatorname{Re} a_1, \operatorname{Re} h_j^{-1} a_j] \big) \cap H_r = \bigcup_{l=2}^{j} h_l^{-1} P \cap H_r \tag{6.5}$$

and S_1, $h_j^{-1} S_j$ are the two (vertical) sides of $\bigcup_{l=2}^{j} h_l^{-1} P$ having ∞ as an endpoint. By iterated application of Lemma 6.1.44, we find that $x_k = x_1$ and $S_k = S_1$. Then $h_k x_1 = x_k = x_1$ and thus $h = \left(\begin{smallmatrix} 1 & n\lambda \\ 0 & 1 \end{smallmatrix}\right)$ for some $n \in \mathbb{Z}$. Since $h_k^{-1} S_1 = h_k^{-1} S_k = S_1$, the set

$$K := \mathrm{pr}_\infty^{-1} \big([\operatorname{Re} a_1, \operatorname{Re} h_k^{-1} a_k] \big) \cap H_r$$

has width $|n|\mu$. With $a := \operatorname{Re} a_1$ and $m := |n| - 1$ we get

$$K = H_r \cap \bigcup_{l=0}^{m} t_\mu^l \overline{\mathcal{F}_\infty(a)}.$$

Now, let l be the minimal element in $\{1, \ldots, k\}$ such that

$$H_r \cap \overline{\mathcal{F}_\infty(a)} \subseteq H_r \cap \bigcup_{j=2}^{l} h_j^{-1} P.$$

By (6.5), to show equality, it suffices to show that the vertical sides of both sets are identical. The geodesic segement $[a+ir,\infty]$, which is contained in S_1, is one of the vertical sides of $H_r \cap \overline{\mathcal{F}_\infty(a)}$ and of $H_r \cap \bigcup_{j=2}^{l} h_j^{-1}P$. The other vertical side of $H_r \cap \overline{\mathcal{F}_\infty(a)}$ is the geodesic segment $b := [a+\mu+ir,\infty]$. Assume for contradiction that b is not a vertical side of $H_r \cap \bigcup_{j=2}^{l} h_j^{-1}P$. Then the minimality of l implies that $(a+\mu+ir,\infty) \subseteq h_l^{-1}P^\circ$. Let $w \in (a+\mu+ir,\infty)$. Then $t_\mu^{-1}w \in S_1$ and $h_l w \in P^\circ$. This means that the orbit Λw contains elements in P° and in ∂P, which is a contradiction to P° being a fundamental domain. Hence b is a vertical side of $H_r \cap \bigcup_{j=2}^{l} h_j^{-1}P$ and

$$H_r \cap \overline{\mathcal{F}_\infty(a)} = H_r \cap \bigcup_{j=2}^{l} h_j^{-1}P. \qquad \square$$

Theorem 6.1.46. *For each $z \in H$, the set $\mathcal{H}_z$ is bounded from above.*

Proof. Fix a geometrically finite, exact, convex fundamental polyhedron P of which ∞ is an infinite vertex. In particular, P is finite-sided. For $r > 0$ set $H_r := \{z \in H \mid \mathrm{ht}(z) \geq r\}$. Prop. 6.1.45 shows that we find $a \in \mathbb{R}$, $r > 0$ and finitely many elements $h_1, \ldots, h_k \in \Lambda$ such that

$$H_r \cap \bigcup_{j=1}^{k} h_j P = H_r \cap \overline{\mathcal{F}_\infty(a)}.$$

Choose $r > 0$ so that, for each $j = 1, \ldots, k$, only the vertical sides of $h_j P$ with endpoint ∞ intersect H_r. Let $\varepsilon > 0$ and set $s := r + \varepsilon$. Obviously,

$$H_s \cap \bigcup_{j=1}^{k} h_j P = H_s \cap \overline{\mathcal{F}_\infty(a)}.$$

Let $z \in H$ and consider

$$\mathrm{HT}_s(z) := \{\mathrm{ht}(gz) \mid g \in \Lambda, \mathrm{ht}(gz) \geq s\}.$$

We will show that $\mathrm{HT}_s(z)$ contains only finitely many elements. More precisely, we will show that $\#\,\mathrm{HT}_r(z) \leq k+2$. Assume for contradiction that there are $k+3$ elements in $\mathrm{HT}_s(z)$, say $b_1, \ldots, b_{k+3}$. Then there exist $g_1, \ldots, g_{k+3} \in \Lambda$ such that $b_l = \mathrm{ht}(g_l z)$. Since the height of a point in H is invariant under Λ_∞, the elements $g_1, \ldots, g_{k+3}$ are pairwise Γ_∞-inequivalent. Moreover, we can suppose that $g_l z \in \overline{\mathcal{F}_\infty(a)}$ for each $l = 1, \ldots, k+3$. Let V be a connected neighborhood of $g_1 z$ such that, for each $l = 1, \ldots, k+3$, the neighborhood $g_l g_1^{-1} V$ of $g_l z$ is contained in H_r and intersects at most two Γ-translates of P. Moreover, for $a \neq b$, suppose $g_a g_1^{-1} V \cap g_b g_1^{-1} V = \emptyset$. By the choice of s, such a V exists. Thus, there are $h_0, h_{k+1} \in \Lambda$ such that

$$g_l g_1^{-1} V \subseteq \bigcup_{j=0}^{k+1} h_j P$$

for each $l = 1, \ldots, k+3$. Fix an element $w \in V$ such that $w \in h_{j_1} P^\circ$ for some $j_1 \in \{0, \ldots, k+1\}$. Then, for each $l = 1, \ldots, k+3$, there exists a (unique) $j_l \in \{0, \ldots, k+1\}$ such that $g_j g_1^{-1} w \in h_{j_l} P^\circ$. Since P° is a fundamental domain for Λ, from $g_a g_1^{-1} w \neq g_b g_1^{-1} w$ for $a \neq b$, it follows that $j_a \neq j_b$. But now

$$\#\{g_l g_1^{-1} w \mid l = 1, \ldots, k+3\} > k+2 = \#\{0, \ldots, k+1\}.$$

This gives the contradiction. Hence $\mathrm{HT}_s(z)$ is finite, which implies that $\mathcal{H}_z$ is bounded from above. $\square$

6.2. Precells in H

Throughout this section let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of which ∞ is a cuspidal point and which satisfies (A1), or, equivalently, let Γ be a geometrically finite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ with ∞ as cuspidal point. To avoid empty statements suppose that the set Rel of relevant isometric spheres is non-empty, or in other words, that $\Gamma \neq \Gamma_\infty$. As before let $\mathcal{K} := \bigcap_{I \in \mathrm{IS}} \operatorname{ext} I$ and suppose that $t_\lambda := \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ is the generator of Γ_∞ with $\lambda > 0$. For $r \in \mathbb{R}$ set $\mathcal{F}_\infty(r) := (r, r+\lambda) + i\mathbb{R}^+$ and $\mathcal{F}(r) := \mathcal{F}_\infty(r) \cap \mathcal{K}$.

This section is devoted to the definition of precells in H and the study of some of their properties. To each vertex of $\mathcal{K}$ we attach one or two precells in H, which are certain convex polyhedrons in H with non-empty interior. Precells in H are the building blocks for cells in H and thus for the geometric cross section. Moreover, precells in H determine the precells in SH and therefore influence the structure of cells in SH, the choice of the reduced cross section and its labeling. There are three types of precells, namely strip precells, which are related to one-sided infinite vertices, cuspidal precells, which are attached to one- and two-sided infinite vertices, and non-cuspidal precells, which are defined for inner vertices. For the definition of strip precells we need to investigate the structure of $\partial_g \mathcal{K}$ in the neighborhood of a one-sided infinite vertex, which we carry out in Sec. 6.2.1. In that section we introduce the notion of boundary intervals, which completely determine the strip precells.

In Sec. 6.2.2 we define all types of precells in H and investigate some of their properties. For this we impose the additional condition (A2) on Γ, which is defined there. In particular, we will introduce the notion of a basal family of precells in H and show its existence. A basal family of precells in H satisfies all properties one would expect from its name. It is a minimal family of precells in H such that each precell in H is a unique Γ_∞-translate of some basal precell. For each precell $\mathcal{A}$ in H there is a basal family of precells containing $\mathcal{A}$, and the cardinality of each basal family of precells in H is finite and independent from the choice of the particular precells contained in the family. Its existence is shown via a decomposition of the closure of the isometric fundamental domain $\mathcal{F}(r)$ for certain parameters r. In Sec. 6.4 these basal families of precells in H are needed to define finite sequences, so-called cycles, of basal precells and elements in $\Gamma \setminus \Gamma_\infty$ which are used for the definition of cells in H.

We end this section with the proof that the family of all Γ-translates of precells in H is a tesselation of H. This fact will show, in Sec. 6.4, that also the family of all Γ-translates of cells in H is a tesselation of H, which in turn will allow to define the base manifold of the geometric cross section.

6.2.1. Boundary intervals

If one considers an isometric sphere as a subset of $\overline{H}^g$, then the set of all isometric spheres need not be locally finite. For example, in the case of the modular group $\mathrm{PSL}(2,\mathbb{Z})$, each neighborhood of 0 in $\overline{H}^g$ contains infinitely many isometric spheres. Therefore, a priori, it is not clear whether the set of all relevant isometric spheres is locally finite in $\overline{H}^g$. This in turn shows that it is not obvious whether or not the set of infinite vertices of $\mathcal{K}$ has accumulation points and if so, whether these accumulation points are infinite vertices. In Prop. 6.2.4 below we will show that if v is a one-sided infinite vertex of $\mathcal{K}$, then there is an interval of the form $\langle v,w\rangle$ in $\partial_g\mathcal{K}$. Moreover, if $\langle v,w\rangle$ is chosen to be maximal, then w is a one-sided infinite vertex as well and w is uniquely determined. The main idea for this fact is to use that the fundamental domains $\mathcal{F}(r)$, $r\in\mathbb{R}$, from Prop. 6.1.36 are finite-sided and that, for an appropriate choice of the parameter r, the infinite vertices of $\mathcal{F}(r)$ in $\mathbb{R}$ coincide with the infinite vertices of $\mathcal{K}$ in the relevant part of $\partial_g H$. Moreover, we will show that the set $\mathrm{pr}_\infty^{-1}(\langle v,w\rangle)\cap H$ is completely contained in $\mathcal{K}$, which will be crucial for the properties of strip precells in H. For the proof of Prop. 6.2.4 we need the following three lemmas.

Lemma 6.2.1. *Let I be an isometric sphere and $z\in\mathrm{int}_\mathbb{R}\big(\mathrm{pr}_\infty(I)\big)$. Then $\mathrm{pr}_\infty^{-1}(z)\cap\partial\mathcal{K}\neq\emptyset$.*

Proof. Suppose that I is the complete geodesic segment $[x,y]$ with $x<y$. Then $\mathrm{int}_\mathbb{R}\big(\mathrm{pr}_\infty(I)\big)$ is the real interval (x,y). Suppose that $z\in(x,y)$. Fix $\varepsilon>0$ such that $(z-\varepsilon,z+\varepsilon)\subseteq(x,y)$. Then $B_\varepsilon(z)\subseteq\mathrm{int}\,I$. Moreover, the geodesic segment $(z,z+i\varepsilon)$ is contained in

$$\bigcup_{J\in\mathrm{IS}}\overline{\mathrm{int}\,J}=\overline{\bigcup_{J\in\mathrm{IS}}\mathrm{int}\,J}=\complement\bigcap_{J\in\mathrm{IS}}\mathrm{ext}\,J=\complement\mathcal{K}.$$

Thus $\mathrm{pr}_\infty^{-1}(z)\cap\complement\mathcal{K}\neq\emptyset$. Lemma 6.1.33 shows that $\mathrm{pr}_\infty^{-1}(z)\cap\mathcal{K}\neq\emptyset$. Since the geodesic segment (z,∞) is connected, it intersects $\partial\mathcal{K}$. $\square$

Lemma 6.2.2. *Let I and J be relevant isometric spheres with relevant parts s_I and s_J. If*

$$\mathrm{pr}_\infty(s_I)\cap\mathrm{int}_\mathbb{R}\big(\mathrm{pr}_\infty(s_J)\big)\neq\emptyset,$$

then $I=J$.

Proof. Since $\mathrm{pr}_\infty(s_I)$ and $\mathrm{pr}_\infty(s_J)$ are intervals in $\mathbb{R}$, the set $\mathrm{pr}_\infty(s_I)\cap\mathrm{int}_\mathbb{R}\big(\mathrm{pr}_\infty(s_J)\big)$ is an open interval, say

$$(a,b):=\mathrm{pr}_\infty(s_I)\cap\mathrm{int}_\mathbb{R}\big(\mathrm{pr}_\infty(s_J)\big).$$

Let $a_I, b_I \in s_I$ resp. $a_J, b_J \in s_J$ such that

$$(a,b) = \big(\operatorname{pr}_\infty(a_I), \operatorname{pr}_\infty(b_I)\big) = \big(\operatorname{pr}_\infty(a_J), \operatorname{pr}_\infty(b_J)\big).$$

Lemma 6.1.35 shows that

$$(a_I, b_I) = \operatorname{pr}_\infty^{-1}\big((a,b)\big) \cap \partial\mathcal{K} = (a_J, b_J).$$

Hence the complete geodesic segments I and J intersect non-trivially, which implies that they are equal. □

Lemma 6.2.3. *Let v be an infinite vertex of $\mathcal{K}$. Then the geodesic segment (v,∞) is contained in $\mathcal{K}$.*

Proof. By the definition of infinite vertices we find a relevant isometric sphere I with relevant part s_I such that v is an endpoint of s_I. Assume for contradiction that there is $z \in (v,\infty)$ such that $z \notin \mathcal{K}$. Then $z \in \complement\mathcal{K} = \bigcup_{J\in \mathrm{IS}} \overline{\operatorname{int} J}$. Pick an isometric sphere $J \in \mathrm{IS}$ such that $z \in \overline{\operatorname{int} J}$. This and $z \in H$ implies that $\operatorname{pr}_\infty(z) \in \operatorname{int}_\mathbb{R}\big(\operatorname{pr}_\infty(J)\big)$. The combination of Lemmas 6.2.1 and 6.1.20 shows that there is a relevant isometric sphere L such that its relevant part s_L intersects (v,∞) in H. Let $s_L = [a,b]$ with $\operatorname{Re} a < \operatorname{Re} b$. We will show that all possible relations between a, b and v lead to a contradiction.

Suppose first $a \in (v,\infty)$. Then a is the intersection point of s_L with (v,∞) and hence in H. But then a is an inner vertex, which implies (see Lemma 6.1.20) that there is a relevant isometric sphere L_2 with relevant part $s_2 := [c,a]$ and $\operatorname{Re} c < \operatorname{Re} a$. Then

$$\operatorname{pr}_\infty(s_2) \cup \operatorname{pr}_\infty(s_L) = [\operatorname{Re} c, \operatorname{Re} b],$$

which contains v in its interior. Hence either $\operatorname{pr}_\infty(s_2) \cap \operatorname{int}_\mathbb{R}\big(\operatorname{pr}_\infty(s_I)\big) \neq \emptyset$ or $\operatorname{pr}_\infty(s_L) \cap \operatorname{int}_\mathbb{R}\big(\operatorname{pr}_\infty(s_I)\big) \neq \emptyset$. By Lemma 6.2.2 either $L_2 = I$ or $L = I$. In each case $v = a$, which contradicts to a being an inner vertex. An analogous argumentation shows that $b \notin (v,\infty)$.

Suppose that $\operatorname{Re} a < v < \operatorname{Re} b$ (which is the last possible constellation). Then there is $\varepsilon > 0$ such that $(v-\varepsilon, v+\varepsilon) \subseteq [\operatorname{Re} a, \operatorname{Re} b] = \operatorname{pr}_\infty(s_L)$. It follows that $\operatorname{pr}_\infty(s_L) \cap \operatorname{int}_\mathbb{R}\big(\operatorname{pr}_\infty(s_I)\big) \neq \emptyset$ and therefore $I = L$. But then s_L cannot intersect (v,∞) in H. This is a contradiction. Hence $(v,\infty) \subseteq \mathcal{K}$. □

Proposition 6.2.4. *Let v be a one-sided infinite vertex of $\mathcal{K}$. Then there exists a unique one-sided infinite vertex w of $\mathcal{K}$ such that the vertical strip $\operatorname{pr}_\infty^{-1}(\langle v,w\rangle) \cap H$ is contained in $\mathcal{K}$. In particular, $\operatorname{pr}_\infty^{-1}(\langle v,w\rangle)$ does not intersect any isometric sphere in H, and, of all vertices of $\mathcal{K}$, $\operatorname{pr}_\infty^{-1}(\langle v,w\rangle)$ contains only v and w.*

Proof. Let I be the relevant isometric sphere with relevant part s_I of which v is an endpoint. W.l.o.g. suppose that I is the complete geodesic segment $[v,x]$ with $v < x$. Consider the fundamental domain $\mathcal{F}(v) = \mathcal{F}_\infty(v) \cap \mathcal{K}$ of Γ in H. Let $\mathcal{V}_\mathbb{R}$ be the set of endpoints in $\mathbb{R}$ of the sides of $\mathcal{F}(v)$. Our first goal is to show that $\mathcal{V}_\mathbb{R}$ is the set $\mathcal{V}_v$ of all infinite vertices of $\mathcal{K}$ in $[v, v+\lambda]$. Lemma 6.2.3 shows that $(v,\infty) \subseteq \mathcal{K}$. Then $(v,\infty) \subseteq \mathcal{K} \cap \partial\mathcal{F}_\infty(v)$, and hence (v,∞) is a vertical side of $\mathcal{F}(v)$ with endpoint v. Recall from Cor. 2.2.15 that $\mathcal{K}$ is Γ_∞-invariant. Therefore $v+\lambda$ is an infinite vertex of $\mathcal{K}$. Analogously to above we see that $v+\lambda \in \mathcal{V}_\mathbb{R}$. Prop. 6.1.34 shows that the elements in $\mathcal{V}_\mathbb{R} \cap (v, v+\lambda)$ are endpoints of non-vertical sides of $\mathcal{F}(v)$ and that the set of non-vertical sides of $\mathcal{F}(v)$ is given by

$$\{J \cap \partial\mathcal{F}(v) \mid J \text{ contributes to } \partial\mathcal{F}(v)\}.$$

Let $w \in \mathcal{V}_\mathbb{R} \cap (v, v+\lambda)$ and $J \in \mathrm{IS}$ such that the side $J \cap \partial\mathcal{F}(v)$ of $\mathcal{F}(v)$ has w as an endpoint. Theorem 6.1.15 implies that

$$J \cap \partial\mathcal{F}(v) = \overline{\mathcal{F}_\infty(v)} \cap J \cap \partial\mathcal{K}.$$

This shows that J is relevant and that w is an endpoint of its relevant part. Hence, $w \in \mathcal{V}_v$. Conversely, suppose that $w \in \mathcal{V}_v \cap (v, v+\lambda)$. Then there is a relevant isometric sphere L such that its relevant part s_L has w as an endpoint. Say $s_L = [a, w]$. Since $\mathcal{F}_\infty(v)$ is the open, convex vertical strip $(v, v+\lambda) + i\mathbb{R}^+$ and $w \in (v, v+\lambda)$, there exists $b \in s_L$ such that the geodesic segment $[b, w]$ is contained in $\overline{\mathcal{F}_\infty(v)}$. Then

$$[b, w] \subseteq \overline{\mathcal{F}_\infty(v)} \cap L \cap \partial\mathcal{K},$$

which shows that w is an endpoint of some side of $\mathcal{F}(v)$. Thus, $w \in \mathcal{V}_\mathbb{R}$ and $\mathcal{V}_\mathbb{R} = \mathcal{V}_v$.

Now we construct the vertex w of $\mathcal{K}$ with the properties of the claim of the proposition. Prop. 6.1.34 states that $\mathcal{F}(v)$ is finite-sided. Thus $\mathcal{V}_\mathbb{R}$ is finite. This and the fact that $\mathcal{V}_\mathbb{R} \smallsetminus \{v\}$ is non-empty show that

$$w := \min \mathcal{V}_\mathbb{R} \smallsetminus \{v\}$$

exists. We claim that $\mathrm{pr}_\infty^{-1}([v, w]) \cap H$ is contained in $\mathcal{K}$. The proof of this claim will also show the other assertions of the proposition. Assume for contradiction that there exists $z \in \mathrm{pr}_\infty^{-1}([v, w]) \cap H$ such that $z \notin \mathcal{K}$. Because $\complement\mathcal{K} = \bigcup_{J \in \mathrm{IS}} \overline{\mathrm{int}\, J}$ by Prop. 6.1.6, we find $J \in \mathrm{IS}$ such that $z \in \overline{\mathrm{int}\, J}$. This and $z \in H$ shows that $\mathrm{pr}_\infty(z) \in \mathrm{int}_\mathbb{R}\big(\mathrm{pr}_\infty(J)\big)$. Lemmas 6.2.1 and 6.1.20 imply that there is a relevant isometric sphere whose relevant part intersects $\mathrm{pr}_\infty^{-1}([v, w])$ in H. By Prop. 6.1.34 there are only finitely many of these, say $I_1, \ldots, I_n$. Suppose that their relevant parts are $s_j := [a_j, b_j]$, $j = 1, \ldots, n$, resp., with $\operatorname{Re} a_j < \operatorname{Re} b_j$ and suppose further that $\operatorname{Re} a_1 < \operatorname{Re} a_k$ for $k = 2, \ldots, n$. We will show that $a_1 \in \mathcal{V}_v \smallsetminus \{v\}$ with $a_1 < w$. By choice, there is $z \in \mathrm{pr}_\infty^{-1}([v, w]) \cap H$ such that $z \in [a_1, b_1]$. Lemma 6.2.3 shows that $\mathrm{pr}_\infty^{-1}(v) \cap H$ and $\mathrm{pr}_\infty^{-1}(w) \cap H$ are contained in $\mathcal{K}$. Since $s_1 \subseteq \partial\mathcal{K}$ and $\mathcal{K}$ is open, $s_1 \cap \mathrm{pr}_\infty^{-1}(\{v, w\}) \cap H = \emptyset$. Hence $z \in \mathrm{pr}_\infty^{-1}((v, w)) \cap H$. Since $\mathrm{pr}_\infty^{-1}((v, w)) \cap H$ is connected, it follows that

$$(a_1, b_1) \subseteq \mathrm{pr}_\infty^{-1}\big((v, w)\big) \cap H.$$

Then $a_1 \in \mathrm{pr}_\infty^{-1}([v, w))$. Since v is one-sided, $a_1 \neq v$. As before, we see that $a_1 \notin \mathrm{pr}_\infty^{-1}(v) \cap H$. Therefore $a_1 \in \mathrm{pr}_\infty^{-1}((v, w))$. If a_1 is an inner vertex, then Lemma 6.1.20 shows that there is a relevant isometric sphere I_0 with relevant part $s_0 = [c, a_1]$ and $\operatorname{Re} c < \operatorname{Re} a_1$. Then $s_0 \cap \mathrm{pr}_\infty^{-1}([v, w]) \cap H \neq \emptyset$. Hence

$I_0 \in \{I_1, \ldots, I_n\}$ and thus $\operatorname{Re} c \leq \operatorname{Re} a_1$. This is a contradiction. Thus, a_1 is an infinite vertex of $\mathcal{K}$. Then $a_1 \in \mathcal{V}_\mathbb{R} \setminus \{v\}$ with $a_1 < w = \min \mathcal{V}_\mathbb{R} \setminus \{v\}$. This is a contradiction. Therefore $\mathrm{pr}_\infty^{-1}([v,w]) \cap H$ is contained in $\mathcal{K}$, does not intersect any isometric sphere, w is one-sided and unique and $\mathrm{pr}_\infty^{-1}([v,w]$ contains only v and w of all vertices of $\mathcal{K}$. □

Definition 6.2.5. Suppose that v is a one-sided infinite vertex. Let w be the unique one-sided infinite vertex of $\mathcal{K}$ such that the set $\mathrm{pr}_\infty^{-1}(\langle v,w\rangle)$ does not intersect any isometric sphere in H, which is given by Prop. 6.2.4. The interval $\langle v,w\rangle$ is called a *boundary interval* of $\mathcal{K}$, and w is said to be the one-sided infinite vertex *adjacent* to v.

Example 6.2.6. Recall the group Γ from Example 6.1.21 and the set $\mathcal{K}$ from Example 6.1.23. The boundary intervals of $\mathcal{K}$ are the intervals $[1+4m, 3+4m]$ for each $m \in \mathbb{Z}$.

6.2.2. Precells in H and basal families

In this section we introduce the condition (A2), define the precells in H and investigate some of its properties. In particular, we construct basal families of precells in H. The statement of condition (A2) and the definition of precells in H needs the notion of the summit of an isometric sphere.

Definition 6.2.7. The *summit* of an isometric sphere is its (unique) point of maximal height.

Lemma 6.2.8. *Let* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{R}) \setminus \mathrm{PSL}(2,\mathbb{R})_\infty$. *Then the summit of* $I(g)$ *is*

$$s = -\frac{d}{c} + \frac{i}{|c|},$$

and the summit of $I(g^{-1})$ *is* gs. *Moreover, the geodesic projection* $\mathrm{pr}_\infty(s)$ *of* s *is the center* $g^{-1}\infty$ *of* $I(g)$.

Proof. W.l.o.g. we may assume that the representative $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}(2,\mathbb{R})$ of g is chosen such that $c > 0$. Since

$$I(g) = \left\{ z \in H \;\middle|\; \left|z + \tfrac{d}{c}\right| = \tfrac{1}{c} \right\},$$

we find that $s = -\frac{d}{c} + \frac{i}{c}$ and $\mathrm{pr}_\infty(s) = -\frac{d}{c} = g^{-1}\infty$. Further,

$$gs = \frac{as+b}{cs+d} = \frac{1}{c} \cdot \frac{-ad+bc+ia}{-d+i+d} = \frac{a}{c} + \frac{i}{c}$$

is the summit of $I(g^{-1})$. □

From now on we impose the following condition on Γ:

$$\text{For each relevant isometric sphere, its summit is contained in } \partial\mathcal{K} \text{ but not a vertex of } \mathcal{K}. \tag{A2}$$

The examples in the previous sections show that there are subgroups of $\mathrm{PSL}(2,\mathbb{R})$ satisfying all the requirements we impose on Γ. However, in Sec. 6.3 we provide an example of a geometrically finite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of which ∞ is a cuspidal point but which does not fulfill (A2).

We now define the precells in H. Recall that $\mathcal{K} = \bigcap_{I\in\mathrm{IS}} \operatorname{ext} I$. In the following definition we implicitely make some assertions about the geometry of these precells. These will be discussed in the Remark 6.2.10 just below the definition.

Definition 6.2.9. Let v be a vertex of $\mathcal{K}$. Suppose first that v is an inner vertex or a two-sided infinite vertex. Then (see Lemma 6.1.20 resp. Def. 6.1.22) there are (exactly) two relevant isometric spheres I_1, I_2 with relevant parts $[a_1, v]$ resp. $[v, b_2]$. Let s_1 resp. s_2 be the summit of I_1 resp. I_2.

If v is a two-sided infinite vertex, then define $\mathcal{A}_1$ to be the hyperbolic triangle[1] with vertices v, s_1 and ∞, and $\mathcal{A}_2$ to be the hyperbolic triangle with vertices v, s_2 and ∞. The sets $\mathcal{A}_1$ and $\mathcal{A}_2$ are the *precells in H* attached to v. Precells arising in this way are called *cuspidal*.

If v is an inner vertex, then let $\mathcal{A}$ be the hyperbolic quadrilateral with vertices s_1, v, s_2 and ∞. The set $\mathcal{A}$ is the *precell in H* attached to v. Precells that are constructed in this way are called *non-cuspidal*.

Suppose now that v is a one-sided infinite vertex. Then there exist exactly one relevant isometric sphere I with relevant part $[a, v]$ and a unique one-sided infinite vertex w other than v such that $\mathrm{pr}_\infty^{-1}(\langle v, w\rangle)$ does not contain vertices other than v and w (see Prop. 6.2.4). Let s be the summit of I.

Define $\mathcal{A}_1$ to be the hyperbolic triangle with vertices v, s and ∞, and $\mathcal{A}_2$ to be the vertical strip $\mathrm{pr}_\infty^{-1}(\langle v, w\rangle) \cap H$. The sets $\mathcal{A}_1$ and $\mathcal{A}_2$ are the *precells in H* attached to v. The precell $\mathcal{A}_1$ is called *cuspidal*, and $\mathcal{A}_2$ is called a *strip precell*.

Remark 6.2.10. Let $\mathcal{A}$ be a precell in H. We use the notation from Def. 6.2.9.

Suppose first that $\mathcal{A}$ is a non-cuspidal precell in H attached to the inner vertex v. Condition (A2) implies that $s_1 \neq v \neq s_2$. Therefore $\mathcal{A}$ is indeed a quadrilateral. The precell $\mathcal{A}$ has two vertical sides, namely $[s_1, \infty]$ and $[s_2, \infty]$, and two non-vertical ones, namely $[s_1, v]$ and $[v, s_2]$. Moreover, (A2) states that s_1 is contained in the relevant part of I_1. Hence $[s_1, v]$ is a geodesic subsegment of the relevant part of I_1. Likewise, $[v, s_2]$ is contained in the relevant part of I_2. The geodesic projection of $\mathcal{A}$ from ∞ is

$$\mathrm{pr}_\infty(\mathcal{A}) = \langle \operatorname{Re} s_1, \operatorname{Re} s_2\rangle.$$

Suppose now that $\mathcal{A}$ is a cuspidal precell in H attached to the infinite vertex v. Then $\mathcal{A}$ has two vertical sides, namely $[v, \infty]$ and $[s, \infty]$, and a single non-vertical side, namely $[v, s]$. As for non-cuspidal precells we find that $[v, s]$ is contained in the relevant part of some relevant isometric sphere. The geodesic projection of $\mathcal{A}$ from ∞ is

$$\mathrm{pr}_\infty(\mathcal{A}) = \langle \operatorname{Re} s, v\rangle.$$

[1] We consider the boundary of the triangle in H to belong to it.

Suppose finally that $\mathcal{A}$ is the strip precell $\mathrm{pr}_\infty^{-1}(\langle v, w\rangle) \cap H$. Then $\mathcal{A}$ is attached to the two vertices v and w. It has the two vertical sides $[v,\infty]$ and $[w,\infty]$ and no non-vertical ones. The geodesic projection of $\mathcal{A}$ from ∞ is

$$\mathrm{pr}_\infty(\mathcal{A}) = \langle v, w\rangle.$$

In any case, $\mathcal{A}$ is a convex polyhedron with non-empty interior. Therefore $\overline{\mathcal{A}^\circ} = A$ and $\partial(\mathcal{A}^\circ) = \partial\mathcal{A}$.

Example 6.2.11. The Hecke triangle group G_n from Example 6.1.16 has only one precell $\mathcal{A}$ in H, up to equivalence under $(G_n)_\infty$. Its is given by

$$\mathcal{A} = \{z \in H \mid |z| \geq 1, |z-\lambda_n| \geq 1, 0 \leq \operatorname{Re} z \leq \lambda_n\}.$$

This precell is non-cuspidal.

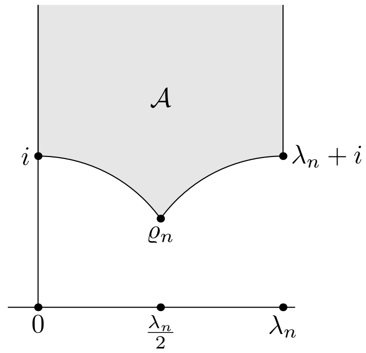


Figure 6.7: The precell $\mathcal{A}$ of G_n.

Example 6.2.12. The precells in H of the congruence group $\mathrm{P}\Gamma_0(5)$ from Example 6.1.17 are indicated in Fig. 6.8 up to $\mathrm{P}\Gamma_0(5)_\infty$-equivalence.

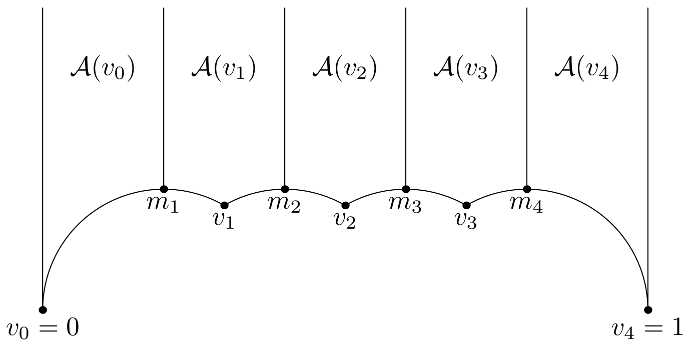


Figure 6.8: Precells in H of $\mathrm{P}\Gamma_0(5)$.

The inner vertices of $\mathcal{K}$ are

$$v_k = \frac{2k+1}{10} + i\frac{\sqrt{3}}{10}, \qquad k = 1, 2, 3,$$

and their translates under $\mathrm{P}\Gamma_0(5)_\infty$. The summits of the indicated isometric spheres are

$$m_k = \frac{k}{5} + \frac{i}{5}, \qquad k = 1, \ldots, 4.$$

The group $\mathrm{P}\Gamma_0(5)$ has cuspidal as well as non-cuspidal precells in H, but no strip precells.

Example 6.2.13. The precells in H of the group Γ from Example 6.1.21 are up to Γ_∞-equivalence one strip precell $\mathcal{A}_1$ and two cuspidal precells $\mathcal{A}_2, \mathcal{A}_3$ as indicated in Fig. 6.9. Here, $v_1 = -3$, $v_2 = -1$, $v_3 = 1$ and $m = i$.

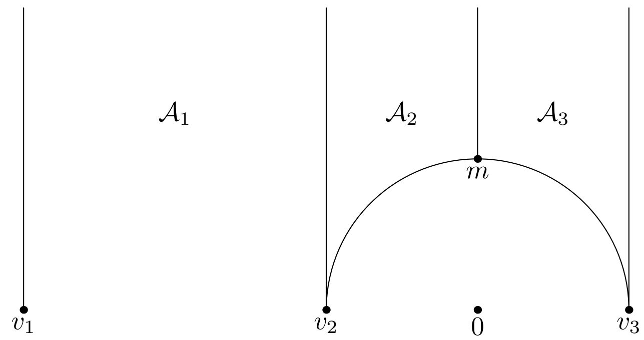


Figure 6.9: Precells in H of Γ.

The following lemma is needed for the proof of Prop. 6.2.15. Beside that, the combination of this lemma and Remark 6.2.10 shows that our definition of non-cuspidal precells in H coincides with that in [Vul99] in presence of condition (A2), whereas cuspidal and strip precells are not precells in the sense of [Vul99].

Lemma 6.2.14. *If $\mathcal{A}$ is a precell in H, then*

$$\mathcal{A} = \mathrm{pr}_\infty^{-1}\left(\mathrm{pr}_\infty(\mathcal{A})\right) \cap \overline{\mathcal{K}} \quad \textit{and} \quad \mathcal{A}^\circ = \mathrm{pr}_\infty^{-1}\left(\mathrm{pr}_\infty(\mathcal{A}^\circ)\right) \cap \mathcal{K}.$$

Proof. For a strip precell, this statement follows immediately from Prop. 6.2.4. Suppose that $\mathcal{A}$ is cuspidal or non-cuspidal. We start with a general observation. Let I be a relevant isometric sphere with relevant part $[a, b]$. Suppose that $c, d \in [a, b]$, $c \neq d$. Lemma 6.1.35 shows that

$$\mathrm{pr}_\infty^{-1}\left(\langle \operatorname{Re} c, \operatorname{Re} d\rangle\right) \cap \partial\mathcal{K} = [c, d],$$

and Lemma 6.1.14 states that for each $e \in [c, d]$ the geodesic segment $[e, \infty)$ is contained in $\overline{\mathcal{K}}$. Hence, $\mathrm{pr}_\infty^{-1}(\langle \operatorname{Re} c, \operatorname{Re} d\rangle) \cap \overline{\mathcal{K}}$ is the hyperbolic triangle with vertices c, d and ∞.

Suppose that $\mathcal{A}$ is a cuspidal precell with vertices v, s and ∞, where v is an (infinite) vertex of $\mathcal{K}$. Let I be the relevant isometric sphere whose relevant part has v as an endpoint and of which s is the summit. By (A2), $s \in \partial\mathcal{K}$. Then

Lemma 6.1.20 implies that $[v, s]$ is contained in the relevant part of I. Our observation from above shows that $\mathrm{pr}_\infty^{-1}(\langle v, \mathrm{Re}\, s\rangle) \cap \overline{\mathcal{K}}$ is the hyperbolic triangle with vertices v, s and ∞. Since $\mathrm{pr}_\infty(\mathcal{A}) = \langle v, \mathrm{Re}\, s\rangle$, the first claim follows. For the second claim note that $\mathrm{pr}_\infty^{-1}(v) \cap \overline{\mathcal{K}}$ and $\mathrm{pr}_\infty^{-1}(s) \cap \overline{\mathcal{K}}$ are the vertical sides of $\mathcal{A}$ and that the non-vertical sides of $\mathcal{A}$ are contained in $\partial\mathcal{K}$. Since $\mathcal{K}$ is open, the second claim follows.

Suppose that $\mathcal{A}$ is a non-cuspidal precell with vertices s_1, v, s_2 and ∞, where v is an (inner) vertex of $\mathcal{K}$. For $j = 1, 2$, let I_j be the relevant isometric sphere with summit s_j and relevant part of which v is an endpoint. As before, we deduce that $\mathrm{pr}_\infty^{-1}(\langle \mathrm{Re}\, v, \mathrm{Re}\, s_j\rangle) \cap \overline{\mathcal{K}}$ is the hyperbolic triangle with vertices v, s_j and ∞. Now

$$\left(\mathrm{pr}_\infty^{-1}\left(\langle \mathrm{Re}\, v, \mathrm{Re}\, s_1\rangle\right) \cap \overline{\mathcal{K}}\right) \cap \left(\mathrm{pr}_\infty^{-1}\left(\langle \mathrm{Re}\, v, \mathrm{Re}\, s_2\rangle\right) \cap \overline{\mathcal{K}}\right) = [v, \infty).$$

Hence

$$\begin{aligned}
&\left(\mathrm{pr}_\infty^{-1}\left(\langle \mathrm{Re}\, v, \mathrm{Re}\, s_1\rangle\right) \cap \overline{\mathcal{K}}\right) \cup \left(\mathrm{pr}_\infty^{-1}\left(\langle \mathrm{Re}\, v, \mathrm{Re}\, s_2\rangle\right) \cap \overline{\mathcal{K}}\right) \\
&= \left(\mathrm{pr}_\infty^{-1}\left(\langle \mathrm{Re}\, v, \mathrm{Re}\, s_1\rangle\right) \cup \mathrm{pr}_\infty^{-1}\left(\langle \mathrm{Re}\, v, \mathrm{Re}\, s_2\rangle\right)\right) \cap \overline{\mathcal{K}} \\
&= \mathrm{pr}_\infty^{-1}\left(\langle \mathrm{Re}\, s_1, \mathrm{Re}\, s_2\rangle\right) \cap \overline{\mathcal{K}}
\end{aligned}$$

is the hyperbolic quadrilateral with endpoints v, s_1, s_2, and ∞. Then $\mathrm{pr}_\infty(\mathcal{A}) = \langle \mathrm{Re}\, s_1, \mathrm{Re}\, s_2\rangle$ implies the first claim. The second claims follows as for cuspidal precells. This completes the proof. $\square$

Proposition 6.2.15. *If two precells in H have a common point, then either they are identical or they coincide exactly at a common vertical side.*

Proof. Let $\mathcal{A}_1, \mathcal{A}_2$ be two non-identical precells in H that have a common point. Suppose first that $\mathcal{A}_1$ and $\mathcal{A}_2$ are both strip precells in H and suppose that $\mathcal{A}_1 = \mathrm{pr}_\infty^{-1}([v_1, v_2]) \cap H$ and $\mathcal{A}_2 = \mathrm{pr}_\infty^{-1}([v_3, v_4]) \cap H$ where $v_1 < v_3$. From $\mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$ it follows that

$$\emptyset \neq \mathrm{pr}_\infty(\mathcal{A}_1) \cap \mathrm{pr}_\infty(\mathcal{A}_2) = [v_1, v_2] \cap [v_3, v_4].$$

Then $v_3 \leq v_2$. If $v_3 < v_2$, then $\mathrm{pr}_\infty^{-1}([v_1, v_2])$ contains the vertex v_3 of $\mathcal{K}$ which is not v_1 or v_2. This contradicts Prop. 6.2.4. If $v_3 = v_2$, then

$$\mathrm{pr}_\infty^{-1}\left([v_1, v_4]\right) \cap H = \left(\mathrm{pr}_\infty^{-1}\left([v_1, v_2]\right) \cap H\right) \cup \left(\mathrm{pr}_\infty^{-1}\left([v_2, v_4]\right) \cap H\right)$$

does not intersect any isometric sphere in H. But then v_2 is not a vertex of $\mathcal{K}$. This is a contradiction. Hence, strip precells are either identical or disjoint.

Suppose now that $\mathcal{A}_2$ is not a strip precell. Let $z \in \mathcal{A}_1 \cap \mathcal{A}_2$. Remark 6.2.10 shows that for $j = 1, 2$, the set $\mathrm{pr}_\infty(\mathcal{A}_j)$ is a closed interval in $\mathbb{R}$ and $\mathrm{pr}_\infty(\mathcal{A}_j^\circ) = \mathrm{int}_\mathbb{R}\left(\mathrm{pr}_\infty(\mathcal{A}_j)\right)$ is an open interval in $\mathbb{R}$. Assume for contradiction that z is not contained in a vertical side of $\mathcal{A}_1$. Then $\mathrm{pr}_\infty(z) \in \mathrm{pr}_\infty(\mathcal{A}_1^\circ) \cap \mathrm{pr}_\infty(\mathcal{A}_2)$ and hence $\mathrm{pr}_\infty(\mathcal{A}_1^\circ) \cap \mathrm{pr}_\infty(\mathcal{A}_2^\circ) \neq \emptyset$. Lemma 6.1.33 and Lemma 6.2.14 show that

$$\emptyset \neq \mathrm{pr}_\infty^{-1}\left(\mathrm{pr}_\infty(\mathcal{A}_1^\circ) \cap \mathrm{pr}_\infty(\mathcal{A}_2^\circ)\right) \cap \mathcal{K} = \mathcal{A}_1^\circ \cap \mathcal{A}_2^\circ.$$

Analogously, we see that $\mathcal{A}_1^\circ \cap \mathcal{A}_2^\circ \neq \emptyset$ if z ist not contained in a vertical side of $\mathcal{A}_2$. We can find $w \in \mathcal{A}_1^\circ \cap \mathcal{A}_2^\circ$ such that $\mathrm{pr}_\infty(w) \neq \mathrm{pr}_\infty(v)$ for each vertex v of $\mathcal{K}$. There is a non-vertical side S of $\mathcal{A}_2$ such that $\mathrm{pr}_\infty(w) \in \mathrm{int}_\mathbb{R}\big(\mathrm{pr}_\infty(S)\big)$.

Suppose first that $\mathcal{A}_1$ is a strip precell. Then $\mathrm{pr}_\infty(w) \in \mathrm{pr}_\infty(\mathcal{A}_1^\circ) \cap \mathrm{pr}_\infty(S)$. Recall that S is contained in the relevant part of some relevant isometric sphere. By Lemma 6.1.35 and the definition of strip precell we find

$$\emptyset \neq \mathrm{pr}_\infty^{-1}\big(\mathrm{pr}_\infty(\mathcal{A}_1^\circ) \cap \mathrm{pr}_\infty(S)\big) \cap \overline{\mathcal{K}} = \mathcal{A}_1^\circ \cap S.$$

Hence, there is an isometric sphere intersecting $\mathcal{A}_1^\circ$. This contradicts Prop. 6.2.4. Therefore $\mathcal{A}_1^\circ \cap \mathcal{A}_2^\circ = \emptyset$ and z is contained in a vertical side of $\mathcal{A}_1$, say in (w, ∞), and in a vertical side of $\mathcal{A}_2$, say in (a, ∞). Remark 6.2.10 shows that a is either the summit of some isometric sphere or a is an infinite vertex of $\mathcal{K}$.

If a is a summit, then a is not a vertex of $\mathcal{K}$ by (A2). Therefore, there is an isometric sphere I such that $\mathrm{pr}_\infty(a) \in \mathrm{int}_\mathbb{R}\big(\mathrm{pr}_\infty(I)\big)$. As before, I intersects $\mathcal{A}_1^\circ$, which contradicts Prop. 6.2.4. Hence a is an infinite vertex, in which case $\mathcal{A}_2$ is cuspidal and $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide exactly at the common vertical side (w, ∞).

Suppose now that $\mathcal{A}_1$ is not a strip precell. Let T be the non-vertical side of $\mathcal{A}_1$ such that $\mathrm{pr}_\infty(w) \in \mathrm{int}_\mathbb{R}\big(\mathrm{pr}_\infty(T)\big)$. We will show that S and T intersect non-trivially. We have that

$$(a, b) := \mathrm{int}_\mathbb{R}\big(\mathrm{pr}_\infty(S)\big) \cap \mathrm{int}_\mathbb{R}\big(\mathrm{pr}_\infty(T)\big)$$

is a non-empty interval in $\mathbb{R}$. Lemma 6.1.35 shows that

$$\mathrm{pr}_\infty^{-1}((a, b)) \cap \partial\mathcal{K} \subseteq S \cap T,$$

hence S and T intersect non-trivially. Recall that the non-vertical sides of precells in H are determined by a vertex v of $\mathcal{K}$ and the summit s of a relevant isometric sphere I such that $[s, v]$ is contained in the relevant part of I. Therefore, $S = T$. This implies that $\mathcal{A}_1$ and $\mathcal{A}_2$ have in common the vertices s, v and ∞, which completely determine $\mathcal{A}_1$ and $\mathcal{A}_2$. Therefore $\mathcal{A}_1 = \mathcal{A}_2$, which contradicts to our hypothesis that $\mathcal{A}_1 \neq \mathcal{A}_2$.

Thus, z is contained in a vertical side of $\mathcal{A}_1$, say in (a_1, ∞), and in a vertical side of $\mathcal{A}_2$, say in (a_2, ∞). Moreover, a_1 is contained in a (unique) non-vertical side S_1 of $\mathcal{A}_1$ and a_2 is contained in a (unique) non-vertical side S_2 of $\mathcal{A}_2$. The sides S_1 and S_2 intersect at most trivially. For $j = 1, 2$ let I_j be the relevant isometric sphere with relevant part s_j such that $S_j \subseteq s_j$. If a_1 is the summit of I_1, then

$$\mathrm{pr}_\infty(a_2) = \mathrm{pr}_\infty(z) = \mathrm{pr}_\infty(a_1) \in \mathrm{int}_\mathbb{R}\big(\mathrm{pr}_\infty(s_1)\big) \cap \mathrm{pr}_\infty(s_2).$$

Lemma 6.2.2 shows that $I_1 = I_2$. Then a_1 is an endpoint of S_2, hence $a_1 = a_2$ and $(a_1, \infty) = (a_2, \infty)$. Hence $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide exactly at the common vertical side (a_1, ∞). The same argumentation applies if a_2 is the summit of I_2.

Suppose now that a_1 and a_2 are endpoint of I_1 resp. I_2. Then (see Remark 6.2.10) a_1 and a_2 are infinite vertices of $\mathcal{K}$ and $\mathcal{A}_1$ and $\mathcal{A}_2$ are cuspidal. Then $a_1 = a_2$, and $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide exactly at the common vertical side (a_1, ∞). □

Proposition 6.2.16. *The set $\overline{\mathcal{K}}$ is the essentially disjoint union of all precells in H,*

$$\overline{\mathcal{K}} = \bigcup \{\mathcal{A} \mid \mathcal{A} \textit{ precell in } H\},$$

and $\mathcal{K}$ contains the disjoint union of the interiors of all precells in H,

$$\bigcup \{\mathcal{A}^\circ \mid \mathcal{A} \textit{ precell in } H\} \subseteq \mathcal{K}.$$

Proof. Let $\mathbb{A} := \{\mathcal{A} \mid \mathcal{A} \text{ precell in } H\}$. Lemma 6.2.14 implies that $\bigcup \mathbb{A} \subseteq \overline{\mathcal{K}}$. Conversely, let $z \in \overline{\mathcal{K}}$. Suppose first that $\mathrm{pr}_\infty^{-1}(\mathrm{pr}_\infty(z)) \cap \partial\mathcal{K} \neq \emptyset$. Let $w \in \mathrm{pr}_\infty^{-1}(\mathrm{pr}_\infty(z)) \cap \partial\mathcal{K}$. Then w is contained in the relevant part $[a,b]$ of some relevant isometric sphere I. Let s be the summit of I. By (A2), $s \in [a,b]$. By definition, the points a, b are vertices of $\mathcal{K}$. Then $[a,s]$ and $[s,b]$ are non-vertical sides of some precells in H. Since $w \in [a,s]$ or $w \in [s,b]$, the point w is contained in some precell, say $w \in \mathcal{A}$. Since $\mathrm{pr}_\infty(w) = \mathrm{pr}_\infty(z)$, Lemma 6.2.14 shows that

$$z \in \mathrm{pr}_\infty^{-1}\left(\mathrm{pr}_\infty(\mathcal{A})\right) \cap \overline{\mathcal{K}} = \mathcal{A}.$$

Suppose now that $\mathrm{pr}_\infty^{-1}(\mathrm{pr}_\infty(z)) \cap \partial\mathcal{K} = \emptyset$. Then $\mathrm{pr}_\infty(z)$ is an infinite vertex of $\mathcal{K}$ or $\mathrm{pr}_\infty(z)$ is contained in some boundary interval of $\mathcal{K}$. In the first case, $\mathrm{pr}_\infty(z)$ is the endpoint of a vertical side of some cuspidal precell $\mathcal{A}$. Then $z \in \mathrm{pr}_\infty^{-1}(\mathrm{pr}_\infty(z)) \subseteq \mathcal{A}$. In the latter case, z is contained in the strip precell determined by the boundary interval. Therefore, $\overline{\mathcal{K}} \subseteq \bigcup \mathbb{A}$. The remaining assertions follow directly from Prop. 6.2.15, Lemma 6.2.14 and the fact that $\mathcal{K}$ is open. $\square$

Recall that $t_\lambda := \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ is the generator of Γ_∞ with $\lambda > 0$ and that for $r \in \mathbb{R}$ we defined $\mathcal{F}_\infty(r) = (r, r+\lambda) + i\mathbb{R}^+$ and $\mathcal{F}(r) = \mathcal{F}_\infty(r) \cap \mathcal{K}$.

Lemma 6.2.17. *If $r \in \mathbb{R}$ is the center of some relevant isometric sphere, then $\partial\mathcal{F}(r)$ contains the summits of all relevant isometric spheres contributing to $\partial\mathcal{F}(r)$, and only these.*

Proof. Recall the boundary structure of $\mathcal{F}(r)$ from Prop. 6.1.34. Let I be a relevant isometric sphere with relevant part s_I. Suppose that r is the center of I and s its summit. Lemma 6.2.8 shows that $\mathrm{pr}_\infty(s) = r$. Hence $s \in \partial\mathcal{F}_\infty(r)$. Since $s \in \partial\mathcal{K}$ by (A2), (s, ∞) is a vertical side of $\mathcal{F}(r)$. Thus, $s \in \partial\mathcal{F}(r)$. By (A2), s is contained in s_I but not an endpoint. Since $\overline{\mathcal{F}_\infty(r)}$ is convex, all of its sides are vertical and s_I non-vertical, we find that s_I intersects $\mathcal{F}_\infty(r) \cap \partial\mathcal{K}$ non-trivially. Theorem 6.1.15 implies that s_I intersects $\partial\mathcal{F}(r)$ non-trivially, which shows that I contributes to $\partial\mathcal{F}(r)$.

The other vertical side of $\mathcal{F}(r)$ is $(s+\lambda, \infty)$. The Γ_∞-invariance of $\mathcal{K}$ shows that $I + \lambda$ is a relevant isometric sphere with relevant part $s_I + \lambda$ and summit $s + \lambda$. Analogously to above, we see that $I + \lambda$ contributes to $\partial\mathcal{F}(r)$.

Finally, all other (relevant) isometric spheres that contribute to $\partial\mathcal{F}(r)$ have their centers in $(r, r+\lambda)$, and all other summits of relevant isometric spheres contained in $\partial\mathcal{F}(r)$ arise from relevant isometric spheres with center in $(r, r+\lambda)$.

Since $\mathcal{F}_\infty(r)$ is the vertical strip $(r, r+\lambda) + i\mathbb{R}^+$, Lemma 6.1.20 implies that each relevant isometric sphere with center in $(r, r+\lambda)$ contributes to $\partial\mathcal{F}(r)$. If J is a relevant isometric sphere with center in $(r, r+\lambda)$, then (A2) shows that its summit s is contained in $\partial\mathcal{K}$. Hence, $s \in \partial\mathcal{K} \cap \mathcal{F}_\infty(r)$, which means that $s \in \partial\mathcal{F}(r)$. □

Remark 6.2.18. Let r be the center of some relevant isometric sphere I. Further let $[a, b]$ be its relevant part with $\operatorname{Re} a < \operatorname{Re} b$ and s its summit. Consider the fundamental domain $\mathcal{F}(r)$. Lemma 6.1.20 and 6.2.17 show that the boundary of $\mathcal{F}(r)$ decomposes into the following sides: There are two vertical sides, namely $[s, i\infty]$ and $[s+\lambda, i\infty]$, and several non-vertical sides, namely $[s, b]$, $[a+\lambda, s+\lambda]$ and the relevant parts of all those relevant isometric spheres with center in $(r, r+\lambda)$.

Definition 6.2.19. Let Λ be a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. A subset $\mathcal{F}$ of H is called a *closed* fundamental region for Λ in H if $\mathcal{F}$ is closed and $\mathcal{F}^\circ$ is a fundamental region for Λ in H. If, in addition, $\mathcal{F}^\circ$ is connected, then $\mathcal{F}$ is said to be a *closed fundamental domain* for Λ in H.

Note that if $\mathcal{F}$ is a fundamental region for Γ in H, then $\overline{\mathcal{F}}$ can happen to be a closed fundamental domain.

Theorem 6.2.20. *There exists a set $\{\mathcal{A}_j \mid j \in J\}$, indexed by J, of precells in H such that the (essentially disjoint) union $\bigcup_{j\in J} \mathcal{A}_j$ is a closed fundamental region for Γ in H. The set J is finite and its cardinality does not depend on the choice of the specific set of precells. The set $\{\mathcal{A}_j \mid j \in J\}$ can be chosen such that $\bigcup_{j\in J} \mathcal{A}_j$ is a closed fundamental domain for Γ in H. In each case, the (disjoint) union $\bigcup_{j\in J} \mathcal{A}_j^\circ$ is a fundamental region for Γ in H.*

Proof. By Prop. 6.2.15 the union of each family of pairwise different precells in H is essentially disjoint. Let r be the center of some relevant isometric sphere I. The boundary structure of $\mathcal{F}(r)$ (see Remark 6.2.18) and Prop. 6.2.16 imply that $\overline{\mathcal{F}(r)}$ decomposes into a set $\mathbb{A} := \{\mathcal{A}_j \mid j \in J\}$ of precells in H. By Prop. 6.1.34 $\overline{\mathcal{F}(r)}$ is finite-sided. Therefore also the set $\mathcal{V}$ of vertices of $\mathcal{K}$ that are contained in $\overline{\mathcal{F}(r)}^g$ is finite. Each vertex of $\mathcal{K}$ determines at most two precells in H. Hence J is finite. Moreover, $\big(\overline{\mathcal{F}(r)}\big)^\circ = \mathcal{F}(r)$. Therefore $\overline{\mathcal{F}(r)}$ is a closed fundamental domain.

Let $\mathbb{A}_2 := \{\mathcal{A}_k \mid k \in K\}$ be a set of precells in H such that $\mathcal{F} := \bigcup_{k\in K} \mathcal{A}_k$ is a closed fundamental region for Γ in H. Cor. 2.3.5 implies that

$$\overline{\mathcal{K}} = \bigcup \{h\mathcal{A} \mid h \in \Gamma_\infty,\ \mathcal{A} \in \mathbb{A}\}.$$

Let $\mathcal{A}_k \in \mathbb{A}_2$ and pick $z \in \mathcal{A}_k^\circ$. Then there exists $h_k \in \Gamma_\infty$ and $j_k \in J$ such that $h_k z \in \mathcal{A}_{j_k}$. Therefore $h_k\mathcal{A}_k^\circ \cap \mathcal{A}_{j_k} \neq \emptyset$. The Γ_∞-invariance of $\mathcal{K}$ shows that $h_k\mathcal{A}_k$ is a precell in H. Then Prop. 6.2.15 implies that $h_k\mathcal{A}_k = \mathcal{A}_{j_k}$, and in turn h_k and j_k are unique. We will show that the map $\varphi\colon K \to J$, $k \mapsto j_k$ is a bijection. To show that φ is injective suppose that there are $l, k \in K$ such that $j_k = j_l =: j$. Then $h_l\mathcal{A}_l = \mathcal{A}_j = h_k\mathcal{A}_k$, hence $h_k^{-1}h_l\mathcal{A}_l = \mathcal{A}_k$. In particular,

$h_k^{-1}h_l\mathcal{A}_l^\circ \cap \mathcal{A}_k^\circ \neq \emptyset$. Since $\bigcup_{h\in K}\mathcal{A}_h^\circ \subseteq \mathcal{F}^\circ$ and $\mathcal{F}^\circ$ is a fundamental region, it follows that $h_k^{-1}h_l = \mathrm{id}$ and $l = k$. Thus, φ is injective. To show surjectivity let $j \in J$ and $z \in \mathcal{A}_j^\circ$. Then there exists $g \in \Gamma$ and $k \in K$ such that $gz \in \mathcal{A}_k$. On the other hand, $\mathcal{A}_k = h_k^{-1}\mathcal{A}_{j_k}$. Hence $h_k g\mathcal{A}_j^\circ \cap \mathcal{A}_{j_k} \neq \emptyset$. Since $\mathcal{A}_j$ and $\mathcal{A}_{j_k}$ are convex polyhedrons, it follows that $h_k g\mathcal{A}_j^\circ \cap \mathcal{A}_{j_k} \neq \emptyset$. Since $\mathcal{F}(r)$ is a fundamental region and $\mathcal{A}_j^\circ, \mathcal{A}_{j_k}^\circ \subseteq \mathcal{F}(r)$, we find that $h_k g = \mathrm{id}$ and $j = j_k$. Hence, φ is surjective. It follows that $\#K = \#J$.

It remains to show that the disjoint union $P := \bigcup_{k\in K}\mathcal{A}_k^\circ$ is a fundamental region for Γ in H. Obviously, P is open and contained in $\mathcal{F}^\circ$. This shows that P satisfies (F1) and (F2). Since $\overline{(\mathcal{A}^\circ)} = \mathcal{A}$ for each precell in H and K is finite, it follows that

$$\overline{P} = \overline{\bigcup_{k\in K}\mathcal{A}_k^\circ} = \bigcup_{k\in K}\mathcal{A}_k = \mathcal{F}.$$

Hence, P satisfies (F3) as well, and thus it is a fundamental region for Γ in H. $\square$

Definition 6.2.21. Each set $\mathbb{A} := \{\mathcal{A}_j \mid j \in J\}$, indexed by J, of precells in H with the property that $\mathcal{F} := \bigcup_{j\in J}\mathcal{A}_j$ is a closed fundamental region is called a *basal family of precells in H* or a *family of basal precells in H*. If, in addition, $\mathcal{F}$ is connected, then $\mathbb{A}$ is called a *connected basal family of precells in H* or a *connected family of basal precells in H*.

Example 6.2.22. Recall the Examples 6.2.11, 6.2.12 and 6.2.13. The set $\{\mathcal{A}\}$ resp. $\{\mathcal{A}(v_0), \ldots, \mathcal{A}(v_4)\}$ resp. $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ of precells in H for G_n resp. $\mathrm{P}\Gamma_0(5)$ resp. Γ is a connected basal family of precells in H for the respective group.

The proof of Theorem 6.2.20 shows the following statements.

Corollary 6.2.23. *Let $\mathbb{A}$ be a basal family of precells in H.*

(i) *For each precell $\mathcal{A}$ in H there exists a unique pair $(\mathcal{A}', m) \in \mathbb{A} \times \mathbb{Z}$ such that $t_\lambda^m\mathcal{A}' = \mathcal{A}$.*

(ii) *For each $\mathcal{A} \in \mathbb{A}$ choose an element $m(\mathcal{A}) \in \mathbb{Z}$. Then $\{t_\lambda^{m(\mathcal{A})}\mathcal{A} \mid \mathcal{A} \in \mathbb{A}\}$ is a basal family of precells in H. For each $\mathcal{A} \in \mathbb{A}$, the precell $t_\lambda^{m(\mathcal{A})}\mathcal{A}$ is of the same type as $\mathcal{A}$.*

(iii) *The set $\overline{\mathcal{K}}$ is the essentially disjoint union $\bigcup\{h\mathcal{A} \mid h \in \Gamma_\infty,\ \mathcal{A} \in \mathbb{A}\}$.*

6.2.3. The tesselation of H by basal families of precells

Suppose that Γ satisfies (A2). The following proposition is crucial for the construction of cells in H from precells in H. Note that the element $g \in \Gamma \smallsetminus \Gamma_\infty$ in this proposition depends not only on $\mathcal{A}$ and b but also on the choice of the basal family $\mathbb{A}$ of precells in H. In this section we will use the proposition as one ingredient for the proof that the family of Γ-translates of all precells in H is a tesselation of H.

Proposition 6.2.24. *Let $\mathbb{A}$ be a basal family of precells in H. Let $\mathcal{A} \in \mathbb{A}$ be a basal precell that is not a strip precell, and suppose that b is a non-vertical side of $\mathcal{A}$. Then there is a unique element $g \in \Gamma \smallsetminus \Gamma_\infty$ such that $b \subseteq I(g)$ and gb is the non-vertical side of some basal precell $\mathcal{A}' \in \mathbb{A}$. If $\mathcal{A}$ is non-cuspidal, then $\mathcal{A}'$ is non-cuspidal, and, if $\mathcal{A}$ is cuspidal, then $\mathcal{A}'$ is cuspidal.*

Proof. Let I be the (relevant) isometric sphere with $b \subseteq I$. We will at first show that there is a generator g of I such that gb is a side of some basal precell. Then $gb \subseteq gI(g) = I(g^{-1})$, which implies that gb is a non-vertical side.

Let $h \in \Gamma \smallsetminus \Gamma_\infty$ be any generator of I, let s be the summit of I and v the vertex of $\mathcal{K}$ that $\mathcal{A}$ is attached to. Then $b = [v, s]$. Further, b is contained in the relevant part of $I = I(h)$. By Prop. 6.1.29, Remark 6.1.30 and Lemma 6.2.8, the set $hb = [hv, hs]$ is contained in the relevant part of the relevant isometric sphere $I(h^{-1})$, the point hv is a vertex of $\mathcal{K}$ and hs is the summit of $I(h^{-1})$. Thus, there is a unique precell $\mathcal{A}_h$ with non-vertical side hb. By Cor. 6.2.23, there is a unique basal precell $\mathcal{A}'$ and a unique $m \in \mathbb{Z}$ such that

$$\mathcal{A}_h = t_\lambda^m \mathcal{A}' = \mathcal{A}' + m\lambda.$$

Then $t_\lambda^{-m} hb$ is a non-vertical side of $\mathcal{A}'$, and $t_\lambda^{-m} hb$ is contained in the relevant part of the relevant isometric sphere $I(h^{-1}) - m\lambda = I(h^{-1} t_\lambda^m) = I((t_\lambda^{-1} h)^{-1})$ (for the first equality see Lemma 6.1.3). Lemma 6.1.2 shows that $g := t_\lambda^{-m} h$ is a generator of I.

To prove the uniqueness of g, let k be any generator of I. By Lemma 6.1.2, there exists a unique $n \in \mathbb{Z}$ such that $k = t_\lambda^n h$. Thus, $kb = t_\lambda^n hb = hb + n\lambda$ and therefore $\mathcal{A}_k = \mathcal{A}_h + n\lambda$. Then

$$\mathcal{A}_k = \mathcal{A}' + m\lambda + n\lambda = t_\lambda^{m+n} \mathcal{A}',$$

and $t_\lambda^{-(m+n)} k$ is a generator of I such that $t_\lambda^{-(m+n)} kb$ is a side of some basal precell. Moreover,

$$t_\lambda^{-(m+n)} k = t_\lambda^{-m} t_\lambda^{-n} k = t_\lambda^{-m} h = g.$$

This shows the uniqueness.

The basal precell $\mathcal{A}'$ cannot be a strip precell, since it has a non-vertical side. Finally, $\mathcal{A}$ is cuspidal if and only if v is an infinite vertex. This is the case if and only if gv is an infinite vertex, which is equivalent to $\mathcal{A}'$ being cuspidal. This completes the proof. □

Lemma 6.2.25. *Let $\mathcal{A}$ be a precell in H. Suppose that S is a vertical side of $\mathcal{A}$. Then there exists a precell $\mathcal{A}'$ in H such that S is a side of $\mathcal{A}'$ and $\mathcal{A}' \neq \mathcal{A}$. In this case, S is a vertical side of $\mathcal{A}'$.*

Proof. We start by showing that each precell in H contains a box of a fixed horizontal width. Let $\mathbb{A}$ be a basal family of precells in H. Theorem 6.2.20 shows that $\mathbb{A}$ contains only finitely many elements. Let λ denote the Lebesgue measure on $\mathbb{R}$. Then

$$m := \min \left\{ \lambda\big(\mathrm{pr}_\infty(\mathcal{A})\big) \;\middle|\; \mathcal{A} \in \mathbb{A} \right\}$$

exists. Cor. 6.2.23 implies that

$$m = \min\left\{\lambda\big(\operatorname{pr}_\infty(\mathcal{A})\big) \;\middle|\; \mathcal{A} \text{ precell in } H\right\}.$$

Lemma 6.1.33 shows that we find $M \geq 0$ such that

$$\{z \in H \mid \operatorname{ht}(z) > M\} \subseteq \mathcal{K}.$$

Let $\widehat{\mathcal{A}}$ be a precell in H and let $S_1 = [a_1, \infty)$ and $S_2 = [a_2, \infty)$ be the vertical sides of $\mathcal{A}$ with $\operatorname{Re} a_1 < \operatorname{Re} a_2$. Remark 6.2.10 and Lemma 6.2.14 imply that

$$K\big(\widehat{\mathcal{A}}\big) := \big\{z \in H \;\big|\; \operatorname{ht}(z) > M,\ \operatorname{Re} z \in [\operatorname{Re} a_1, \operatorname{Re} a_2]\big\} \subseteq \widehat{\mathcal{A}}. \tag{6.6}$$

Our previous consideration shows that $\operatorname{Re} a_2 - \operatorname{Re} a_1 \geq m$.

Now let $\mathcal{A}$ be a precell in H and let $S = [a, \infty)$ be a vertical side of $\mathcal{A}$. W.l.o.g. suppose that $\mathcal{A} \subseteq \{z \in H \mid \operatorname{Re} z \leq \operatorname{Re} a\}$, which means that S is the right vertical side of $\mathcal{A}$. Consider

$$z := \operatorname{Re} a + \tfrac{m}{3} + i(M+1).$$

Then $z \in \mathcal{K}$. By Cor. 6.2.23 there is a precell $\mathcal{A}'$ with $z \in \mathcal{A}'$. Let $T = [b, \infty)$ be the vertical side of $\mathcal{A}'$ such that $\mathcal{A}' \subseteq \{z \in H \mid \operatorname{Re} z \geq \operatorname{Re} b\}$, which means that T is the left vertical side of $\mathcal{A}'$. Since $z \notin \mathcal{A}$, the precells $\mathcal{A}$ and $\mathcal{A}'$ are different. We will show that $\mathcal{A} \cap \mathcal{A}' \neq \emptyset$. Assume for contradiction that $\mathcal{A} \cap \mathcal{A}' = \emptyset$. Then the box

$$K := (\operatorname{Re} a, \operatorname{Re} b) + i(M, \infty)$$

does not intersect $\mathcal{A}$ and $\mathcal{A}'$, but $\partial K \cap \mathcal{A} \neq \emptyset$ and $\partial K \cap \mathcal{A}' \neq \emptyset$. Pick $w \in K$. Then there is a precell $\mathcal{A}''$ in H such that $w \in \mathcal{A}''$. Now $w \in K \cap K(\mathcal{A}'')$. Since $\operatorname{Re} b - \operatorname{Re} a \leq m/3 < m$, the box $K(\mathcal{A}'')$ is not contained in $\overline{K}$. Therefore we have $K(\mathcal{A}'')^\circ \cap \mathcal{A} \neq \emptyset$ or $K(\mathcal{A}'')^\circ \cap \mathcal{A}' \neq \emptyset$. Then Lemma 6.2.14 shows that $(\mathcal{A}'')^\circ \cap \mathcal{A} \neq \emptyset$ or $(\mathcal{A}'')^\circ \cap \mathcal{A}' \neq \emptyset$. By Prop. 6.2.15, $\mathcal{A}'' = \mathcal{A}$ or $\mathcal{A}'' = \mathcal{A}'$. This contradicts to $w \notin \mathcal{A} \cup \mathcal{A}'$. Hence $\mathcal{A} \cap \mathcal{A}' \neq \emptyset$. Prop. 6.2.15 shows that $\mathcal{A}$ and $\mathcal{A}'$ coincide exactly at a common vertical side. If we assume for contradiction that $\operatorname{Re} b < \operatorname{Re} a$, then $K(\mathcal{A}')^\circ \cap K(\mathcal{A}) \neq \emptyset$ (recall that $z \notin K(\mathcal{A})$). But then Lemma 6.2.14 shows that $(\mathcal{A}')^\circ \cap \mathcal{A} \neq \emptyset$, which by Prop. 6.2.15 means that $\mathcal{A}' = \mathcal{A}$. Hence $\operatorname{Re} a = \operatorname{Re} b$ and therefore $S = T$. □

Proposition 6.2.26. *Let $\mathcal{A}_1, \mathcal{A}_2$ be two precells in H and let $g_1, g_2 \in \Gamma$. If $g_1\mathcal{A}_1 \cap g_2\mathcal{A}_2 \neq \emptyset$, then either $g_1\mathcal{A}_1 = g_2\mathcal{A}_2$ and $g_1 g_2^{-1} \in \Gamma_\infty$, or $g_1\mathcal{A}_1 \cap g_2\mathcal{A}_2$ is a common side of $g_1\mathcal{A}_1$ and $g_2\mathcal{A}_2$, or $g_1\mathcal{A}_1 \cap g_2\mathcal{A}_2$ is a point which is the endpoint of some side of $g_1\mathcal{A}_1$ and some side of $g_2\mathcal{A}_2$. If S is a common side of $g_1\mathcal{A}_1$ and $g_2\mathcal{A}_2$, then $g_1^{-1}S$ is a vertical side of $\mathcal{A}_1$ if and only if $g_2^{-1}S$ is a vertical side of $\mathcal{A}_2$.*

Proof. W.l.o.g. $g_1 = \mathrm{id}$. Let $\mathbb{A}$ be a basal family of precells in H. Cor. 6.2.23 shows that we may assume that $\mathcal{A}_1 \in \mathbb{A}$. Let $S_1 = [a_1, \infty]$ and $S_2 = [a_2, \infty]$ be the vertical sides of $\mathcal{A}_1$. If $\mathcal{A}_1$ has non-vertical sides, let these be $S_3 = [b_1, b_2]$ resp. $S_3 = [b_1, b_2]$ and $S_4 = [b_3, b_4]$. Lemma 6.2.25 shows that we find precells

$\mathcal{A}_1'$ and $\mathcal{A}_2'$ such that $\mathcal{A}_1' \neq \mathcal{A}_1 \neq \mathcal{A}_2'$ and S_1 is a vertical side of $\mathcal{A}_1'$ and S_2 is a vertical side of $\mathcal{A}_2'$. Prop. 6.2.24 shows that there exist $(h_3, \mathcal{A}_3') \in \Gamma \times \mathbb{A}$ resp. $(h_3, \mathcal{A}_3'), (h_4, \mathcal{A}_4') \in \Gamma \times \mathbb{A}$ such that $h_3 S_3$ is a non-vertical side of $\mathcal{A}_3'$ and $h_4 S_4$ is a non-vertical side of $\mathcal{A}_4'$ and $h_3\mathcal{A}_1 \neq \mathcal{A}_3'$ and $h_4\mathcal{A}_1 \neq \mathcal{A}_4'$. Recall that each precell in H is a convex polyhedron. Therefore $\mathcal{A}_1 \cup \mathcal{A}_1'$ is a polyhedron with (a_1, ∞) in its interior, and $\mathcal{A}_1 \cup h_3^{-1}\mathcal{A}_3'$ is a polyhedron with (b_1, b_2) in its interior. Likewise for $\mathcal{A}_1 \cup \mathcal{A}_2'$ and $\mathcal{A}_1 \cup h_4^{-1}\mathcal{A}_4'$.

Suppose first that $\mathcal{A}_1^\circ \cap g_2\mathcal{A}_2 \neq \emptyset$. By Cor. 6.2.23 there exists $(h, \mathcal{A}') \in \Gamma_\infty \times \mathbb{A}$ such that $\mathcal{A}_2 = h\mathcal{A}'$. Then $\mathcal{A}_1^\circ \cap g_2 h\mathcal{A}' \neq \emptyset$. Since $\mathcal{A}'$ is a convex polyhedron, $\mathcal{A}_1^\circ \cap g_s h(\mathcal{A}')^\circ \neq \emptyset$. Now $\bigcup\{(\widehat{\mathcal{A}})^\circ \mid \widehat{\mathcal{A}} \in \mathbb{A}\}$ is a fundamental region for Γ in H (see Theorem 6.2.20). Therefore, $g_2 h = \mathrm{id}$ and, by Prop. 6.2.15, $\mathcal{A}_1 = \mathcal{A}'$. Hence $g_2^{-1} \in \Gamma_\infty$ and $\mathcal{A}_1 = g_2\mathcal{A}_2$.

Suppose now that $\mathcal{A}_1^\circ \cap g_2\mathcal{A}_2 = \emptyset$. If $g_2\mathcal{A}_2 \cap (a_1, \infty) \neq \emptyset$, then $g_2\mathcal{A}_2 \cap (\mathcal{A}_1')^\circ \neq \emptyset$ and the argument from above shows that $\mathcal{A}_1' = g_2\mathcal{A}_2$ and $g_2 \in \Gamma_\infty$. From this it follows that S_1 is a vertical side of $\mathcal{A}_2$. If $g_2\mathcal{A}_2 \cap (b_1, b_2) \neq \emptyset$, then $g_2\mathcal{A}_2 \cap h_3^{-1}(\mathcal{A}_3')^\circ \neq \emptyset$. As before, $g_2 h_3 \in \Gamma_\infty$ and $g_2\mathcal{A}_2 = h_3^{-1}\mathcal{A}_3'$. Then S_3 is a non-vertical side of $\mathcal{A}_2$. The argumentation for $g_2\mathcal{A}_2 \cap (a_2, \infty) \neq \emptyset$ and $g_2\mathcal{A}_2 \cap (b_3, b_4) \neq \emptyset$ is analogous.

It remains the case that $g_2\mathcal{A}_2$ intersects $\mathcal{A}_1$ is an endpoint v of some side of $\mathcal{A}_1$. By symmetry of arguments, v is an endpoint of some side of $\mathcal{A}_2$. This completes the proof. $\square$

Definition 6.2.27. A family $\{S_j \mid j \in J\}$ of polyhedrons in H is called a *tesselation* of H if

(T1) $H = \bigcup_{j \in J} S_j$,

(T2) If $S_j \cap S_k \neq \emptyset$ for some $j, k \in J$, then either $S_j = S_k$ or $S_j \cap S_k$ is a common side or vertex of S_j and S_k.

Corollary 6.2.28. *Let $\mathbb{A}$ be a basal family of precells in H. Then*

$$\Gamma \cdot \mathbb{A} = \{g\mathcal{A} \mid g \in \Gamma,\ \mathcal{A} \in \mathbb{A}\}$$

is a tesselation of H which satisfies in addition the property that if $g_1\mathcal{A}_1 = g_2\mathcal{A}_2$, then $g_1 = g_2$ and $\mathcal{A}_1 = \mathcal{A}_2$.

Proof. Let $\mathcal{F} := \bigcup\{\mathcal{A} \mid \mathcal{A} \in \mathbb{A}\}$. Theorem 6.2.20 states that $\mathcal{F}$ is a closed fundamental region for Γ in H, hence $\bigcup_{g \in \Gamma} g\mathcal{F} = H$. This proves (T1). (T2) follows directly from Prop. 6.2.26. Now let $(g_1, \mathcal{A}_1), (g_2, \mathcal{A}_2) \in \Gamma \times \mathbb{A}$ with $g_1\mathcal{A}_1 = g_2\mathcal{A}_2$. Then $g_1\mathcal{A}_1^\circ = g_2\mathcal{A}_2^\circ$. Recalling that $\mathcal{F}^\circ$ is a fundamental region for Γ in H and that $\mathcal{A}_1^\circ, \mathcal{A}_2^\circ \subseteq \mathcal{F}^\circ$, we get that $g_1 = g_2$ and $\mathcal{A}_1 = \mathcal{A}_2$. $\square$

6.3. A group that does not satisfy (A2)

The examples in the previous sections show that there are several subgroups of $\mathrm{PSL}(2, \mathbb{R})$ that are discrete, have ∞ as a cuspidal point and satisfy the conditions

(A1) and (A2). One might speculate that each geometrically finite group with ∞ as cuspidal point fulfills (A2).

However, in the following we provide an example of a geometrically finite subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$ with ∞ as a cuspidal point that does not satisfy (A2). We proceed as follows: The group Γ is given via three generators. We consider a convex polyhedron $\mathcal{F}$ which is of the form of an isometric fundamental domain. We prove, using Poincaré's Theorem, that $\mathcal{F}$ is indeed a fundamental domain for Γ. The shape of $\mathcal{F}$ shows in addition that Γ is cofinite, a property we will not provide a proof for and we will not make use of. One of the generators of Γ is parabolic with ∞ as fixed point, which shows that ∞ is a cuspidal point of Γ. From $\mathcal{F}$, we read off the relevant isometric spheres and their relevant parts. At this point we will see that Γ does not satisfy (A2).

Consider the matrices

$$t := \begin{pmatrix} 1 & \frac{17}{11} \\ 0 & 1 \end{pmatrix}, \quad g_1 := \begin{pmatrix} 18 & -5 \\ 11 & -3 \end{pmatrix} \quad \text{and} \quad g_2 := \begin{pmatrix} 3 & -1 \\ 10 & -3 \end{pmatrix}$$

and let Γ be the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ which is generated by t, g_1 and g_2. Set

$$g_3 := g_1 g_2 = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix},$$

let $\mathcal{F}_\infty := \left(\frac{2}{11}, \frac{19}{11}\right) + i\mathbb{R}^+$ and (see Fig. 6.10)

$$\mathcal{F} := \mathcal{F}_\infty \cap \operatorname{ext} I(g_1) \cap \operatorname{ext} I(g_1^{-1}) \cap \operatorname{ext} I(g_2) \cap \operatorname{ext} I(g_1 g_2) \cap \operatorname{ext} I\big((g_1 g_2)^{-1}\big).$$

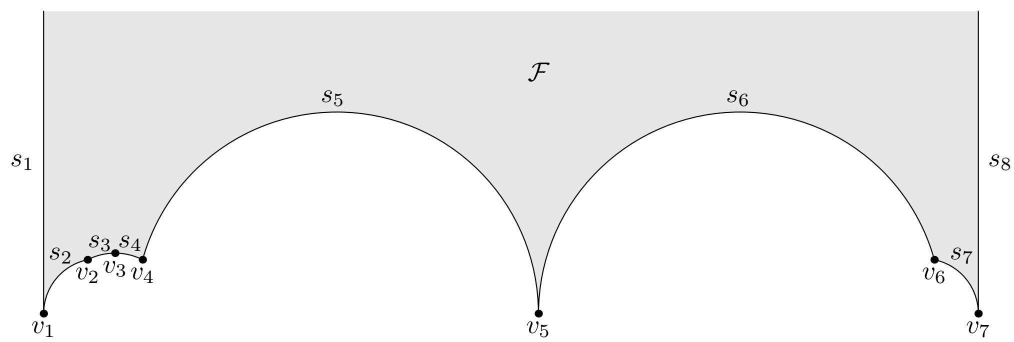


Figure 6.10: The fundamental domain $\mathcal{F}$

We use the following points

$$\begin{array}{lll} v_0 := \infty & v_3 := \frac{3}{10} + \frac{i}{10} & v_6 := \frac{91}{55} + i\frac{\sqrt{24}}{55} \\ v_1 := \frac{2}{11} & v_4 := \frac{19}{55} + i\frac{\sqrt{24}}{55} & v_7 := \frac{19}{11} \\ v_2 := \frac{14}{55} + i\frac{\sqrt{24}}{55} & v_5 := 1 & \end{array}$$

and the geodesic segments

$$\begin{array}{lll} s_1 := [v_0, v_1] & s_4 := [v_3, v_4] & s_7 := [v_6, v_7] \\ s_2 := [v_1, v_2] & s_5 := [v_4, v_5] & s_8 := [v_7, v_0]. \\ s_3 := [v_2, v_3] & s_6 := [v_5, v_6] & \end{array}$$

Lemma 6.3.1. *The sides of $\mathcal{F}$ are $s_1, s_2, s_3 \cup s_4, s_5, s_6, s_7$ and s_8. Further we have the side-pairing $ts_1 = s_8$, $g_1 s_2 = s_7$, $g_2 s_3 = s_4$ and $g_3 s_5 = s_6$, where $tv_0 = v_0$, $tv_1 = v_7$, $g_1 v_1 = v_7$, $g_1 v_2 = v_6$, $g_2 v_2 = v_4$, $g_2 v_3 = v_3$, $g_3 v_4 = v_6$ and $g_3 v_5 = v_5$.*

Proof. We have

$$\begin{aligned}
I(g_1) &= \left\{ z \in H \;\middle|\; \left|z - \tfrac{3}{11}\right| = \tfrac{1}{11} \right\}, \\
I(g_1^{-1}) &= \left\{ z \in H \;\middle|\; \left|z - \tfrac{18}{11}\right| = \tfrac{1}{11} \right\}, \\
I(g_2) &= \left\{ z \in H \;\middle|\; \left|z - \tfrac{3}{10}\right| = \tfrac{1}{10} \right\} = I(g_2^{-1}), \\
I(g_3) &= \left\{ z \in H \;\middle|\; \left|z - \tfrac{2}{3}\right| = \tfrac{1}{3} \right\}, \\
\text{and} \qquad I(g_3^{-1}) &= \left\{ z \in H \;\middle|\; \left|z - \tfrac{4}{3}\right| = \tfrac{1}{3} \right\}.
\end{aligned}$$

These isometric spheres are the following complete geodesic segments:

$$\begin{aligned}
I(g_1) &= \left[\tfrac{2}{11}, \tfrac{4}{11}\right], \quad I(g_2) = \left[\tfrac{1}{5}, \tfrac{2}{5}\right], \quad & I(g_3) &= \left[\tfrac{1}{3}, 1\right] \\
I(g_1^{-1}) &= \left[\tfrac{17}{11}, \tfrac{19}{11}\right], & I(g_3^{-1}) &= \left[1, \tfrac{5}{3}\right].
\end{aligned}$$

This already shows that s_1 and s_8 are the vertical sides of $\mathcal{F}$. Now we determine the non-vertical sides of $\mathcal{F}$ by investigating with parts of the isometric spheres $I(g_1)$, $I(g_1^{-1})$, $I(g_2)$, $I(g_3)$ and $I(g_3^{-1})$ are contained in the interior of some other isometric sphere. The remaining parts then build up the non-vertical sides of $\mathcal{F}$. For that we need to find all intersection points of pairs of these isometric spheres.

One easily calculates that v_2 is the intersection point of $I(g_1)$ and $I(g_2)$. Since $2/11 < 1/5$, the segment $[1/5, v_2)$ of $I(g_2)$ is contained in $\operatorname{int} I(g_1)$. Likewise, since $4/11 < 4/10$, the segment $(v_2, 4/11]$ of $I(g_1)$ is contained in $\operatorname{int} I(g_2)$. Therefore, these two segments cannot contribute to the boundary of $\mathcal{F}$. Analogously, one sees that v_4 is the intersection point of $I(g_2)$ and $I(g_3)$, and therefore $[1/3, v_4) \subseteq \operatorname{int} I(g_2)$ and $(v_4, 4/10] \subseteq \operatorname{int} I(g_3)$. Likewise, v_6 is the intersection point of $I(g_1^{-1})$ and $I(g_3^{-1})$, hence we have $(v_3, 5/3] \subseteq \operatorname{int} I(g_1^{-1})$ and $[17/11, v_6) \subseteq \operatorname{int} I(g_3^{-1})$. Moreover, v_5 is the intersection point of $I(g_3)$ and $I(g_3^{-1})$. The intersection point of $I(g_1)$ and $I(g_3)$ is contained in $\operatorname{int} I(g_2)$ because $\operatorname{Re}(v_2) = 14/55 < 1/3$. Therefore it is not relevant. All other pairs of isometric spheres do not intersect. This implies the claim about the sides of $\mathcal{F}$.

Now one checks by direct calculation the claimed side-pairings. □

Proposition 6.3.2. *The set $\mathcal{F}$ is a fundamental domain for Γ in H.*

Proof. We apply Poincaré's Theorem in the form of [Mas71] to show that $\mathcal{F}$ is a fundamental domain for the group generated by t, g_1, g_2 and g_3. This group is exactly Γ. We use the notions and notations from [Mas71]. In particular, we refer to the conditions (a)-(g) and (f') in [Mas71]. The sides of $\mathcal{F}$ in sense of [Mas71] are the geodesic segments $s_1, \ldots, s_8$. Obviously, $\mathcal{F}$ is a domain and a polygon in the terminology of [Mas71]. Note that v_3 is also called a vertex. The side-pairing of $\mathcal{F}$ is given by Lemma 6.3.1. The conditions (a)-(c) are obviously satisfied. If s is a side of $\mathcal{F}$ and g is the element with which s is mapped to another side, then $s \in I(g)$. Lemma 2.2.12 implies that (d) is fulfilled. The condition (e) is trivially satisfied. Concerning the condition (f') we have three chains of infinite vertices. One is (v_0). An infinite cycle transformation of this chain is t, which is parabolic. Another chain is (v_1, v_7). An infinite cycle transformation is

$$g_1^{-1}t = \begin{pmatrix} -3 & \frac{4}{11} \\ -11 & 1 \end{pmatrix},$$

which is a parabolic element. The third one is (v_5) with an infinite cycle transformation g_3, which is parabolic. Hence (f') is satisfied. Finally we have to show that (g) holds. For the cycle (v_3) this is clearly true since $\alpha(v_3) = \pi$. Consider the cycle (v_2, v_6, v_4) with the cycle transformation $g_2 g_3^{-1} g_1 = \mathrm{id}$. We claim that $\alpha(v_2) + \alpha(v_6) + \alpha(v_5) + \alpha(v_4) = 2\pi$. Note that $\alpha(v_5) = 0$. Let $U = B_\varepsilon(v_2)$ be a Euclidean ball centered at v_2 such that of all sides of $\mathcal{F}$, the set U intersects only s_2 and s_3, the set $g_1 U$, which is a neighborhood of v_6 by Lemma 6.3.1, intersects only the sides s_7 and s_6 and $g_2 U$, which is a neighborhood of v_4, intersects only s_4 and s_5. Suppose further, that $g_1 U$ does not intersect the geodesic segment $[v_5, v_7]$ and that $g_2 U$ does not intersect the geodesic segment $[v_1, v_5]$. Moreover, the sets U, $g_1 U$ and $g_2 U$ should be pairwise disjoint. Let $U_1 := U \cap \overline{\mathcal{F}}$, $U_2 := g_1 U \cap \overline{\mathcal{F}}$ and $U_3 := g_2 U \cap \overline{\mathcal{F}}$. We will show that the union $U_1 \cup g_1^{-1} U_2 \cup g_2 U_3$ is essentially disjoint and equals U. Since $\alpha(v_2)$ is the angle inside U_1 at v_2, and similar for $\alpha(v_6)$ and $\alpha(v_4)$, this then shows that the angle sum is 2π.

Lemma 6.3.1 shows that $g_1^{-1} s_7 = s_2$ and $g_1^{-1} v_6 = v_2$. Then the side s_6 is mapped by g_1^{-1} to the geodesic segment $[v_2, g_1^{-1} 1] = [v_2, 2/7]$. Thus the hyperbolic triangle P_1 with vertices $v_2, v_1, 2/7$ coincides with $\mathcal{F}$ precisely at the side s_1. Note that P_1 is the image under g_1^{-1} of the hyperbolic triangle with vertices v_5, v_6, v_7 which contains U_2. Therefore

$$g_1^{-1} U_2 = g_1^{-1}\big(g_1 U \cap \overline{\mathcal{F}}\big) \cap P_1 = U \cap P_1.$$

Now g_2 maps s_5 to the geodesic segment $[v_2, 2/7]$ and s_4 to s_3. Let P_2 be the hyperbolic triangle with vertices $v_2, v_3, 2/7$. Then

$$g_2 U_3 = g_2\big(g_2 U \cap \overline{\mathcal{F}}\big) \cap P_2 = U \cap P_2.$$

Now P_2 coincides with $\overline{\mathcal{F}}$ exactly at the side s_3 and with P_1 exactly at the side $[v_2, 2/7]$. Thus, the union $U_1 \cup g_1^{-1} U_2 \cup g_2 U_3$ is essentially disjoint and

$$(U \cap \overline{\mathcal{F}}) \cup (U \cap P_1) \cup (U \cap P_2) = U.$$

This shows that the angle sum is indeed 2π. Hence (g) holds. Then Poincaré's Theorem states that F is a fundamental domain for the group generated by t, g_1, g_2 and g_3. $\square$

Proposition 6.3.3. *Γ does not satisfy (A2).*

Proof. Prop. 6.3.2 states that $\mathcal{F}$ is a fundamental domain for Γ in H. Its shape shows that it is an isometric fundamental domain. Therefore, the isometric sphere $I(g_1)$ is relevant and s_1 is its relevant part. The summit of $I(g_1)$ is $s := \frac{3+i}{11}$. One easily calculates that $s \in \operatorname{int} I(g_2)$. Therefore, Γ does not satisfy (A2). $\square$

Remark 6.3.4. In [Vul99], Vulakh states that each geometrically finite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ for which ∞ is a cuspidal point satisfies (A2). The previous example shows that this statement is not right. This property is crucial for the results in [Vul99]. Thus, Vulakh's constructions do not apply to such a huge class of groups as he claims.

6.4. Cells in H

Let Γ be a geometrically finite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of which ∞ is a cuspidal point and which satisfies (A2). Suppose that the set of relevant isometric spheres is non-empty. Let $\mathbb{A}$ be a basal family of precells in H. To each basal precell in H we assign a cell in H, which is an essentially disjoint union of certain Γ-translates of certain basal precells. More precisely, using Prop. 6.2.24 we define so-called cycles in $\mathbb{A} \times \Gamma$. These are certain finite sequences of pairs $(\mathcal{A}, h) \in \mathbb{A} \times \Gamma \setminus \Gamma_\infty$ such that each cycle is determined up to cyclic permutation by any pair which belongs to it. Moreover, if $(\mathcal{A}, h_\mathcal{A})$ is an element of some cycle, then $h_\mathcal{A}$ is an element in $\Gamma \setminus \Gamma_\infty$ assigned to $\mathcal{A}$ by Prop. 6.2.24 (or $h_\mathcal{A} = \mathrm{id}$ if $\mathcal{A}$ is a strip precell). Conversely, if $h_\mathcal{A}$ is an element assigned to $\mathcal{A}$ by Prop. 6.2.24, then $(\mathcal{A}, h_\mathcal{A})$ determines a cycle in $\mathbb{A} \times \Gamma \setminus \Gamma_\infty$.

One of the crucial properties of each cell in H is that it is a convex polyhedron with non-empty interior of which each side is a complete geodesic segment. This fact is mainly due to the condition (A2) of Γ. The other two important properties of cells in H are that each non-vertical side of a cell is a Γ-translate of some vertical side of some cell in H and that the family of Γ-translates of all cells in H is a tesselation of H. To prove these facts, we devote a substantial part of this section to the study of boundaries of cells.

6.4.1. Cycles in $\mathbb{A} \times \Gamma$

Remark and Definition 6.4.1. Let $\mathcal{A} \in \mathbb{A}$ be a non-cuspidal precell in H. The definition of precells shows that $\mathcal{A}$ is attached to a unique (inner) vertex v of $\mathcal{K}$, and $\mathcal{A}$ is the unique precell attached to v. Therefore we set $\mathcal{A}(v) := \mathcal{A}$. Further, $\mathcal{A}$ has two non-vertical sides b_1 and b_2. Let $\{k_1(\mathcal{A}), k_2(\mathcal{A})\}$ be the two elements in $\Gamma \setminus \Gamma_\infty$ given by Prop. 6.2.24 such that $b_j \in I(k_j(\mathcal{A}))$ and $k_j(\mathcal{A})b_j$ is a non-vertical side of some basal precell. Necessarily, the isometric spheres $I(k_1(\mathcal{A}))$ and $I(k_2(\mathcal{A}))$ are different, therefore $k_1(\mathcal{A}) \neq k_2(\mathcal{A})$. The set $\{k_1(\mathcal{A}), k_2(\mathcal{A})\}$ is uniquely determined by Prop. 6.2.24, the assignment $\mathcal{A} \mapsto k_1(\mathcal{A})$ clearly depends on the enumeration of the non-vertical sides of $\mathcal{A}$. By Remark 6.1.30, $w := k_j(\mathcal{A})v$ is an inner vertex. Let $\mathcal{A}(w)$ be the (unique non-cuspidal) basal precell attached to w. Since one non-vertical side of $\mathcal{A}(w)$ is $k_j(\mathcal{A})b_j$, which is contained in the relevant isometric sphere $I(k_j(\mathcal{A})^{-1})$, and $k_j(\mathcal{A})^{-1}k_j(\mathcal{A})b_j = b_j$ is a non-vertical side of some basal precell, namely of $\mathcal{A}$, one of the elements in $\Gamma\diagdown\Gamma_\infty$ assigned to $\mathcal{A}(w)$ by Prop. 6.2.24 is $k_j(\mathcal{A})^{-1}$.

Construction 6.4.2. Let $\mathcal{A} \in \mathbb{A}$ be a non-cuspidal precell and suppose that $\mathcal{A} = \mathcal{A}(v)$ is attached to the vertex v of $\mathcal{K}$. We assign to $\mathcal{A}$ two sequences (h_j) of elements in $\Gamma\diagdown\Gamma_\infty$ using the following algorithm:

(step 1) Let $v_1 := v$ and let h_1 be either $k_1(\mathcal{A})$ or $k_2(\mathcal{A})$. Set $g_1 := \mathrm{id}$, $g_2 := h_1$ and carry out (step 2).

(step j) Set $v_j := g_j(v)$ and $\mathcal{A}_j := \mathcal{A}(v_j)$. Let h_j be the element in $\Gamma\diagdown\Gamma_\infty$ such that $\{h_j, h_{j-1}^{-1}\} = \{k_1(\mathcal{A}_j), k_2(\mathcal{A}_j)\}$. Set $g_{j+1} := h_j g_j$. If $g_{j+1} = \mathrm{id}$, then the algorithm stops. If $g_{j+1} \neq \mathrm{id}$, then carry out (step $j+1$).

Example 6.4.3. Recall the Hecke triangle group G_n and its basal family $\mathbb{A} = \{\mathcal{A}\}$ of precells in H from Example 6.2.11. Let

$$U_n = T_n S = \begin{pmatrix} \lambda_n & -1 \\ 1 & 0 \end{pmatrix}.$$

The two sequences assigned to $\mathcal{A}$ are $(U_n)_{j=1}^n$ and $\left(U_n^{-1}\right)_{j=1}^n$.

Proposition 6.4.4. *Let $\mathcal{A} = \mathcal{A}(v)$ be a non-cuspidal basal precell.*

(i) *The sequences from Constr. 6.4.2 are finite. In other words, the algorithm for the construction of the sequences always terminates.*

(ii) *Both sequences have same length, say $k \in \mathbb{N}$.*

(iii) *Let $(a_j)_{j=1,\ldots,k}$ and $(b_j)_{j=1,\ldots,k}$ be the two sequences assigned to $\mathcal{A}$. Then they are inverse to each other in the following sense: For each $j = 1, \ldots, k$ we have $a_j = b_{k-j+1}^{-1}$.*

(iv) *For $j = 1, \ldots, k$ set $c_{j+1} := a_j a_{j-1} \cdots a_2 a_1$, $d_{j+1} := b_j b_{j-1} \cdots b_2 b_1$ and $c_1 := \mathrm{id} =: d_1$. Then*

$$\mathcal{B} := \bigcup_{j=1}^{k} c_j^{-1}\mathcal{A}(c_j v) = \bigcup_{j=1}^{k} d_j^{-1}\mathcal{A}(d_j v).$$

Further, both unions are essentially disjoint, and $\mathcal{B}$ is the polyhedron with the (pairwise distinct) vertices $c_1^{-1}\infty$, $c_2^{-1}\infty$, ..., $c_k^{-1}\infty$ resp. $d_1^{-1}\infty$, $d_2^{-1}\infty$, ..., $d_k^{-1}\infty$.

Proof. Suppose that $\mathcal{A} = \mathcal{A}(v)$. Let $(h_j)_{j\in J}$ be one of the sequences assigned to $\mathcal{A}$ by Constr. 6.4.2. As in Constr. 6.4.2 we set $g_1 := \mathrm{id}$, $g_{j+1} := h_j h_{j-1} \cdots h_2 h_1$, $v_j := g_j(v)$ and $\mathcal{A}_j := \mathcal{A}(v_j)$ for $j \in J$. Let s_j denote the summit of $I(h_j)$ for

$j \in J$. Then $\mathcal{A}_j \cap I(h_j) = [v_j, s_j]$. Let $j \in J$ such that also $j+1 \in J$. Then the non-vertical sides of $\mathcal{A}_{j+1}$ are

$$[v_{j+1}, h_j s_j] = \mathcal{A}_{j+1} \cap h_j I(h_j) = \mathcal{A}_{j+1} \cap I\big(h_j^{-1}\big)$$

and $[v_{j+1}, s_{j+1}]$. Hence $\mathcal{A}_{j+1}$ is the hyperbolic quadrilateral with vertices $h_j s_j$, v_{j+1}, s_{j+1}, ∞. Since $g_{j+1}^{-1} h_j s_j = g_j^{-1} s_j$ and $g_{j+1}^{-1} v_{j+1} = v$, the set $g_{j+1}^{-1}\mathcal{A}_{j+1}$ is the hyperbolic quadrilateral with vertices $g_j^{-1}s_j$, v, $g_{j+1}^{-1}s_{j+1}$, $g_{j+1}^{-1}\infty$. Thus, $g_j^{-1}\mathcal{A}_j$ and $g_{j+1}^{-1}\mathcal{A}_{j+1}$ have at least the side $[g_j^{-1}s_j, v]$ in common. Since $\mathcal{A}_j$ and $\mathcal{A}_{j+1}$ are both basal precells and $h_j \neq \mathrm{id}$, the sets $g_j^{-1}\mathcal{A}_j$ and $g_{j+1}^{-1}\mathcal{A}_{j+1} = g_j^{-1}h_j^{-1}\mathcal{A}_{j+1}$ intersect at most at a common side (see Cor. 6.2.28). Hence

$$g_j^{-1}\mathcal{A}_j \cap g_{j+1}^{-1}\mathcal{A}_{j+1} = [g_j^{-1}s_j, v].$$

Recall from Lemma 6.2.8 that $\mathrm{pr}_\infty(s_j) = h_j^{-1}\infty$. Hence s_j is contained in the (complete) geodesic segment $[h_j^{-1}\infty, \infty]$. Therefore, the sides $[g_j^{-1}\infty, g_j^{-1}s_j] = g_j^{-1}[\infty, s_j]$ of $g_j^{-1}\mathcal{A}_j$ and $[g_j^{-1}s_j, g_{j+1}^{-1}\infty] = g_j^{-1}[s_j, h_j^{-1}\infty]$ of $g_{j+1}^{-1}\mathcal{A}_{j+1}$ add up to the complete geodesic segment $[g_j^{-1}\infty, g_{j+1}^{-1}\infty]$. Further, since $\mathcal{A}_1$ and $\mathcal{A}_{j+1}$ are basal and $g_{j+1} \neq \mathrm{id}$, the sets $\mathcal{A}_1$ and $g_{j+1}^{-1}\mathcal{A}_{j+1}$ have at most one side in common.

For simplicity of exposition suppose that $\operatorname{Re} s_1 < \operatorname{Re} v$, and let s_1, v, t, ∞ denote the vertices of $\mathcal{A} = \mathcal{A}_1$. By the previous arguments, we find that $\mathcal{A}_1 \cup g_2^{-1}\mathcal{A}_2$ is the hyperbolic pentagon with vertices $\infty, g_2^{-1}\infty, g_2^{-1}s_2, v, t$ (counter clockwise). Using that each $\mathcal{A}_j$ is connected, we successively see that for each $k \in J$ the union $T_k := \bigcup_{j=1}^k g_j^{-1}\mathcal{A}_j$ is essentially disjoint. Further, T_k is either the polyhedron with (pairwise distinct) vertices $g_2^{-1}\infty < g_3^{-1}\infty < \ldots < g_k^{-1}\infty$, $g_k^{-1}s_k$, v, t and ∞ (counter clockwise), or $k \geq 3$ and $g_k^{-1}\mathcal{A}_k$ intersects $\mathcal{A}_1$ in more than the point v.

We will show that for k large enough, the set T_k is of the second kind. By Cor. 6.1.37, there is some $c > 0$ such that for each $j \in J$, the angle α_j inside $\mathcal{A}_j$ at v_j enclosed by the two non-vertical sides of $\mathcal{A}_j$ is bounded below by c. Thus, for the angle at v we get

$$2\pi \geq \sum_{j\in J} \alpha_j \geq c|J|.$$

Therefore, J is finite.

Suppose that T_k is of the first kind. We will show that for some $l > k$, the set T_l is of the second kind. To that end we first show that $g_{k+1} \neq \mathrm{id}$. Assume for contradiction that $h_k g_k = g_{k+1} = \mathrm{id}$, hence $h_k = g_k^{-1}$. Then

$$h_k[v_k, s_k] = g_k^{-1}[v_k, s_k] = [v, g_k^{-1}s_k]$$

is not a non-vertical side of some basal cell, but h_k was chosen to be the unique generator of the isometric sphere $I(h_k)$ such that $h_k(\mathcal{A}_k \cap I(h_k)) = h_k[v_k, s_k]$ is the non-vertical side of some basal precell. Thus, $g_{k+1} \neq \mathrm{id}$ and therefore $k+1 \in J$. Since J is finite, for some $l \in \mathbb{N}$, the set T_l must be of the second kind.

Suppose now that T_k is of the second kind. Then

$$g_k^{-1}\mathcal{A}_k \cap \mathcal{A}_1 = g_k^{-1}[v_k, s_k] \cap [v,t]$$

and $g_k^{-1}[v_k, s_k] \cap [v,t]$ is a geodesic segment of positive length. By Cor. 6.2.28 $g_k^{-1}[v_k, s_k] = [v,t]$ and therefore $I(g_k^{-1}) = I(h_k)$. From the choice of h_k now follows that $g_k^{-1} = h_k$. Thus, $g_{k+1} = h_k g_k = \mathrm{id}$. This shows that $J = \{1, \ldots, k\}$. Moreover, the set T_k is the polyhedron with vertices $g_2^{-1}\infty < g_3^{-1}\infty < \ldots < g_k^{-1}\infty$ and ∞ (counter clockwise).

Further, this argument shows that $\{h_1, h_k^{-1}\} = \{k_1(\mathcal{A}), k_2(\mathcal{A})\}$. Let (a_j) and (b_j) be the two sequences assigned to $\mathcal{A}$ by Constr. 6.4.2 and suppose that $a_1 = h_1$ and $b_1 = h_k^{-1}$. Then the sequence (a_j) has length k and $a_j = h_j$ for $j = 1, \ldots, k$. Note that b_1 maps $[v,t]$ to the non-vertical side $h_k^{-1}[v,t] = [v_k, s_k]$ of $\mathcal{A}_k$. Now b_2 is determined by b_1 via $\{b_2, b_1^{-1}\} = \{b_2, h_k\} = \{k_1(\mathcal{A}_k), k_2(\mathcal{A}_k)\}$, thus $b_2 = h_{k-1}^{-1}$. Recursively, we see that $b_j = h_{k-j+1}^{-1} = a_{k-j+1}^{-1}$ for $j = 1, \ldots, k$, and

$$b_k b_{k-1} \cdots b_1 = h_1^{-1} h_2^{-1} \cdots h_k^{-1} = g_{k+1}^{-1} = \mathrm{id}\,.$$

For each $j = 1, \ldots, k$ we have

$$\begin{aligned} d_{j+1} := b_j b_{j-1} \cdots b_2 b_1 &= h_{k-j+1}^{-1} h_{k-j+2}^{-1} \cdots h_{k-1}^{-1} h_k^{-1} \\ &= h_{k-j} \cdots h_1 h_1^{-1} \cdots h_{k-j}^{-1} h_{k-j+1}^{-1} \cdots h_k^{-1} = g_{k-j+1} g_{k+1}^{-1} \\ &= g_{k-j+1}. \end{aligned}$$

Since $d_{j+1} = g_{k-j+1} \neq \mathrm{id}$ for $j = 1, \ldots, k-1$, but $d_{k+1} = \mathrm{id}$, also the sequence (b_j) has length k. Let $d_1 := \mathrm{id}$. Then

$$\bigcup_{j=1}^{k} d_j^{-1}\mathcal{A}(d_j v) = \bigcup_{j=1}^{k} g_{k-j+1}^{-1}\mathcal{A}(g_{k-j+1}v) = \bigcup_{j=1}^{k} g_j^{-1}\mathcal{A}(g_j v).$$

□

Definition 6.4.5. Let $\mathcal{A} \in \mathbb{A}$ be a non-cuspidal precell and suppose that $\mathcal{A}$ is attached to the vertex v of $\mathcal{K}$. Let $h_{\mathcal{A}}$ be one of the elements in $\Gamma \setminus \Gamma_\infty$ assigned to $\mathcal{A}$ by Prop. 6.2.24. Let $(h_j)_{j=1,\ldots,k}$ be the sequence assigned to $\mathcal{A}$ by Constr. 6.4.2 with $h_1 = h_{\mathcal{A}}$. For $j = 1, \ldots, k$ set $g_1 := \mathrm{id}$ and $g_{j+1} := h_j g_j$. Then the (finite) sequence $\big((\mathcal{A}(g_j v), h_j)\big)_{j=1,\ldots,k}$ is called the *cycle in* $\mathbb{A}\times\Gamma$ *determined by* $(\mathcal{A}, h_{\mathcal{A}})$.

Let $\mathcal{A} \in \mathbb{A}$ be a cuspidal precell. Suppose that b is the non-vertical side of $\mathcal{A}$ and let $h_{\mathcal{A}}$ be the element in $\Gamma\setminus\Gamma_\infty$ assigned to $\mathcal{A}$ by Prop. 6.2.24. Let $\mathcal{A}'$ be the (cuspidal) basal precell with non-vertical side $h_{\mathcal{A}} b$. Then the (finite) sequence $\big((\mathcal{A}, h_{\mathcal{A}}), (\mathcal{A}', h_{\mathcal{A}}^{-1})\big)$ is called the *cycle in* $\mathbb{A} \times \Gamma$ *determined by* $(\mathcal{A}, h_{\mathcal{A}})$.

Let $\mathcal{A} \in \mathbb{A}$ be a strip precell. Set $h_{\mathcal{A}} := \mathrm{id}$. Then $\big((\mathcal{A}, h_{\mathcal{A}})\big)$ is called the *cycle in* $\mathbb{A} \times \Gamma$ *determined by* $(\mathcal{A}, h_{\mathcal{A}})$.

Example 6.4.6.

(i) Recall Example 6.4.3. The cycle in $\mathbb{A} \times G_n$ determined by $(\mathcal{A}, U_n)$ is $\big((\mathcal{A}, U_n)\big)_{j=1}^{n}$.

(ii) Recall the group $P\Gamma_0(5)$ and the basal family $\mathbb{A} = \{\mathcal{A}(v_0), \ldots, \mathcal{A}(v_4)\}$ from Example 6.2.12. The element in $P\Gamma_0(5)\diagdown P\Gamma_0(5)_\infty$ assigned to $\mathcal{A}(v_0)$ is $h := \left(\begin{smallmatrix} 4 & -1 \\ 5 & -1 \end{smallmatrix}\right)$. The cycle in $\mathbb{A} \times P\Gamma_0(5)$ determined by $(\mathcal{A}(v_0), h)$ is $\big((\mathcal{A}(v_0), h), (\mathcal{A}(v_4), h^{-1})\big)$. Further let $h_1 := h$, $h_2 := \left(\begin{smallmatrix} 3 & -2 \\ 5 & -3 \end{smallmatrix}\right)$ and $h_3 := \left(\begin{smallmatrix} 2 & -1 \\ 5 & -2 \end{smallmatrix}\right)$. The cycle in $\mathbb{A} \times P\Gamma_0(5)$ determined by $(\mathcal{A}(v_1), h_1)$ is

$$\big((\mathcal{A}(v_1), h_1), (\mathcal{A}(v_3), h_2), (\mathcal{A}(v_2), h_3)\big).$$

(iii) Recall the group Γ and the basal family $\mathbb{A} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ from Example 6.2.13. The cycle in $\mathbb{A} \times \Gamma$ determined by $(\mathcal{A}_2, S)$ is $\big((\mathcal{A}_2, S), (\mathcal{A}_3, S)\big)$.

Proposition 6.4.7. *Let $\mathcal{A} \in \mathbb{A}$ be a non-cuspidal precell in H and suppose that $h_\mathcal{A}$ is one of the elements in $\Gamma\diagdown\Gamma_\infty$ assigned to $\mathcal{A}$ by Prop. 6.2.24. Let $\big((\mathcal{A}_j, h_j)\big)_{j=1,\ldots,k}$ be the cycle in $\mathbb{A}\times\Gamma$ determined by $(\mathcal{A}, h_\mathcal{A})$. Let $j \in \{1, \ldots, k\}$ and define the sequence $\big((\mathcal{A}'_l, a_l)\big)_{l=1,\ldots,k}$ by*

$$a_l := \begin{cases} h_{l+j-1} & \text{for } l = 1, \ldots, k-j+1, \\ h_{l+j-1-k} & \text{for } l = k-j+2, \ldots, k, \end{cases}$$

and

$$\mathcal{A}'_l := \begin{cases} \mathcal{A}_{l+j-1} & \text{for } l = 1, \ldots, k-j+1, \\ \mathcal{A}_{l+j-1-k} & \text{for } l = k-j+2, \ldots, k. \end{cases}$$

Then $a_1 = h_j$ is one of the elements in $\Gamma\diagdown\Gamma_\infty$ assigned to $\mathcal{A}_j$ by Prop. 6.2.24 and $\big((\mathcal{A}'_l, a_l)\big)_{l=1,\ldots,k}$ is the cycle in $\mathbb{A}\times\Gamma$ determined by $(\mathcal{A}_j, h_j)$.

Proof. We first show that $\{h_1, h_k^{-1}\} = \{k_1(\mathcal{A}), k_2(\mathcal{A})\}$. Suppose that $h_1 = k_1(\mathcal{A})$. Prop. 6.4.4(iii) states that $h_k = k_2(\mathcal{A})^{-1}$. This shows that $\{a_l, a_{l-1}^{-1}\} = \{k_1(\mathcal{A}'_l), k_2(\mathcal{A}'_l)\}$ for $l = 2, \ldots, k$. For $l = 1, \ldots, k$ set $c_1 := \mathrm{id}$ and $c_{l+1} := a_l c_l$. It remains to show that $c_l \neq \mathrm{id}$ for $l = 2, \ldots, k$ and $c_{k+1} = \mathrm{id}$. For $p = 1, \ldots, k$ set $g_1 := \mathrm{id}$ and $g_{p+1} := h_p g_p$. Then

$$c_l = \begin{cases} g_{l+j-1} g_j^{-1} & \text{for } l = 1, \ldots, k-j+1, \\ g_{l+j-1-k} g_j^{-1} & \text{for } l = k-j+2, \ldots, k+1. \end{cases}$$

Obviously, $c_{k+1} = g_j g_j^{-1} = \mathrm{id}$. Let $l \in \{2, \ldots, k-j+1\}$. Then $l+j-1 \neq j$ and by Prop. 6.4.4(iv) $g_{l+j-1} \neq g_j$. Hence $c_l \neq \mathrm{id}$. Analogously, we see that $c_l \neq \mathrm{id}$ for $l \in \{k-j+2, \ldots, k\}$. This completes the proof. $\square$

The proof of the next proposition follows immediately from the definition of $h_\mathcal{A}$.

Proposition 6.4.8. *Let $\mathcal{A} \in \mathbb{A}$ be a cuspidal precell in H and let $h_\mathcal{A}$ be the element in $\Gamma\diagdown\Gamma_\infty$ assigned to $\mathcal{A}$ by Prop. 6.2.24. Let $\big((\mathcal{A}, h_\mathcal{A}), (\mathcal{A}', h_\mathcal{A}^{-1})\big)$ be the cycle in $\mathbb{A} \times \Gamma$ determined by $(\mathcal{A}, h_\mathcal{A})$. Then $h_\mathcal{A}^{-1}$ is the element in $\Gamma\diagdown\Gamma_\infty$ assigned to $\mathcal{A}'$ by Prop. 6.2.24 and $\big((\mathcal{A}', h_\mathcal{A}^{-1}), (\mathcal{A}, h_\mathcal{A})\big)$ is the cycle in $\mathbb{A} \times \Gamma$ determined by $(\mathcal{A}', h_\mathcal{A}^{-1})$.*

6.4.2. Cells in H and their properties

Now we can define a cell in H for each basal precell in H.

Construction 6.4.9. Let $\mathcal{A}$ be a basal strip precell in H. Then we set

$$\mathcal{B}(\mathcal{A}) := \mathcal{A}.$$

Let $\mathcal{A}$ be a cuspidal basal precell in H. Suppose that g is the element in $\Gamma\smallsetminus\Gamma_\infty$ assigned to $\mathcal{A}$ by Prop. 6.2.24 and let $\big((\mathcal{A},g),(\mathcal{A}',g^{-1})\big)$ be the cycle in $\mathbb{A}\times\Gamma$ determined by $(\mathcal{A},g)$. Define

$$\mathcal{B}(\mathcal{A}) := \mathcal{A}\cup g^{-1}\mathcal{A}'.$$

The set $\mathcal{B}(\mathcal{A})$ is well-defined because g is uniquely determined.

Let $\mathcal{A}$ be a non-cuspidal basal precell in H and fix an element $h_\mathcal{A}$ in $\Gamma\smallsetminus\Gamma_\infty$ assigned to $\mathcal{A}$ by Prop. 6.2.24. Let $\big((\mathcal{A}_j,h_j)\big)_{j=1,\ldots,k}$ be the cycle in $\mathbb{A}\times\Gamma$ determined by $(\mathcal{A},h_\mathcal{A})$. For $j=1,\ldots,k$ set $g_1 := \mathrm{id}$ and $g_{j+1} := h_j g_j$. Set

$$\mathcal{B}(\mathcal{A}) := \bigcup_{j=1}^{k} g_j^{-1}\mathcal{A}_j.$$

By Prop. 6.4.4, the set $\mathcal{B}(\mathcal{A})$ does not depend on the choice of $h_\mathcal{A}$. The family $\mathbb{B} := \{\mathcal{B}(\mathcal{A}) \mid \mathcal{A}\in\mathbb{A}\}$ is called the *family of cells in H assigned to $\mathbb{A}$*. Each element of $\mathbb{B}$ is called a *cell in H*.

Note that the family $\mathbb{B}$ of cells in H depends on the choice of $\mathbb{A}$. If we need to distinguish cells in H assigned to the basal family $\mathbb{A}$ of precells in H from those assigned to the basal family $\mathbb{A}'$ of precells in H, we will call the first ones $\mathbb{A}$-*cells* and the latter ones $\mathbb{A}'$-*cells*.

Example 6.4.10. Recall the Example 6.4.6. For the Hecke triangle group G_5, Fig. 6.11 shows the cell assigned to the family $\mathbb{A}=\{\mathcal{A}\}$ from Example 6.2.11. For the group $\mathrm{P}\Gamma_0(5)$, the family of cells in H assigned to $\mathbb{A}$ is indicated in

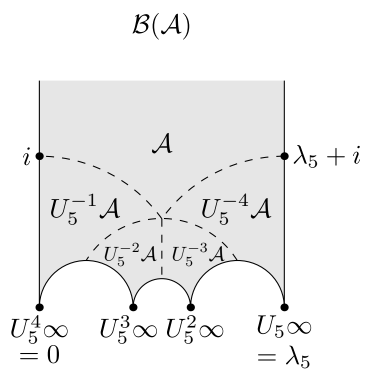


Figure 6.11: The cell $\mathcal{B}(\mathcal{A})$ for G_5.

Fig. 6.12. Fig. 6.13 shows the family of cells in H assigned to the basal family $\mathbb{A}$ of precells of Γ.

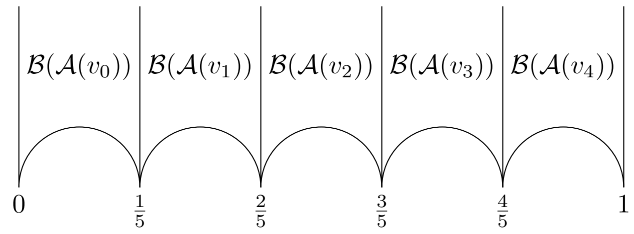


Figure 6.12: The family of cells in H assigned to $\mathbb{A}$ for $\mathrm{P}\Gamma_0(5)$.

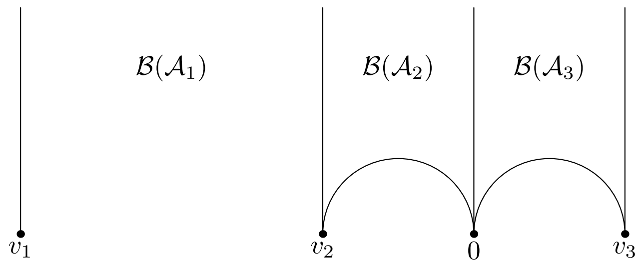


Figure 6.13: The family of cells in H assigned to $\mathbb{A}$ for Γ.

In the series of the following six propositions we investigate the structure of cells and their relations to each other. This will allow to show that the family of Γ-translates of cells in H is a tesselation of H, and it will be of interest for the labeling of the cross section.

Proposition 6.4.11. *Let $\mathcal{A}$ be a non-cuspidal basal precell in H. Suppose that $h_{\mathcal{A}}$ is an element in $\Gamma\smallsetminus\Gamma_\infty$ assigned to $\mathcal{A}$ by Prop. 6.2.24 and let $\big((\mathcal{A}_j, h_j)\big)_{j=1,\ldots,k}$ be the cycle in $\mathbb{A}\times\Gamma$ determined by $(\mathcal{A}, h_{\mathcal{A}})$. For $j = 1,\ldots,k$ set $g_1 := \mathrm{id}$ and $g_{j+1} := h_j g_j$. Then the following assertions hold true.*

(i) *The set $\mathcal{B}(\mathcal{A})$ is the convex polyhedron with vertices $g_1^{-1}\infty, g_2^{-1}\infty,\ldots,g_k^{-1}\infty$.*

(ii) *The boundary of $\mathcal{B}(\mathcal{A})$ consists precisely of the union of the images of the vertical sides of $\mathcal{A}_j$ under g_j^{-1}, $j = 1,\ldots,k$. More precisely, if s_j denotes the summit of $I(h_j)$ for $j = 1,\ldots,k$, then*

$$\partial\mathcal{B}(\mathcal{A}) = \bigcup_{j=1}^{k} g_j^{-1}[s_j,\infty] \cup \bigcup_{j=2}^{k+1} g_j^{-1}[h_{j-1}s_{j-1},\infty].$$

(iii) *For each $j = 1,\ldots,k$ we have $g_j\mathcal{B}(\mathcal{A}) = \mathcal{B}(\mathcal{A}_j)$. In particular, the side $[g_j^{-1}\infty, g_{j+1}^{-1}\infty]$ of $\mathcal{B}(\mathcal{A})$ is the image of the vertical side $[\infty, h_j^{-1}\infty]$ of $\mathcal{B}(\mathcal{A}_j)$ under g_j^{-1}.*

(iv) *Let $\widehat{\mathcal{A}}$ be a basal precell in H and $h\in\Gamma$ such that $h\widehat{\mathcal{A}}\cap\mathcal{B}(\mathcal{A})^\circ \neq \emptyset$. Then there exists a unique $j\in\{1,\ldots,k\}$ such that $h = g_j^{-1}$ and $\widehat{\mathcal{A}} = \mathcal{A}_j$. In particular, $\widehat{\mathcal{A}}$ is non-cuspidal and $h\mathcal{B}(\widehat{\mathcal{A}}) = \mathcal{B}(\mathcal{A})$.*

Proof. By Prop. 6.4.4, $\mathcal{B}(\mathcal{A})$ is the polyhedron with vertices $g_1^{-1}\infty, g_2^{-1}\infty, \ldots, g_k^{-1}\infty$. Since each of its sides is a complete geodesic segment, $\mathcal{B}(\mathcal{A})$ is convex. This shows (i). The statement (ii) follows from the proof of Prop. 6.4.4.

To prove (iii), fix $j \in \{1, \ldots, k\}$ and recall from Prop. 6.4.7 the cycle $\big((\mathcal{A}'_l, a_l)\big)_{l=1,\ldots,k}$ in $\mathbb{A} \times \Gamma$ determined by $(\mathcal{A}_j, h_j)$. For $l = 1, \ldots, k$ set $c_1 := \mathrm{id}$ and $c_{l+1} := a_l c_l$. Then

$$c_l = \begin{cases} g_{l+j-1}g_j^{-1} & \text{for } l = 1, \ldots, k-j+1 \\ g_{l+j-k-1}g_j^{-1} & \text{for } l = k-j+2, \ldots, k. \end{cases}$$

Hence

$$\begin{aligned} \mathcal{B}(\mathcal{A}_j) &= \bigcup_{l=1}^{k} c_l^{-1}\mathcal{A}'_l = \bigcup_{l=1}^{k} g_j(c_l g_j)^{-1}\mathcal{A}'_l \\ &= g_j \bigcup_{l=1}^{k-j+1} g_{l+j-1}^{-1}\mathcal{A}_{l+j-1} \cup g_j \bigcup_{l=k-j+2}^{k} g_{l+j-k-1}^{-1}\mathcal{A}_{l+j-k-1} \\ &= g_j \bigcup_{l=1}^{k} g_l^{-1}\mathcal{A}_l = g_j\mathcal{B}(\mathcal{A}). \end{aligned}$$

This immediately implies that the side $[g_j^{-1}\infty, g_{j+1}^{-1}\infty]$ of $\mathcal{B}(\mathcal{A})$ maps to the side $g_j[g_j^{-1}\infty, g_{j+1}^{-1}\infty] = [\infty, h_j^{-1}\infty]$ of $\mathcal{B}(\mathcal{A}_j)$, which is vertical.

To prove (iv), fix $z \in h\widehat{\mathcal{A}} \cap \mathcal{B}(\mathcal{A})^\circ$. Then there exists $l \in \{1, \ldots, k\}$ such that $z \in h\widehat{\mathcal{A}} \cap g_l^{-1}\mathcal{A}_l$. Let $b := h\widehat{\mathcal{A}} \cap g_l^{-1}\mathcal{A}_l$. By Prop. 6.2.26 there are three possibilities for b.

If $b = h\widehat{\mathcal{A}} = g_l^{-1}\mathcal{A}_l$, then $g_l h\widehat{\mathcal{A}} = \mathcal{A}_l$. Since $\widehat{\mathcal{A}}$ and $\mathcal{A}_l$ are both basal, it follows that $h = g_l^{-1}$ and $\widehat{\mathcal{A}} = \mathcal{A}_l$.

Suppose that v is the vertex of $\mathcal{K}$ to which $\mathcal{A}$ is attached. If b is a common side of $h\widehat{\mathcal{A}}$ and $g_l^{-1}\mathcal{A}_l$, then, since $z \in \mathcal{B}(\mathcal{A})^\circ$, $g_l b$ must be a non-vertical side of $\mathcal{A}_l$ (see (ii)). This implies that $v \in b$. In turn, there is a neighborhood U of v such that $U \subseteq \mathcal{B}(\mathcal{A})$ and $U \cap h(\widehat{\mathcal{A}})^\circ \neq \emptyset$. Hence, $h(\widehat{\mathcal{A}})^\circ \cap \mathcal{B}(\mathcal{A})^\circ \neq \emptyset$. Thus there exists $j \in \{1, \ldots, k\}$ such that $h(\widehat{\mathcal{A}})^\circ \cap g_j^{-1}\mathcal{A}_j \neq \emptyset$. Prop. 6.2.26 implies that $\widehat{\mathcal{A}} = \mathcal{A}_j$ and $h = g_j^{-1}$.

If b is a point, then $b = z$ must be the endpoint of some side of $g_l^{-1}\mathcal{A}_l$. From $z \in \mathcal{B}(\mathcal{A})^\circ$ it follows that $z = v$. Now the previous argument applies.

To show the uniqueness of $j \in \{1, \ldots, k\}$ with $\widehat{\mathcal{A}} = \mathcal{A}_j$ and $h = g_j^{-1}$, suppose that there is $p \in \{1, \ldots, k\}$ with $\widehat{\mathcal{A}} = \mathcal{A}_p$ and $h = g_p^{-1}$. Then $g_j = g_p$. By Prop. 6.4.4(iv) $j = p$. The remaining parts of (iv) follow from (iii). $\square$

Proposition 6.4.12. *Let $\mathcal{A}$ be a cuspidal basal precell in H which is attached to the vertex v of $\mathcal{K}$. Suppose that g is the element in $\Gamma \setminus \Gamma_\infty$ assigned to $\mathcal{A}$ by Prop. 6.2.24. Let $\big((\mathcal{A}, g), (\mathcal{A}', g^{-1})\big)$ be the cycle in $\mathbb{A} \times \Gamma$ determined by $(\mathcal{A}, g)$. Then we have the following properties.*

(i) *The set $\mathcal{B}(\mathcal{A})$ is the hyperbolic triangle with vertices v, $g^{-1}\infty$, ∞.*

(ii) *The boundary of $\mathcal{B}(\mathcal{A})$ is the union of the vertical sides of $\mathcal{A}$ with the images of the vertical sides of $\mathcal{A}'$ under g^{-1}.*

(iii) *The sets $g\mathcal{B}(\mathcal{A})$ and $\mathcal{B}(\mathcal{A}')$ coincide. In particular, the non-vertical side $[v, g^{-1}\infty]$ of $\mathcal{B}(\mathcal{A})$ is the image of the vertical side $[gv, \infty]$ of $\mathcal{A}'$ under g^{-1}.*

(iv) *Suppose that $\widehat{\mathcal{A}}$ is a basal precell in H and $h \in \Gamma$ such that $h\widehat{\mathcal{A}} \cap \mathcal{B}(\mathcal{A})^\circ \neq \emptyset$. Then either $h = \mathrm{id}$ and $\widehat{\mathcal{A}} = \mathcal{A}$ or $h = g^{-1}$ and $\widehat{\mathcal{A}} = \mathcal{A}'$. In particular, $\widehat{\mathcal{A}}$ is cuspidal and $h\mathcal{B}(\widehat{\mathcal{A}}) = \mathcal{B}(\mathcal{A})$.*

Proof. Let s be the summit of $I(g)$ and denote the non-vertical side of $\mathcal{A}$ by b. Then $b = [v, s]$ and $gb = [gv, gs]$ is a non-vertical side of $\mathcal{A}'$. By Prop. 6.2.24, $\mathcal{A}'$ is cuspidal. Hence $\mathcal{A}'$ is the hyperbolic triangle with vertices gs, gv, ∞. Since $g^{-1}\infty$ is the center of $I(g)$ and $\mathrm{pr}_\infty(s) = g^{-1}\infty$, the cell $\mathcal{B}(\mathcal{A}) = \mathcal{A} \cup g^{-1}\mathcal{A}'$ is the hyperbolic triangle with vertices v, $g^{-1}\infty$, ∞. Moreover,

$$\partial\mathcal{B}(\mathcal{A}) = g^{-1}[gv, \infty] \cup g^{-1}[gs, \infty] \cup [s, \infty] \cup [v, \infty]$$

as claimed. Now g^{-1} is the element assigned to $\mathcal{A}'$ by Prop. 6.2.24. Hence

$$\mathcal{B}(\mathcal{A}') = \mathcal{A}' \cup g\mathcal{A} = g\mathcal{B}(\mathcal{A}).$$

The remaining assertions are proved analogously to the corresponding statements of Prop. 6.4.11. □

Proposition 6.4.13. *Let $\mathcal{A}$ be a basal strip precell in H. Let $\widehat{\mathcal{A}}$ be a basal precell in H and $h \in \Gamma$ such that $h\widehat{\mathcal{A}} \cap \mathcal{B}(\mathcal{A})^\circ \neq \emptyset$. Then $h = \mathrm{id}$ and $\widehat{\mathcal{A}} = \mathcal{A}$.*

Proof. This follows from $\mathcal{B}(\mathcal{A}) = \mathcal{A}$ and Cor. 6.2.28. □

Corollary 6.4.14. *The map $\mathbb{A} \to \mathbb{B}$, $\mathcal{A} \mapsto \mathcal{B}(\mathcal{A})$ is a bijection.*

Proof. Let

$$\varphi\colon \begin{cases} \mathbb{A} & \to & \mathbb{B} \\ \mathcal{A} & \mapsto & \mathcal{B}(\mathcal{A}). \end{cases}$$

By definition of $\mathbb{B}$, the map φ is surjective. To show injectivity, let $\mathcal{A}_1, \mathcal{A}_2$ be basal precells in H such that $\mathcal{B}(\mathcal{A}_1) = \mathcal{B}(\mathcal{A}_2)$. From $\mathcal{A}_2 \subseteq \mathcal{B}(\mathcal{A}_2) = \mathcal{B}(\mathcal{A}_1)$ it follows that $\mathcal{A}_2 \cap \mathcal{B}(\mathcal{A}_1)^\circ = \emptyset$. Then Prop. 6.4.11(iv) resp. 6.4.12(iv) resp. 6.4.13 states that $\mathcal{A}_2 = \mathcal{A}_1$. □

Proposition 6.4.15. *Let $\mathcal{A}$ be a non-cuspidal basal precell in H. Suppose that $(h_j)_{j=1,\ldots,k}$ is a sequence in $\Gamma \smallsetminus \Gamma_\infty$ assigned to $\mathcal{A}$ by Constr. 6.4.2. For $j = 1, \ldots, k$ set $g_1 := \mathrm{id}$ and $g_{j+1} := h_j g_j$. Let $\mathcal{A}'$ be a basal precell in H and $g \in \Gamma$ such that $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}') \neq \emptyset$. Then we have the following properties.*

(i) *Either $\mathcal{B}(\mathcal{A}) = g\mathcal{B}(\mathcal{A}')$, or $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}')$ is a common side of $\mathcal{B}(\mathcal{A})$ and $g\mathcal{B}(\mathcal{A}')$.*

(ii) *If $\mathcal{B}(\mathcal{A}) = g\mathcal{B}(\mathcal{A}')$, then $g = g_j^{-1}$ for a unique $j \in \{1, \ldots, k\}$. In particular, $\mathcal{A}'$ is non-cuspidal.*

(iii) *If $\mathcal{B}(\mathcal{A}) \neq g\mathcal{B}(\mathcal{A}')$, then $\mathcal{A}'$ is cuspidal or non-cuspidal. If $\mathcal{A}'$ is cuspidal and $k \in \Gamma\smallsetminus\Gamma_\infty$ is the element assigned to $\mathcal{A}'$ by Prop. 6.2.24, then we have $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}') = g[k^{-1}\infty, \infty]$.*

Proof. Since $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}') \neq \emptyset$, there exist a basal precell $\widehat{\mathcal{A}}$ and an element $h \in \Gamma$ such that $h\widehat{\mathcal{A}} \subseteq \mathcal{B}(\mathcal{A}')$ and $gh\widehat{\mathcal{A}} \cap \mathcal{B}(\mathcal{A}) \neq \emptyset$. From Prop. 6.4.11(iv) it follows that $h\mathcal{B}(\widehat{\mathcal{A}}) = \mathcal{B}(\mathcal{A}')$.

If $gh\widehat{\mathcal{A}} \cap \mathcal{B}(\mathcal{A})^\circ \neq \emptyset$, then Prop. 6.4.11(iv) shows that $\mathcal{B}(\mathcal{A}) = gh\mathcal{B}(\widehat{\mathcal{A}}) = g\mathcal{B}(\mathcal{A}')$. Moreover, $\widehat{\mathcal{A}}$ is non-cuspidal and $g = g_j^{-1}$ for a unique $j \in \{1, \ldots, k\}$.

Suppose that $gh\widehat{\mathcal{A}} \cap \mathcal{B}(\mathcal{A})^\circ = \emptyset$. Then $gh\widehat{\mathcal{A}} \cap \mathcal{B}(\mathcal{A}) \subseteq \partial\mathcal{B}(\mathcal{A})$. Let v be the vertex of $\mathcal{K}$ to which $\mathcal{A}$ is attached. Then there exists $j \in \{1, \ldots, k\}$ such that $gh\widehat{\mathcal{A}} \cap g_j^{-1}\mathcal{A}(g_j v) \neq \emptyset$. Let $b := gh\widehat{\mathcal{A}} \cap g_j^{-1}\mathcal{A}(g_j v)$. The boundary structure of $\mathcal{B}(\mathcal{A})$ (see Prop. 6.4.11(ii)) implies that $g_j b$ is contained in a vertical side of $\mathcal{A}(g_j v)$. In particular, b is not a complete geodesic segment. By Cor. 6.2.28, b is either a common side of $gh\widehat{\mathcal{A}}$ and $g_j^{-1}\mathcal{A}(g_j v)$ or a point which is the endpoint of some side of $gh\widehat{\mathcal{A}}$ and some side of $g_j^{-1}\mathcal{A}(g_j v)$. Each case excludes that $\widehat{\mathcal{A}}$ is a strip precell.

Suppose that $\widehat{\mathcal{A}}$ is a cuspidal precell, attached to the vertex w of $\mathcal{K}$. Then $(gh)^{-1}b \cap [w, \infty] = \emptyset$ because b is not a complete geodesic segment. Let $[w, a]$ be the non-vertical side of $\widehat{\mathcal{A}}$. By Prop. 6.4.12(ii), $h(w, a) \subseteq \mathcal{B}(\mathcal{A}')^\circ$. If we had $(gh)^{-1}b \cap (w, a) \neq \emptyset$, then $g\mathcal{B}(\mathcal{A}')^\circ \cap \mathcal{B}(\mathcal{A}) \neq \emptyset$. Since $g\mathcal{B}(\mathcal{A}')$ and $\mathcal{B}(\mathcal{A})$ are both convex polyhedrons, it follows that $g\mathcal{B}(\mathcal{A}')^\circ \cap \mathcal{B}(\mathcal{A})^\circ \neq \emptyset$. The very first case shows that then $g\mathcal{B}(\mathcal{A}') = \mathcal{B}(\mathcal{A})$ and $\mathcal{A}'$ is non-cuspidal, which is a contradiction to $\widehat{\mathcal{A}}$ being cuspidal. Thus, $(gh)^{-1}b \subseteq [a, \infty]$ and therefore

$$g^{-1}b = h(gh)^{-1}b \subseteq [ha, h\infty] \subseteq [k^{-1}\infty, \infty].$$

On the other hand, b is contained is some side c of $\mathcal{B}(\mathcal{A})$. Since both $g[k^{-1}\infty, \infty]$ and c are complete geodesic segments which are not disjoint but do not intersect transversely, they are identical. Hence $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}') = g[k^{-1}\infty, \infty]$.

Suppose that $\widehat{\mathcal{A}}$ is a non-cuspidal precell which is attached to the vertex w of $\mathcal{K}$ and let $[a, w], [w, c]$ be its two non-vertical sides. If $(gh)^{-1}b \cap \big((a, w] \cup [w, c)\big) \neq \emptyset$, then, as before, $g\mathcal{B}(\mathcal{A}') \cap \mathcal{B}(\mathcal{A})^\circ \neq \emptyset$ and by the very first case, $g\mathcal{B}(\mathcal{A}') = \mathcal{B}(\mathcal{A})$, which contradicts to $gh\widehat{\mathcal{A}} \cap \mathcal{B}(\mathcal{A})^\circ = \emptyset$. Therefore $(gh)^{-1}b$ is contained in a vertical side of $\widehat{\mathcal{A}}$ and thus in a vertical side of $\mathcal{B}(\mathcal{A}')$. As in the discussion of a cuspidal $\widehat{\mathcal{A}}$ it follows that $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}')$ is a common side of $\mathcal{B}(\mathcal{A})$ and $g\mathcal{B}(\mathcal{A}')$. □

The proofs of the following two propositions go along the lines of the proof of Prop. 6.4.15.

Proposition 6.4.16. *Let $\mathcal{A}$ be a cuspidal basal precell in H which is attached to the vertex v of $\mathcal{K}$. Suppose that $h \in \Gamma\smallsetminus\Gamma_\infty$ is the element assigned to $\mathcal{A}$ by Prop. 6.2.24. Let $\mathcal{A}'$ be a basal precell in H and $g \in \Gamma$ such that we have $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}') \neq \emptyset$. Then the following assertions hold true.*

(i) *Either* $\mathcal{B}(\mathcal{A}) = g\mathcal{B}(\mathcal{A}')$, *or* $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}')$ *is a common side of* $\mathcal{B}(\mathcal{A})$ *and* $g\mathcal{B}(\mathcal{A}')$.

(ii) *If* $\mathcal{B}(\mathcal{A}) = g\mathcal{B}(\mathcal{A}')$, *then either* $g = \mathrm{id}$ *or* $g = h^{-1}$. *In particular,* $\mathcal{A}'$ *is cuspidal.*

(iii) *If* $\mathcal{B}(\mathcal{A}) \neq g\mathcal{B}(\mathcal{A}')$, *then* $\mathcal{A}'$ *is cuspidal or non-cuspidal or a strip precell. If* $\mathcal{A}'$ *is a strip precell, then* $[h^{-1}\infty, \infty] \neq \mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}')$. *If* $\mathcal{A}'$ *is a cuspidal precell attached to the vertex* w *of* $\mathcal{K}$ *and* $k \in \Gamma \smallsetminus \Gamma_\infty$ *is the element assigned to* $\mathcal{A}'$ *by Prop. 6.2.24, then* $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}')$ *is either* $[v, \infty] = g[w, \infty]$ *or* $[v, \infty] = g[w, k^{-1}\infty]$ *or* $[v, h^{-1}\infty] = g[w, \infty]$ *or* $[v, h^{-1}\infty] = g[w, k^{-1}\infty]$ *or* $[h^{-1}\infty, \infty] = g[k^{-1}\infty, \infty]$. *If* $\mathcal{A}'$ *is a non-cuspidal precell, then* $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}') = [h^{-1}\infty, \infty]$.

Proposition 6.4.17. *Let* $\mathcal{A}$ *be a basal strip precell in* H. *Let* $\mathcal{A}'$ *be a basal precell in* H *and* $g \in \Gamma$ *such that* $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}') \neq \emptyset$. *Then the following statements hold.*

(i) *Either* $\mathcal{B}(\mathcal{A}) = g\mathcal{B}(\mathcal{A}')$, *or* $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}')$ *is a common side of* $\mathcal{B}(\mathcal{A})$ *and* $g\mathcal{B}(\mathcal{A}')$.

(ii) *If* $\mathcal{B}(\mathcal{A}) = g\mathcal{B}(\mathcal{A}')$, *then* $g = \mathrm{id}$ *and* $\mathcal{A} = \mathcal{A}'$.

(iii) *If* $\mathcal{B}(\mathcal{A}) \neq g\mathcal{B}(\mathcal{A}')$, *then* $\mathcal{A}'$ *is a cuspidal or strip precell. If* $\mathcal{A}'$ *is cuspidal and* $k \in \Gamma \smallsetminus \Gamma_\infty$ *is the element assigned to* $\mathcal{A}'$ *by Prop. 6.2.24, then we have* $\mathcal{B}(\mathcal{A}) \cap g\mathcal{B}(\mathcal{A}') \neq g[k^{-1}\infty, \infty]$.

Corollary 6.4.18. *The family of* Γ*-translates of all cells provides a tesselation of* H. *In particular, if* $\mathcal{B}$ *is a cell in* H *and* S *a side of* $\mathcal{B}$, *then there exists a pair* $(\mathcal{B}', g) \in \mathbb{B} \times \Gamma$ *such that* $S = \mathcal{B} \cap g\mathcal{B}'$. *Moreover,* $(\mathcal{B}', g)$ *can be chosen such that* $g^{-1}S$ *is a vertical side of* $\mathcal{B}'$.

Proof. For each precell $\mathcal{A}$ in H we have $\mathcal{A} \subseteq \mathcal{B}(\mathcal{A})$. Therefore, the covering property of the family of all Γ-translates of the cells in H follows directly from Cor. 6.2.28. The property (T2) is proven by Prop. 6.4.16, 6.4.15 and 6.4.17. □

6.5. The base manifold of the cross sections

Let Γ be a geometrically finite subgroup of $\mathrm{PSL}(2, \mathbb{R})$ of which ∞ is a cuspidal point and which satisfies (A2), and suppose that $\mathrm{Rel} \neq \emptyset$. In this section we define the set $\widehat{\mathrm{CS}}$ which will turn out, in Sec. 6.7.1, to be a cross section for the geodesic flow on $Y = \Gamma\backslash H$ w.r.t. to certain measures μ, which will be characterized in Sec. 6.7.1 (see Remark 6.5.5). Here we will already see that $\widehat{\mathrm{CS}}$ satisfies (C2) by showing that $\mathrm{pr}(\widehat{\mathrm{CS}})$ is a totally geodesic suborbifold of Y of codimension one and that $\widehat{\mathrm{CS}}$ is the set of unit tangent vectors based on $\mathrm{pr}(\widehat{\mathrm{CS}})$ but not tangent to it. To achieve this, we start at the other end. We fix a basal family $\mathbb{A}$ of precells in H and consider the family $\mathbb{B}$ of cells in H assigned to $\mathbb{A}$.

We define $\mathrm{BS}(\mathbb{B})$ to be the set of Γ-translates of the sides of these cells. Then we show that the set $\mathrm{BS} := \mathrm{BS}(\mathbb{B})$ is in fact independent of the choice of $\mathbb{A}$. We proceed to prove that BS is a totally geodesic submanifold of H of codimension one and define CS to be the set of unit tangent vectors based on BS but not tangent to it. Then $\widehat{\mathrm{CS}} := \pi(\mathrm{CS})$ is our (future) geometric cross section and $\mathrm{pr}(\widehat{\mathrm{CS}}) = \widehat{\mathrm{BS}} := \pi(\mathrm{BS})$. This construction shows in particular that the set $\widehat{\mathrm{CS}}$ does not depend on the choice of $\mathbb{A}$. For future purposes we already define the sets $\mathrm{NC}(\mathbb{B})$ and $\mathrm{bd}(\mathbb{B})$ and show that also these are independent of the choice of $\mathbb{A}$.

Definition 6.5.1. Let $\mathbb{A}$ be a basal family of precells in H and let $\mathbb{B}$ be the family of cells in H assigned to $\mathbb{A}$. For $\mathcal{B} \in \mathbb{B}$ let $\mathrm{Sides}(\mathcal{B})$ be the set of sides of $\mathcal{B}$. Then set

$$\mathrm{Sides}(\mathbb{B}) := \bigcup_{\mathcal{B}\in\mathbb{B}} \mathrm{Sides}(\mathcal{B})$$

and define

$$\mathrm{BS}(\mathbb{B}) := \bigcup \Gamma \cdot \mathrm{Sides}(\mathbb{B}) = \bigcup \{gS \mid g \in \Gamma,\ S \in \mathrm{Sides}(\mathbb{B})\}.$$

For $\mathcal{B} \in \mathbb{B}$ define $\mathrm{bd}(\mathcal{B}) := \partial_g \mathcal{B}$ and let $\mathrm{NC}(\mathcal{B})$ be the set of geodesics on Y which have a representative on H both endpoints of which are contained in $\mathrm{bd}(\mathcal{B})$. Further set

$$\mathrm{bd}(\mathbb{B}) := \bigcup_{g\in\Gamma} \bigcup_{\mathcal{B}\in\mathbb{B}} g \cdot \mathrm{bd}(\mathcal{B})$$

and

$$\mathrm{NC}(\mathbb{B}) := \bigcup_{\mathcal{B}\in\mathbb{B}} \mathrm{NC}(\mathcal{B}).$$

Proposition 6.5.2. *Let $\mathbb{A}$ and $\mathbb{A}'$ be two basal families of precells in H and suppose that $\mathbb{B}$ resp. $\mathbb{B}'$ are the families of corresponding cells in H assigned to $\mathbb{A}$ resp. $\mathbb{A}'$. There exists a unique map $\mathbb{A} \to \mathbb{Z}$, $\mathcal{A} \mapsto m(\mathcal{A})$ such that*

$$\begin{cases} \mathbb{A} & \to & \{\text{precells in } H\} \\ \mathcal{A} & \mapsto & t_\lambda^{m(\mathcal{A})}\mathcal{A} \end{cases}$$

is a bijection from $\mathbb{A}$ to $\mathbb{A}'$. Then

$$\begin{cases} \mathbb{B} & \to & \mathbb{B}' \\ \mathcal{B}(\mathcal{A}) & \mapsto & t_\lambda^{m(\mathcal{A})}\mathcal{B}(\mathcal{A}) \end{cases}$$

is a bijection as well. Further we have that $\mathrm{BS}(\mathbb{B}) = \mathrm{BS}(\mathbb{B}')$, $\mathrm{NC}(\mathbb{B}) = \mathrm{NC}(\mathbb{B}')$ and $\mathrm{bd}(\mathbb{B}) = \mathrm{bd}(\mathbb{B}')$.

Proof. Cor. 6.2.23 shows that for each precell $\mathcal{A} \in \mathbb{A}$ there exists a unique pair $(\mathcal{A}', -m(\mathcal{A})) \in \mathbb{A}' \times \mathbb{Z}$ such that $t_\lambda^{m(\mathcal{A})}\mathcal{A} = \mathcal{A}'$. Conversely, again by Cor. 6.2.23, for each $\mathcal{A}' \in \mathbb{A}'$ there exists a unique pair $(\mathcal{A}, s) \in \mathbb{A} \times \mathbb{Z}$ such that $t_\lambda^s \mathcal{A} = \mathcal{A}'$. Hence, the map

$$\psi\colon \begin{cases} \mathbb{A} & \to \quad \{\text{precells in } H\} \\ \mathcal{A} & \mapsto \quad t_\lambda^{m(\mathcal{A})}\mathcal{A} \end{cases}$$

maps into $\mathbb{A}'$ and is surjective. Since $\mathbb{A}$ and $\mathbb{A}'$ have the same finite cardinality (see Theorem 6.2.20), the map ψ is a bijection.

We will now show that $t_\lambda^{m(\mathcal{A})}\mathcal{B}(\mathcal{A}) = \mathcal{B}\big(t_\lambda^{m(\mathcal{A})}\mathcal{A}\big)$ for each $\mathcal{A} \in \mathbb{A}$. From this it follows that the map

$$\chi\colon \begin{cases} \mathbb{B} & \to \quad \{U \mid U \subseteq H\} \\ \mathcal{B}(\mathcal{A}) & \mapsto \quad t_\lambda^{m(\mathcal{A})}\mathcal{B}(\mathcal{A}) \end{cases}$$

maps into $\mathbb{B}'$. Since ψ is a bijection and the maps $\mathbb{A} \to \mathbb{B}$, $\mathcal{A} \mapsto \mathcal{B}(\mathcal{A})$ and $\mathbb{A}' \to \mathbb{B}'$, $\mathcal{A}' \mapsto \mathcal{B}(\mathcal{A}')$ are bijections (see Cor. 6.4.14), χ is a bijection as well. Recall that for $\mathcal{A} \in \mathbb{A}$, the $\mathbb{A}$-cell $\mathcal{B}(\mathcal{A})$ is constructed w.r.t. $\mathbb{A}$ and the $\mathbb{A}'$-cell $\mathcal{B}\big(t_\lambda^{m(\mathcal{A})}\mathcal{A}\big)$ is constructed w.r.t. $\mathbb{A}'$.

Let $\mathcal{A} \in \mathbb{A}$. Suppose that $\mathcal{A}$ is a strip precell. Then $t_\lambda^{m(\mathcal{A})}\mathcal{A}$ is a strip precell in H (see Cor. 6.2.23). It follows that

$$\mathcal{B}\big(t_\lambda^{m(\mathcal{A})}\mathcal{A}\big) = t_\lambda^{m(\mathcal{A})}\mathcal{A} = t_\lambda^{m(\mathcal{A})}\mathcal{B}(\mathcal{A}).$$

Suppose that $\mathcal{A}$ is a cuspidal precell. Let b be the non-vertical side of $\mathcal{A}$ and $g \in \Gamma\smallsetminus\Gamma_\infty$ the element that is assigned to $\mathcal{A}$ by Prop. 6.2.24 w.r.t. $\mathbb{A}$. Let $\mathcal{A}_1 \in \mathbb{A}$ be the (cuspidal) precell in H with non-vertical side gb. Set $\mathcal{A}' := t_\lambda^{m(\mathcal{A})}\mathcal{A}$ and $\mathcal{A}_1' := t_\lambda^{m(\mathcal{A}_1)}\mathcal{A}_1$. By Cor. 6.2.23, $\mathcal{A}'$ and $\mathcal{A}_1'$ are cuspidal precells. The non-vertical side of $\mathcal{A}'$ is $t_\lambda^{m(\mathcal{A})}b$ and that of $\mathcal{A}_1'$ is $t_\lambda^{m(\mathcal{A}_1)}gb$. Hence the element $h := t_\lambda^{m(\mathcal{A}_1)}gt_\lambda^{-m(\mathcal{A})}$ maps the non-vertical side of $\mathcal{A}'$ to that of $\mathcal{A}_1'$. Moreover, by Lemmas 6.1.2 and 6.1.3,

$$I(h) = I\big(t_\lambda^{m(\mathcal{A}_1)}gt_\lambda^{-m(\mathcal{A})}\big) = I(g) + m(\mathcal{A})\lambda.$$

Since $b \subseteq I(g)$ by the choice of g, we have $t_\lambda^{m(\mathcal{A})}b \subseteq I(h)$, which shows that h is the element assigned to $\mathcal{A}'$ by Prop. 6.2.24 w.r.t. $\mathbb{A}'$. Then

$$\begin{aligned}
\mathcal{B}\big(t_\lambda^{m(\mathcal{A})}\mathcal{A}\big) &= \mathcal{B}(\mathcal{A}') = \mathcal{A}' \cup h^{-1}\mathcal{A}_1' \\
&= t_\lambda^{m(\mathcal{A})}\mathcal{A} \cup t_\lambda^{m(\mathcal{A})}g^{-1}t_\lambda^{-m(\mathcal{A}_1)}t_\lambda^{m(\mathcal{A}_1)}\mathcal{A}_1 \\
&= t_\lambda^{m(\mathcal{A})}\big(\mathcal{A} \cup g^{-1}\mathcal{A}_1\big) \\
&= t_\lambda^{m(\mathcal{A})}\mathcal{B}(\mathcal{A}).
\end{aligned}$$

Suppose that $\mathcal{A}$ is a non-cuspidal precell. Let a_1 be one of the elements in $\Gamma\smallsetminus\Gamma_\infty$ assigned to $\mathcal{A}$ by Prop. 6.2.24 w.r.t. $\mathbb{A}$ and let $\big((\mathcal{A}_j, a_j)\big)_{j=1,\ldots,k}$ be the cycle in $\mathbb{A} \times \Gamma$ determined by $(\mathcal{A}, a_1)$. For $j = 1, \ldots, k$ set $\mathcal{A}_j' := t_\lambda^{m(\mathcal{A}_j)}\mathcal{A}_j$, let b_j denote the non-vertical side of $\mathcal{A}_j$ for which $b_j \subseteq I(a_j)$. Recall that for $j = 1, \ldots, k-1$, the geodesic segment a_jb_j is the non-vertical side of $\mathcal{A}_{j+1}$ which is different from b_{j+1}, and that a_kb_k is the non-vertical side of $\mathcal{A}_1$ which is not b_1. For $j = 1, \ldots, k-1$ set $c_j := t_\lambda^{m(\mathcal{A}_{j+1})} a_j t_\lambda^{-m(\mathcal{A}_j)}$ and $c_k := t_\lambda^{m(\mathcal{A}_1)} a_k t_\lambda^{-m(\mathcal{A}_k)}$. Then c_j maps the non-vertical side $t_\lambda^{m(\mathcal{A}_j)} b_j$ of $\mathcal{A}'_j$ to the non-vertical side $t_\lambda^{m(\mathcal{A}_{j+1})} a_j b_j$ of $\mathcal{A}'_{j+1}$ for $j = 1, \ldots, k-1$, and c_k maps the non-vertical side $t_\lambda^{m(\mathcal{A}_k)} b_k$ of $\mathcal{A}'_k$ to the non-vertical side $t_\lambda^{m(\mathcal{A}_1)} a_k b_k$ of $\mathcal{A}'_1$. As before, for $j = 1, \ldots, k$, we have $I(c_j) = I(a_j) + m(\mathcal{A}_j)\lambda$ and $t_\lambda^{m(\mathcal{A}_j)} b_j \subseteq I(c_j)$. This implies that c_1 is an element assigned $\mathcal{A}'_1$ by Prop. 6.2.24 w. r. t. $\mathbb{A}'$ and that $\{c_j, c_{j-1}^{-1}\} = \{k_1(\mathcal{A}'_j), k_2(\mathcal{A}'_j)\}$ for $j = 2, \ldots, k$. We will show that $\big((\mathcal{A}'_j, c_j)\big)_{j=1,\ldots,k}$ is the cycle in $\mathbb{A}' \times \Gamma$ determined by $(\mathcal{A}'_1, c_1)$. For $j = 1, \ldots, k$ set $d_1 := \mathrm{id}$, $d_{j+1} := a_j d_j$, $e_1 := \mathrm{id}$ and $e_{j+1} := c_j e_j$. Then

$$e_{j+1} = t_\lambda^{m(\mathcal{A}_{j+1})} d_{j+1} t_\lambda^{-m(\mathcal{A}_1)}$$

for $j = 1, \ldots, k-1$ and $e_{k+1} = t_\lambda^{m(\mathcal{A}_1)} d_{k+1} t_\lambda^{-m(\mathcal{A}_1)} = \mathrm{id}$. Assume for contradiction that $e_{j+1} = \mathrm{id}$ for some $j \in \{1, \ldots, k-1\}$. Then $d_{j+1} = t_\lambda^{m(\mathcal{A}_1) - m(\mathcal{A}_{j+1})}$ is an element in Γ_∞. Prop. 6.4.4(iv) states that $d_{j+1}^{-1}\infty \neq \infty$, hence $d_{j+1} \notin \Gamma_\infty$. This shows that $e_{j+1} \neq \mathrm{id}$ for $j = 1, \ldots, k-1$ and hence $\big((\mathcal{A}'_j, c_j)\big)_{j=1,\ldots,k}$ is the cycle in $\mathbb{A}' \times \Gamma$ determined by $(\mathcal{A}'_1, c_1)$. Therefore

$$\begin{aligned}
\mathcal{B}\big(t_\lambda^{m(\mathcal{A})}\mathcal{A}\big) = \mathcal{B}(\mathcal{A}'_1) &= \bigcup_{j=1}^{k} e_j^{-1} \mathcal{A}'_j \\
&= \bigcup_{j=1}^{k} t_\lambda^{m(\mathcal{A}_1)} d_j^{-1} t_\lambda^{-m(\mathcal{A}_j)} t_\lambda^{m(\mathcal{A}_j)} \mathcal{A}_j \\
&= t_\lambda^{m(\mathcal{A}_1)} \bigcup_{j=1}^{k} d_j^{-1} \mathcal{A}_j \\
&= t_\lambda^{m(\mathcal{A})} \mathcal{B}(\mathcal{A}).
\end{aligned}$$

This shows that χ is a bijection.

Let $\mathcal{A} \in \mathbb{A}$. Then the sides of $\mathcal{B}(\mathcal{A})$ are the $t_\lambda^{-m(\mathcal{A})}$-translates of the sides of $\mathcal{B}\big(t_\lambda^{m(\mathcal{A})}\mathcal{A}\big)$ and $\mathrm{bd}(\mathcal{A}) = t_\lambda^{-m(\mathcal{A})}\,\mathrm{bd}\big(t_\lambda^{m(\mathcal{A})}\mathcal{A}\big)$. This shows that $\mathrm{BS}(\mathbb{B}) = \mathrm{BS}(\mathbb{B}')$ and $\mathrm{bd}(\mathbb{B}) = \mathrm{bd}(\mathbb{B}')$. Now let $\widehat{\gamma}$ be a geodesic on Y which belongs to $\mathrm{NC}\big(\mathcal{B}(\mathcal{A})\big)$. This means that $\widehat{\gamma}$ has a representative, say γ, on H such that $\gamma(\pm\infty) \in \mathrm{bd}\big(\mathcal{B}(\mathcal{A})\big)$. Then $t_\lambda^{m(\mathcal{A})}\gamma$ is also a representative of $\widehat{\gamma}$ on H and

$$t_\lambda^{m(\mathcal{A})}\gamma(\pm\infty) \in t_\lambda^{m(\mathcal{A})}\,\mathrm{bd}\big(\mathcal{B}(\mathcal{A})\big) = \mathrm{bd}\big(t_\lambda^{m(\mathcal{A})}\mathcal{B}(\mathcal{A})\big) = \mathrm{bd}\big(\mathcal{B}\big(t_\lambda^{m(\mathcal{A})}\mathcal{A}\big)\big).$$

Hence $\widehat{\gamma} \in \mathrm{NC}\big(\mathcal{B}\big(t_\lambda^{m(\mathcal{A})}\mathcal{A}\big)\big)$. Therefore $\mathrm{NC}(\mathbb{B}) \subseteq \mathrm{NC}(\mathbb{B}')$ and by interchanging the roles of $\mathbb{A}$ and $\mathbb{A}'$ we find $\mathrm{NC}(\mathbb{B}) = \mathrm{NC}(\mathbb{B}')$. □

We set

$$\mathrm{BS} := \mathrm{BS}(\mathbb{B}), \quad \mathrm{bd} := \mathrm{bd}(\mathbb{B}) \quad \text{and} \quad \mathrm{NC} := \mathrm{NC}(\mathbb{B})$$

for the family $\mathbb{B}$ of cells in H assigned to an arbitrary family $\mathbb{A}$ of precells in H. Prop. 6.5.2 shows that BS, bd and NC are well-defined.

Proposition 6.5.3. *The set* BS *is a totally geodesic submanifold of H of codimension one.*

Proof. Let $\mathbb{A}$ be a basal family of precells in H. Let $\mathbb{B}$ be the family of cells in H assigned to $\mathbb{A}$. Let $\mathcal{B} \in \mathbb{B}$. Prop. 6.4.11(i) resp. 6.4.12(i) resp. Remark 6.2.10 shows that the set of sides of $\mathcal{B}$ is a finite disjoint union of complete geodesic segments. Since $\mathbb{B}$ is finite and Γ is countable (see [Rat06, Cor. 3 of Thm. 5.3.2]), BS is a countable union of complete geodesic segments. Cor. 6.4.18 states that the family of Γ-translates of all cells is a tesselation of H. Therefore, BS is a disjoint countable union of complete geodesic segments. Hence, if BS is a submanifold of H of codimension one, then it is totally geodesic. Now let $z \in \mathrm{BS}$. Suppose that $z \in gS$ for some $g \in \Gamma$ and $S \in \mathrm{Sides}(\mathbb{B})$. By the tesselation property there exist $(\mathcal{B}_1, g_1), (\mathcal{B}_2, g_2) \in \mathbb{B} \times \Gamma$ such that S is a side of $g_1\mathcal{B}_1$ and $g_2\mathcal{B}_2$ and $g_1\mathcal{B}_1 \neq g_2\mathcal{B}_2$. Since each cell is a convex polyhedron with non-empty interior, we find $\varepsilon > 0$ such that $B_\varepsilon(g^{-1}z) \cap g_j\mathcal{B}_j \subseteq g_j\mathcal{B}_j^\circ \cup S$ for $j = 1, 2$. Hence $B_\varepsilon(g^{-1}z) \cap \mathrm{BS}$ is an open subset of S. Since S is a submanifold of H of codimension one, so is BS. This completes the proof. □

Let CS denote the set of unit tangent vectors in SH that are based on BS but not tangent to BS. Recall that Y denotes the orbifold $\Gamma\backslash H$ and recall the canonical projections $\pi\colon H \to Y$, $\pi\colon SH \to SY$ from Sec. 4. Set $\widehat{\mathrm{BS}} := \pi(\mathrm{BS})$ and $\widehat{\mathrm{CS}} := \pi(\mathrm{CS})$.

Proposition 6.5.4. *The set $\widehat{\mathrm{BS}}$ is a totally geodesic suborbifold of Y of codimension one, $\widehat{\mathrm{CS}}$ is the set of unit tangent vectors based on $\widehat{\mathrm{BS}}$ but not tangent to $\widehat{\mathrm{BS}}$ and $\widehat{\mathrm{CS}}$ satisfies (C2).*

Proof. Since BS is Γ-invariant by definition, we see that $\mathrm{BS} = \pi^{-1}(\widehat{\mathrm{BS}})$. Therefore, $\widehat{\mathrm{BS}}$ is a totally geodesic suborbifold of Y of codimension one. Moreover, $\mathrm{CS} = \pi^{-1}(\widehat{\mathrm{CS}})$ and hence $\widehat{\mathrm{CS}}$ is indeed the set of unit tangent vectors based on $\widehat{\mathrm{BS}}$ but not tangent to $\widehat{\mathrm{BS}}$. Finally, $\mathrm{pr}(\widehat{\mathrm{CS}}) = \widehat{\mathrm{BS}}$. By Sec. 5 the set $\widehat{\mathrm{CS}}$ satisfies (C2). □

Remark 6.5.5. Let NIC be the set of geodesics on Y of which at least one endpoint is contained in $\pi(\mathrm{bd})$. Here, $\pi\colon \overline{H}^g \to \Gamma\backslash\overline{H}^g$ denotes the extension of the canonical projection $H \to Y$ to $\overline{H}^g$. In Sec. 6.7.1 we will show that $\widehat{\mathrm{CS}}$ is a cross section for the geodesic flow on Y w.r.t. any measure μ on the space of geodesics on Y for which $\mu(\mathrm{NIC}) = 0$.

We end this section with a short explanation of the acronyms. Obviously, CS stands for "cross section" and BS for "base of (cross) section". Then bd is for "boundary" in sense of geodesic boundary, and $\mathrm{bd}(\mathcal{B})$ is the geodesic boundary of the cell $\mathcal{B}$. Moreover, which will become more sense in Sec. 6.7.2 (see Remark 6.7.34), NC stands for "not coded" and $\mathrm{NC}(\mathcal{B})$ for "not coded due to the cell $\mathcal{B}$". Finally, NIC is for "not infinitely often coded".

6.6. Precells and cells in SH

Let Γ be a geometrically finite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ which satisfies (A2). Suppose that ∞ is a cuspidal point of Γ and that the set of relevant isometric spheres is non-empty. In this section we define the precells and cells in SH and study their properties. The purpose of precells and cells in SH is to get very detailed information about the set $\widehat{\mathrm{CS}}$ from Sec. 6.5 and its relation to the geodesic flow on Y, see Sec. 6.7.1. To each precell in H we assign a precell in SH in an easy, geometric way. Then we fix a basal family $\mathbb{A}$ of precells in H and a set of choices $\mathbb{S}$, that is, a set of generators of equivalence classes of cycles in $\mathbb{A}\times\Gamma$. Let $\widetilde{\mathbb{A}}$ be the family of precells in SH that correspond to the elements in $\mathbb{A}$. In an effective "cut-and-paste" procedure we construct a finite family $\widetilde{\mathbb{B}}_{\mathbb{S}}$ of cells in SH assigned to $\mathbb{A}$ and $\mathbb{S}$ by partitioning the elements in $\widetilde{\mathbb{A}}$, translating some subsets in these partitions by certain elements in Γ and afterwards reunion them in a specific way. However, this procedure involves some choices, which are unimportant for all further applications of $\widetilde{\mathbb{B}}_{\mathbb{S}}$.

The union of the elements in $\widetilde{\mathbb{B}}_{\mathbb{S}}$ is a fundamental set for Γ in SH, and each cell in SH is related to some cell in H in a specific way. We will see that the cycles in $\mathbb{A}\times\Gamma$ play a crucial rôle in the construction of cells in SH as well as in the proofs of the relations between cells in SH and cells in H. We end this section with the definition of the notion of a shift map for $\widetilde{\mathbb{B}}_{\mathbb{S}}$.

Definition 6.6.1. Let U be a subset of H and $z\in\overline{U}$. A unit tangent vector v at z is said to *point into* U if the geodesic γ_v determined by v runs into U, i. e., if there exists $\varepsilon>0$ such that $\gamma_v((0,\varepsilon))\subseteq U$. The unit tangent vector v is said to *point along the boundary of* U if there exists $\varepsilon>0$ such that $\gamma_v((0,\varepsilon))\subseteq\partial U$. It is said to *point out of* U if it points into $H\smallsetminus U$.

Definition 6.6.2. Let $\mathcal{A}$ be a precell in H. Define $\widetilde{\mathcal{A}}$ to be the set of unit tangent vectors that are based on $\mathcal{A}$ and point into $\mathcal{A}^\circ$. The set $\widetilde{\mathcal{A}}$ is called the *precell in* SH *corresponding to* $\mathcal{A}$. If $\mathcal{A}$ is attached to the vertex v of $\mathcal{K}$, we call $\widetilde{\mathcal{A}}$ a *precell in* SH *attached to* v.

Recall the projection $\mathrm{pr}\colon SH\to H$ on base points.

Remark 6.6.3. Let $\mathcal{A}$ be a precell in H and $\widetilde{\mathcal{A}}$ the corresponding precell in SH. Since $\mathcal{A}$ is a convex polyhedron with non-empty interior (see Remark 6.2.10), at each point $x\in\mathcal{A}$ there is based a unit tangent vector which points into $\mathcal{A}^\circ$. Hence $\mathrm{pr}(\widetilde{\mathcal{A}})=\mathcal{A}$. From this it follows that if $\widetilde{\mathcal{A}}$ is a precell in SH, then $\mathrm{pr}(\widetilde{\mathcal{A}})$ is the precell in H to which $\widetilde{\mathcal{A}}$ corresponds. Thus, we have a canonical bijection between precells in H and precells in SH.

Lemma 6.6.4. *Let $\mathcal{A}_1,\mathcal{A}_2$ be two different precells in H. Then the precells $\widetilde{\mathcal{A}_1}$ and $\widetilde{\mathcal{A}_2}$ in SH are disjoint.*

Proof. This is an immediate consequence of Prop. 6.2.15 and Def. 6.6.2. ☐

Definition 6.6.5. Let $\mathcal{A}$ be a precell in H and $\widetilde{\mathcal{A}}$ the corresponding precell in SH. The set $\mathrm{vb}(\widetilde{\mathcal{A}})$ of unit tangent vectors based on $\partial\mathcal{A}$ and pointing along $\partial\mathcal{A}$

is called the *visual boundary of* $\widetilde{\mathcal{A}}$. Further, $\mathrm{vc}(\widetilde{\mathcal{A}}) := \widetilde{\mathcal{A}} \cup \mathrm{vb}(\widetilde{\mathcal{A}})$ is said to be the *visual closure of* $\widetilde{\mathcal{A}}$.

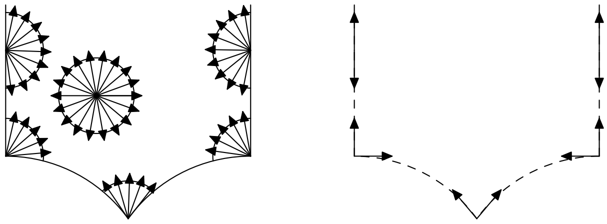


Figure 6.14: The precell in SH and its visual boundary for a non-cuspidal precell in H.

The next lemma is clear from the definitions.

Lemma 6.6.6. *Let $\mathcal{A}$ be a precell in H and $\widetilde{\mathcal{A}}$ be the corresponding precell in SH. Then $\mathrm{vc}(\widetilde{\mathcal{A}})$ is the disjoint union of $\widetilde{\mathcal{A}}$ and $\mathrm{vb}(\widetilde{\mathcal{A}})$.*

The following lemma shows that the visual boundary and the visual closure of a precell $\widetilde{\mathcal{A}}$ in SH is a proper subset of the topological boundary resp. closure of $\widehat{\mathcal{A}}$ in SH.

Lemma 6.6.7. *Let $\widetilde{\mathcal{A}}$ be a precell in SH corresponding to the precell $\mathcal{A}$ in H. The topological boundary $\partial\widetilde{\mathcal{A}}$ is the set of unit tangent vectors based on $\partial\mathcal{A}$.*

Proof. The topology on SH implies that the projection $\mathrm{pr}\colon SH \to H$ on base points is continuous and open. The set $\mathrm{pr}^{-1}(\mathcal{A}^\circ)$ of all unit tangent vectors based on $\mathcal{A}^\circ$ is obviously open. Since $\mathcal{A}$ is convex, it is an (open) subset of $\widetilde{\mathcal{A}}$. We claim that $(\widetilde{\mathcal{A}})^\circ = \mathrm{pr}^{-1}(\mathcal{A}^\circ)$. For contradiction assume that $\mathrm{pr}^{-1}(\mathcal{A}^\circ)$ does not equal $(\widetilde{\mathcal{A}})^\circ$, hence $\mathrm{pr}^{-1}(\mathcal{A}^\circ) \subsetneqq (\widetilde{\mathcal{A}})^\circ$. Then $(\widetilde{\mathcal{A}})^\circ$ contains unit tangent vectors that are based on $\partial\mathcal{A}$. But then $\mathrm{pr}((\widetilde{\mathcal{A}})^\circ)$ is not open, in contradition to pr being an open map. Hence $\mathrm{pr}^{-1}(\mathcal{A}^\circ) = (\widetilde{\mathcal{A}})^\circ$. An analogous argumentation shows that $\mathrm{pr}^{-1}(\complement\mathcal{A}) = \big(\complement\widetilde{\mathcal{A}}\big)^\circ$. Thus $\complement\,\mathrm{cl}\,\big(\widetilde{\mathcal{A}}\big) = \complement\,\mathrm{pr}^{-1}(\mathcal{A})$, which shows that

$$\partial\widetilde{\mathcal{A}} = \mathrm{cl}\,\big(\widetilde{\mathcal{A}}\big)\smallsetminus\big(\widetilde{\mathcal{A}}\big)^\circ = \mathrm{pr}^{-1}(\mathcal{A})\smallsetminus\mathrm{pr}^{-1}(\mathcal{A}^\circ) = \mathrm{pr}^{-1}(\mathcal{A}\smallsetminus\mathcal{A}^\circ) = \mathrm{pr}^{-1}(\partial\mathcal{A})$$

is the set of unit tangent vectors based on $\partial\mathcal{A}$. $\square$

Proposition 6.6.8. *Let $\{\mathcal{A}_j \mid j \in J\}$ be a basal family of precells in H and let $\{\widetilde{\mathcal{A}_j} \mid j \in J\}$ be the set of corresponding precells in SH. Then there is a fundamental set $\widetilde{\mathcal{F}}$ for Γ in SH such that*

$$\bigcup_{j\in J}\widetilde{\mathcal{A}_j} \subseteq \widetilde{\mathcal{F}} \subseteq \bigcup_{j\in J}\mathrm{vc}\,\big(\widetilde{\mathcal{A}_j}\big). \tag{6.7}$$

Moreover, $\mathrm{pr}(\widetilde{\mathcal{F}}) = \bigcup_{j\in J}\mathcal{A}_j$. If $\widetilde{\mathcal{F}}$ is a fundamental set for Γ in SH such that $\widetilde{\mathcal{F}} \subseteq \bigcup_{j\in J}\mathrm{vc}(\widetilde{\mathcal{A}_j})$, then $\widetilde{\mathcal{F}}$ satisfies (6.7). Conversely, if $\{\widetilde{\mathcal{A}_j} \mid j \in J\}$ is a set, *indexed by J, of precells in SH such that (6.7) holds for some fundamental set $\widetilde{\mathcal{F}}$ for Γ in SH, then the family $\{\mathrm{pr}(\widetilde{\mathcal{A}}_j) \mid j \in J\}$ is a basal family of precells in H.*

Proof. Let $\mathbb{A} := \{\mathcal{A}_j \mid j \in J\}$ be a basal family of precells in H and suppose that $\{\widetilde{\mathcal{A}}_j \mid j \in J\}$ is the set of corresponding precells in SH. Set $P := \bigcup_{j\in J} \mathcal{A}_j^\circ$. Recall from Theorem 6.2.20 that P is a fundamental region for Γ in H. Further set $V := \bigcup_{j\in J} \widetilde{\mathcal{A}}_j$ and $W := \bigcup_{j\in J} \mathrm{vc}(\widetilde{\mathcal{A}}_j)$.

At first we show that SH is covered by the Γ-translates of W. To that end let $v \in SH$. Since P is a fundamental region for Γ in H, we find $g \in \Gamma$ such that $g\,\mathrm{pr}(v) \in \overline{P} = \bigcup_{j\in J} \mathcal{A}_j$. W.l.o.g. we may and shall assume that $g = \mathrm{id}$. Pick a basal precell $\mathcal{A}$ in H such that $\mathrm{pr}(v) \in \mathcal{A}$. Since $\mathcal{A}$ is a convex polyhedron, v either points into $\mathcal{A}^\circ$ or along $\partial\mathcal{A}$ or out of $\mathcal{A}$. In the first two cases, $v \in \mathrm{vc}(\widetilde{\mathcal{A}}) \subseteq W$. In the latter case, since the Γ-translates of the elements in $\mathbb{A}$ provide a tesselation of H (see Cor. 6.2.28), there is some $h \in \Gamma$ and some basal precell $\mathcal{A}_1$ in H such that hv points into $h\mathcal{A}_1^\circ$ or along $h\partial\mathcal{A}_1$. Then $h^{-1}v \in \mathrm{vc}(\widetilde{\mathcal{A}}_1) \subseteq W$. This shows that $\Gamma \cdot W = SH$.

Now we show that each nontrivial Γ-translate of V is disjoint from V. To that end consider any $g \in \Gamma\setminus\{\mathrm{id}\}$ and $v \in V$. For contradiction assume that $gv \in V$. Let γ be the geodesic determined by v. Then $g\gamma$ is the geodesic determined by gv. By definition, there are some $\varepsilon > 0$ and basal precells $\mathcal{A}_1, \mathcal{A}_2$ in H such that $\gamma((0,\varepsilon)) \subseteq \mathcal{A}_1^\circ$ and $g\gamma((0,\varepsilon)) \subseteq \mathcal{A}_2^\circ$. Hence, $\gamma(\varepsilon/2) \in \mathcal{A}_1^\circ$ and $g\gamma(\varepsilon/2) \in \mathcal{A}_2^\circ$, which is a contradiction to P being a fundamental region for Γ in H. Thus, $gV \cap V = \emptyset$.

This shows that there is a fundamental set $\widetilde{\mathcal{F}}$ for Γ in SH such that $V \subseteq \widetilde{\mathcal{F}} \subseteq W$. From

$$\bigcup_{j\in J} \mathcal{A}_j = \bigcup_{j\in J} \mathrm{pr}\left(\widetilde{\mathcal{A}}_j\right) = \mathrm{pr}(V) \subseteq \mathrm{pr}\left(\widetilde{\mathcal{F}}\right) \subseteq \mathrm{pr}(W) = \bigcup_{j\in J} \mathrm{pr}\left(\mathrm{vc}\left(\widetilde{\mathcal{A}}_j\right)\right) = \bigcup_{j\in J} \mathcal{A}_j$$

it follows that $\mathrm{pr}(\widetilde{\mathcal{F}}) = \bigcup_{j\in J} \mathcal{A}_j$.

Now let $\widetilde{\mathcal{F}}$ be a fundamental set for Γ in SH such that $\widetilde{\mathcal{F}} \subseteq \bigcup_{j\in J} \mathrm{vc}(\widetilde{\mathcal{A}}_j)$. To prove that $\widetilde{\mathcal{F}}$ satisfies (6.7), it suffices to show that for each $j \in J$ no unit tangent vector in $\mathrm{vb}(\widetilde{\mathcal{A}}_j)$ is Γ-equivalent to some element in V. Let $v \in \mathrm{vb}(\widetilde{\mathcal{A}}_j)$ and assume for contradiction that there exists $(g,k) \in \Gamma \times J$ such that $gv \in \widetilde{\mathcal{A}}_k$. Let η be the geodesic determined by v. Then $g\eta$ is the geodesic determined by gv. By definition we find $\varepsilon > 0$ such that $\eta((0,\varepsilon)) \subseteq \partial\mathcal{A}_j$ and $g\eta((0,\varepsilon)) \subseteq \mathcal{A}_k^\circ$. Then $\eta(\varepsilon/2) \in \partial\mathcal{A}_j \cap g^{-1}\mathcal{A}_k^\circ$, which contradicts to Cor. 6.2.28. This shows that $V \subseteq \widetilde{\mathcal{F}}$.

Finally, let $\{\widetilde{\mathcal{A}}_j \mid j \in J\}$ be a set, indexed by J, of precells in SH and $\widetilde{\mathcal{F}}$ a fundamental set for Γ in SH such that

$$\bigcup_{j\in J} \widetilde{\mathcal{A}}_j \subseteq \widetilde{\mathcal{F}} \subseteq \bigcup_{j\in J} \mathrm{vc}\left(\widetilde{\mathcal{A}}_j\right).$$

Recall from Remark 6.6.3 that $\mathcal{A}_j := \mathrm{pr}(\widetilde{\mathcal{A}}_j)$ is the precell in H to which $\widetilde{\mathcal{A}}_j$ corresponds. Set $\mathcal{F} := \bigcup_{j\in J} \mathcal{A}_j$ and let $z \in H$. Pick any $v \in SH$ such that $\mathrm{pr}(v) = z$. Then there exists $g \in \Gamma$ such that $gv \in \widetilde{\mathcal{F}}$. Now

$$\mathcal{F} = \bigcup_{j\in J} \mathcal{A}_j = \mathrm{pr}\big(\bigcup_{j\in J} \widetilde{\mathcal{A}}_j\big) \subseteq \mathrm{pr}\big(\widetilde{\mathcal{F}}\big) \subseteq \mathrm{pr}\big(\bigcup_{j\in J} \mathrm{vc}\,(\widetilde{\mathcal{A}}_j)\big) = \bigcup_{j\in J} \mathcal{A}_j = \mathcal{F},$$

hence $\mathrm{pr}(\widetilde{\mathcal{F}}) = \mathcal{F}$. This implies that $gz = \mathrm{pr}(gv) \in \mathcal{F}$. Therefore, $\Gamma \cdot \mathcal{F} = H$. Moreover, since $\widetilde{\mathcal{F}}$ is a fundamental set, Lemma 6.6.4 implies that $\mathcal{A}_j \neq \mathcal{A}_k$ for $j,k \in J$, $j \neq k$. Thus, the union $\bigcup_{j\in J} \mathcal{A}_j$ is essentially disjoint. Finally, let $z \in \mathcal{F}^\circ$ and $g \in \Gamma$. Suppose that $gz \in \mathcal{F}^\circ$. We will show that $g = \mathrm{id}$. Pick $j \in J$ such that $z \in \mathcal{A}_j$. Fix an open neighborhood U of z such that $U \subseteq \mathcal{F}^\circ$ and $gU \subseteq \mathcal{F}^\circ$. Since $\mathcal{A}_j$ is a non-empty convex polyhedron, $U \cap \mathcal{A}_j^\circ \neq \emptyset$. Choose $w \in U \cap \mathcal{A}_j^\circ$ and pick $k \in J$ such that $gw \in \mathcal{A}_k$. Then $\mathcal{A}_j^\circ \cap g^{-1}\mathcal{A}_k \neq \emptyset$, which, by Prop. 6.2.26, means that $\mathcal{A}_j = g^{-1}\mathcal{A}_k$. In turn, $\widetilde{\mathcal{A}}_j = g^{-1}\widetilde{\mathcal{A}}_k$. Since $\widetilde{\mathcal{A}}_j, \widetilde{\mathcal{A}}_k \subseteq \widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}$ is a fundamental region for Γ in SH, it follows that $g = \mathrm{id}$. Therefore $\mathcal{F}$ is a closed fundamental region for Γ in H, which means that $\{\mathcal{A}_j \mid j \in J\}$ is a basal family of precells in H. □

Remark 6.6.9. Recall from Theorem 6.2.20 that each basal family of precells in H contains the same finite number of precells, say m. Prop. 6.6.8 shows that if $\{\widetilde{\mathcal{A}}_k \mid k \in K\}$ is a set of precells in SH, indexed by K, such that (6.7) holds for some fundamental set $\widetilde{\mathcal{F}}$ for Γ in SH, then $\#K = m$.

Definition 6.6.10. Let $\mathbb{A} := \{\mathcal{A}_j \mid j \in J\}$ be a basal family of precells in H. Then the set $\widetilde{\mathbb{A}} := \{\widetilde{\mathcal{A}_j} \mid j \in J\}$ of corresponding precells in SH is called a *basal family of precells in* SH or a *family of basal precells in* SH. If $\mathbb{A}$ is a connected family of basal precells in H, then $\widetilde{\mathbb{A}}$ is said to be a *connected family of basal precells in* SH or a *connected basal family of precells in* SH.

Let $\mathbb{A} := \{\mathcal{A}_j \mid j \in J\}$ be a basal family of precells in H and let $\widetilde{\mathbb{A}} := \{\widetilde{\mathcal{A}_j} \mid j \in J\}$ be the corresponding basal family of precells in SH.

Definition and Remark 6.6.11. We call two cycles c_1, c_2 in $\mathbb{A}\times\Gamma$ *equivalent* if there exists a basal precell $\mathcal{A} \in \mathbb{A}$ and elements $g_1, g_2 \in \Gamma\smallsetminus\Gamma_\infty$ such that $(\mathcal{A}, g_1)$ is an element of c_1 and $(\mathcal{A}, g_2)$ is an element of c_2. Obviously, equivalence of cycles is an equivalence relation (on the set of all cycles). If $[c]$ is an equivalence class of cycles in $\mathbb{A}\times\Gamma$, then each element $(\mathcal{A}, h_\mathcal{A}) \in \mathbb{A}\times\Gamma$ in any representative c of $[c]$ is called a *generator* of $[c]$.

Lemma 6.6.12. *Let $\mathcal{A}$ be a non-cuspidal basal precell in H and suppose that $h_\mathcal{A}$ is an element in $\Gamma\smallsetminus\Gamma_\infty$ assigned to $\mathcal{A}$ by Prop. 6.2.24. Let $\big((\mathcal{A}_j, h_j)\big)_{j=1,\ldots,k}$ be the cycle in $\mathbb{A}\times\Gamma$ determined by $(\mathcal{A}, h_\mathcal{A})$. If $\mathcal{A} = \mathcal{A}_l$ for some $l \in \{2,\ldots,k\}$, then $h_l = h_\mathcal{A}$. Moreover, if*

$$q := \min\big\{l \in \{1,\ldots,k-1\} \;\big|\; \mathcal{A}_{l+1} = \mathcal{A}\big\}$$

exists, then q does not depend on the choice of $h_\mathcal{A}$, k is a multiple of q, and $(\mathcal{A}_{l+q}, h_{l+q}) = (\mathcal{A}_l, h_l)$ for $l \in \{1,\ldots,k-q\}$.

Proof. We start by showing that $\mathcal{A}_1 = \mathcal{A}_2$ implies $h_1 = h_2$. If $\mathcal{A}_1 = \mathcal{A}_2$, then Constr. 6.4.2 and Prop. 6.4.4(iii) yield that

$$\{h_2, h_1^{-1}\} = \{k_1(\mathcal{A}_2), k_2(\mathcal{A}_2)\} = \{k_1(\mathcal{A}_1), k_2(\mathcal{A}_1)\} = \{h_1, h_k^{-1}\}.$$

Assume for contradiction that $h_2 = h_k^{-1}$. Then $h_1^{-1} = h_1$. Let v be the vertex of $\mathcal{K}$ to which $\mathcal{A}$ is attached and let s denote the summit of $I(h_1)$. It follows that

$$[v, s] = I(h_1) \cap \mathcal{A}_1 = I(h_1^{-1}) \cap \mathcal{A}_2 = h_1[v, s].$$

Thus $h_1 s = s$ and $h_1 v = v$. But then h_1 fixes two points in H, which shows that $h_1 = \mathrm{id}$. This is a contradiction to $h_1 \in \Gamma \smallsetminus \Gamma_\infty$. Hence $h_2 = h_1$. From Prop. 6.4.7 now follows that $\mathcal{A}_l = \mathcal{A}_{l+1}$ implies $h_l = h_{l+1}$ and $h_l \neq h_l^{-1}$ for $l \in \{1, \ldots, k-1\}$ and also that $\mathcal{A}_k = \mathcal{A}_1$ implies $h_k = h_1$.

Suppose now that there is $l \in \{2, \ldots, k\}$ such that $\mathcal{A} = \mathcal{A}_l$. If $l = k$, then our previous considerations show that $h_\mathcal{A} = h_1 = h_k$. Suppose that $l < k$. Then

$$\{h_l, h_{l-1}^{-1}\} = \{k_1(\mathcal{A}_l), k_2(\mathcal{A}_l)\} = \{k_1(\mathcal{A}), k_2(\mathcal{A})\} = \{h_1, h_k^{-1}\}.$$

Assume for contradiction that $h_l = h_k^{-1}$. Then $(\mathcal{A}_l, h_l) = (\mathcal{A}, h_k^{-1})$. Prop. 6.4.4(iii) implies that

$$(\mathcal{A}_{l+q}, h_{l+q}) = (\mathcal{A}_{k-(q-1)}, h_{k-q}^{-1})$$

for all $1 \leq q \leq k - l$. If $l = k - 2p$ for some $p \in \mathbb{N}$, then

$$(\mathcal{A}_{l+p}, h_{l+p}) = (\mathcal{A}_{l+p+1}, h_{l+p}^{-1}).$$

Hence $\mathcal{A}_{l+p} = \mathcal{A}_{l+p+1}$ and $h_{l+p} = h_{l+p}^{-1}$, which is a contradiction according to our previous considerations. Hence $h_l = h_1$. If $l = k - 2p - 1$ for some $p \in \mathbb{N}_0$, then

$$(\mathcal{A}_{l+p+1}, h_{l+p+1}) = (\mathcal{A}_{l+p+1}, h_{l+p}^{-1}).$$

Hence $h_{l+p}^{-1} = h_{l+p+1}$. This contradicts to

$$\#\{h_{l+p}^{-1}, h_{l+p+1}\} = \#\{k_1(\mathcal{A}_{l+p+1}), k_2(\mathcal{A}_{l+p+1})\} = 2.$$

Thus, $h_l = h_1$. This proves the first statement of the lemma.

Now suppose that

$$q := \min\{l \in \{1, \ldots, k-1\} \mid \mathcal{A}_{l+1} = \mathcal{A}\}$$

exists. Since $(\mathcal{A}_{l+1}, h_{l+1})$ is determined by $(\mathcal{A}_l, h_l)$, it follows that $(\mathcal{A}_{l+q}, h_{l+q}) = (\mathcal{A}_l, h_l)$ for $l \in \{1, \ldots, k-q\}$. Moreover,

$$\{h_1, h_k^{-1}\} = \{k_1(\mathcal{A}_1), k_2(\mathcal{A}_1)\} = \{k_1(\mathcal{A}_{q+1}), k_2(\mathcal{A}_{q+1})\} = \{h_1, h_q^{-1}\}.$$

Therefore $h_k = h_q$ and in turn $\mathcal{A}_k = \mathcal{A}_q$. Thus k is a multiple of q. The independence of q from the choice of $h_\mathcal{A}$ is an immediate consequence of Prop. 6.4.4(iii). $\square$

Definition 6.6.13. Let $\mathcal{A}$ be a non-cuspidal basal precell in H and let $h_{\mathcal{A}}$ be an element in $\Gamma\smallsetminus\Gamma_\infty$ assigned to $\mathcal{A}$ by Prop. 6.2.24. Suppose that $\big((\mathcal{A}_j, h_j)\big)_{j=1,\ldots,k}$ is the cycle in $\mathbb{A}\times\Gamma$ determined by $(\mathcal{A}, h_{\mathcal{A}})$. We set

$$\operatorname{cyl}(\mathcal{A}) := \min\big(\{l \in \{1,\ldots,k-1\} \mid \mathcal{A}_{l+1} = \mathcal{A}\} \cup \{k\}\big).$$

Lemma 6.6.12 shows that $\operatorname{cyl}(\mathcal{A})$ is well-defined. Moreover, it implies that $\operatorname{cyl}(\mathcal{A})$ does not depend on the choice of the generator $(\mathcal{A}, h_{\mathcal{A}})$ of an equivalence class of cycles. For a cuspidal basal precell $\mathcal{A}$ in H we set

$$\operatorname{cyl}(\mathcal{A}) := 3,$$

and for a basal strip precell $\mathcal{A}$ in H we define

$$\operatorname{cyl}(\mathcal{A}) := 2.$$

Example 6.6.14. Recall Example 6.4.6. For the Hecke triangle group G_n and its basal precell $\mathcal{A}$ in H we have $\mathcal{A} = \mathcal{A}_2$ and hence $\operatorname{cyl}(\mathcal{A}) = 1$. In contrast, the basal precell $\mathcal{A}(v_1)$ in H of the congruence group $\mathrm{P}\Gamma_0(5)$ appears only once in the cycle in $\mathbb{A}\times\mathrm{P}\Gamma_0(5)$ and therefore $\operatorname{cyl}\big(\mathcal{A}(v_1)\big) = 3$.

Construction and Definition 6.6.15. Set $\mathcal{F} := \bigcup_{j\in J}\mathcal{A}_j$. Pick a fundamental set $\widetilde{\mathcal{F}}$ for Γ in SH such that

$$\bigcup_{j\in J}\widetilde{\mathcal{A}_j} \subseteq \widetilde{\mathcal{F}} \subseteq \bigcup_{j\in J}\operatorname{vc}\big(\widetilde{\mathcal{A}_j}\big),$$

which is possible by Prop. 6.6.8. For each basal precell $\mathcal{A}\in\mathbb{A}$ and each $z\in\mathcal{F}$ let $\widetilde{E}_z(\mathcal{A})$ denote the set of unit tangent vectors in $\widetilde{\mathcal{F}}\cap\operatorname{vc}(\widetilde{\mathcal{A}})$ based at z. Fix any enumeration of the index set J of $\mathbb{A}$, say $J = \{j_1,\ldots,j_k\}$. For $z\in\mathcal{F}$ and $l\in\{1,\ldots,k\}$ set

$$\widetilde{\mathcal{F}}_z(\mathcal{A}_{j_1}) := \widetilde{E}_z(\mathcal{A}_{j_1}) \quad\text{and}\quad \widetilde{\mathcal{F}}_z(\mathcal{A}_{j_l}) := \widetilde{E}_z(\mathcal{A}_{j_l}) \smallsetminus \bigcup_{m=1}^{l-1}\widetilde{E}_z(\mathcal{A}_{j_m}).$$

Further set

$$\widetilde{\mathcal{F}}(\mathcal{A}) := \bigcup_{z\in\mathcal{A}}\widetilde{\mathcal{F}}_z(\mathcal{A})$$

for $\mathcal{A}\in\mathbb{A}$. Recall from Prop. 6.6.8 that $\operatorname{pr}(\widetilde{\mathcal{F}}) = \mathcal{F}$. Thus,

$$\bigcup_{z\in\mathcal{F}}\bigcup_{\mathcal{A}\in\mathbb{A}}\widetilde{\mathcal{F}}_z(\mathcal{A}) = \widetilde{\mathcal{F}}, \tag{6.8}$$

and the union is disjoint. For each equivalence class of cycles in $\mathbb{A}\times\Gamma$ fix a generator and let $\mathbb{S}$ denote the set of chosen generators. Let $(\mathcal{A}, h_{\mathcal{A}})\in\mathbb{S}$.

Suppose that $\mathcal{A}$ is a non-cuspidal precell in H and let v be the vertex of $\mathcal{K}$ to which $\mathcal{A}$ is attached. Let $\big((\mathcal{A}_j, h_j)\big)_{j=1,\ldots,k}$ be the cycle in $\mathbb{A}\times\Gamma$ determined by $(\mathcal{A}, h_{\mathcal{A}})$. For $j = 1,\ldots,k$ set $g_1 := \mathrm{id}$ and $g_{j+1} := h_j g_j$, and let s_j be the summit of $I(h_j)$. Further, for convenience, set $h_0 := h_k$ and $s_0 := s_k$. In the following we partition certain $\widetilde{\mathcal{F}}(\mathcal{A}_j)$ into k subsets. More precisely, we partition each element of the set $\{\widetilde{\mathcal{F}}(\mathcal{A}_j) \mid j = 1, \ldots, k\}$ into k subsets. Let $j \in \{1, \ldots, \operatorname{cyl}(\mathcal{A})\}$.

For each $z \in \mathcal{A}_j^\circ \cup (h_{j-1}s_{j-1}, g_j v] \cup [g_j v, s_j)$ we pick any partition of $\widetilde{\mathcal{F}}_z(\mathcal{A}_j)$ into k non-empty disjoint subsets $W_{j,z}^{(1)}, \ldots, W_{j,z}^{(k)}$.

For $z \in [s_j, \infty)$ we set $W_{j,z}^{(1)} := \widetilde{\mathcal{F}}_z(\mathcal{A}_j)$ and $W_{j,z}^{(2)} = \ldots = W_{j,z}^{(k)} := \emptyset$.

For $z \in [h_{j-1}s_{j-1}, \infty)$ we set $W_{j,z}^{(1)} := \emptyset$, $W_{j,z}^{(2)} := \widetilde{\mathcal{F}}_z(\mathcal{A}_j)$ and $W_{j,z}^{(3)} = \ldots = W_{j,z}^{(k)} := \emptyset$.

For $m \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, \operatorname{cyl}(\mathcal{A})\}$ we set

$$\widetilde{\mathcal{A}}_{j,m} := \bigcup_{z \in \mathcal{A}_j} W_{j,z}^{(m)}$$

and

$$\widetilde{\mathcal{B}}_j(\mathcal{A}, h_{\mathcal{A}}) := \bigcup_{l=1}^{k} g_j g_l^{-1} \widetilde{\mathcal{A}}_{l,l-j+1}$$

where the first part (l) of the subscript of $\widetilde{\mathcal{A}}_{l,l-j+1}$ is calculated modulo $\operatorname{cyl}(\mathcal{A})$ and the second part ($l - j + 1$) is calculated modulo k.

Suppose that $\mathcal{A}$ is a cuspidal precell in H. Let $\big((\mathcal{A}_1, h_1), (\mathcal{A}_2, h_2)\big)$ be the cycle in $\mathbb{A} \times \Gamma$ determined by $(\mathcal{A}, h_{\mathcal{A}})$. Set $g := h_1 = h_{\mathcal{A}}$, $g_1 := \mathrm{id}$ and $g_2 := h_1 = g$. Suppose that v is the vertex of $\mathcal{K}$ to which A is attached, and let s be the summit of $I(g)$. Let $j \in \{1, 2\}$. We partition $\widetilde{\mathcal{F}}(\mathcal{A}_j)$ into three subsets as follows.

For $z \in \mathcal{A}_j^\circ \cup (g_j v, g_j s)$ we pick any partition of $\widetilde{\mathcal{F}}_z(\mathcal{A}_j)$ into three non-empty disjoint subsets $W_{j,z}^{(1)}$, $W_{j,z}^{(2)}$ and $W_{j,z}^{(3)}$.

For $z \in (g_j v, \infty)$ we set $W_{j,z}^{(1)} := \widetilde{\mathcal{F}}_z(\mathcal{A}_j)$ and $W_{j,z}^{(2)} = W_{j,z}^{(3)} := \emptyset$.

For $z \in [g_j s, \infty)$ we set $W_{j,z}^{(1)} = W_{j,z}^{(2)} := \emptyset$ and $W_{j,z}^{(3)} := \widetilde{\mathcal{F}}_z(\mathcal{A}_j)$.

For $m \in \{1, 2, 3\}$ and $j \in \{1, 2\}$ we set

$$\widetilde{\mathcal{A}}_{j,m} := \bigcup_{z \in \mathcal{A}_j} W_{j,z}^{(m)}.$$

Then we define

$$\begin{aligned}
\widetilde{\mathcal{B}}_1(\mathcal{A}, h_{\mathcal{A}}) &:= \widetilde{\mathcal{A}}_{1,1} \cup g^{-1} \widetilde{\mathcal{A}}_{2,2}, \\
\widetilde{\mathcal{B}}_2(\mathcal{A}, h_{\mathcal{A}}) &:= g \widetilde{\mathcal{A}}_{1,2} \cup \widetilde{\mathcal{A}}_{2,1}, \\
\widetilde{\mathcal{B}}_3(\mathcal{A}, h_{\mathcal{A}}) &:= \widetilde{\mathcal{A}}_{1,3} \cup g^{-1} \widetilde{\mathcal{A}}_{2,3}.
\end{aligned}$$

Suppose that $\mathcal{A}$ is a strip precell in H. Let v_1, v_2 be the two (infinite) vertices of $\mathcal{K}$ to which $\mathcal{A}$ is attached and suppose that $v_1 < v_2$. We partition $\widetilde{\mathcal{F}}(\mathcal{A})$ into two subsets as follows.

For $z \in \mathcal{A}^\circ$ we pick any partition of $\widetilde{\mathcal{F}}_z(\mathcal{A})$ into two non-empty disjoint subsets $W_z^{(1)}$ and $W_z^{(2)}$.

For $z \in (v_1, \infty)$ we set $W_z^{(1)} := \widetilde{\mathcal{F}}_z(\mathcal{A})$ and $W_z^{(2)} := \emptyset$. For $z \in (v_2, \infty)$ we set $W_z^{(1)} := \emptyset$ and $W_z^{(2)} := \widetilde{\mathcal{F}}_z(\mathcal{A})$.

For $m \in \{1, 2\}$ we define

$$\widetilde{\mathcal{B}}_1(\mathcal{A}, h_{\mathcal{A}}) := \bigcup_{z \in \mathcal{A}} W_z^{(1)} \quad \text{and} \quad \widetilde{\mathcal{B}}_2(\mathcal{A}, h_{\mathcal{A}}) := \bigcup_{z \in \mathcal{A}} W_z^{(2)}.$$

The set $\mathbb{S}$ ("selection") is called a *set of choices associated to* $\mathbb{A}$. The family

$$\widetilde{\mathbb{B}}_{\mathbb{S}} := \left\{ \widetilde{\mathcal{B}}_j(\mathcal{A}, h_{\mathcal{A}}) \,\middle|\, (\mathcal{A}, h_{\mathcal{A}}) \in \mathbb{S}, j = 1, \ldots, \mathrm{cyl}(\mathcal{A}) \right\}$$

is called the *family of cells in* SH *associated to* $\mathbb{A}$ *and* $\mathbb{S}$. The elements in this family are subject to the choice of the fundamental set $\widetilde{\mathcal{F}}$, the enumeration of J and some choices for partitions in unit tangent bundle. However, all further applications of $\widetilde{\mathbb{B}}_{\mathbb{S}}$ are invariant under these choices. This justifies to call $\widetilde{\mathbb{B}}_{\mathbb{S}}$ *the* family of cells in SH associated to $\mathbb{A}$ and $\mathbb{S}$. Each element in $\widetilde{\mathbb{B}}_{\mathbb{S}}$ is called a *cell in* SH.

Example 6.6.16. For the Hecke triangle group G_5 from Example 6.4.6 we choose $\mathbb{S} = \{(\mathcal{A}, U_5)\}$. Here we have $k = 5$ and $\mathrm{cyl}(\mathcal{A}) = 1$. The first figure in Fig. 6.15 indicates a possible partition of $\widetilde{\mathcal{F}}(\mathcal{A})$ into the sets $\widetilde{\mathcal{A}}_{1,1}, \widetilde{\mathcal{A}}_{1,2}, \ldots, \widetilde{\mathcal{A}}_{1,5}$.

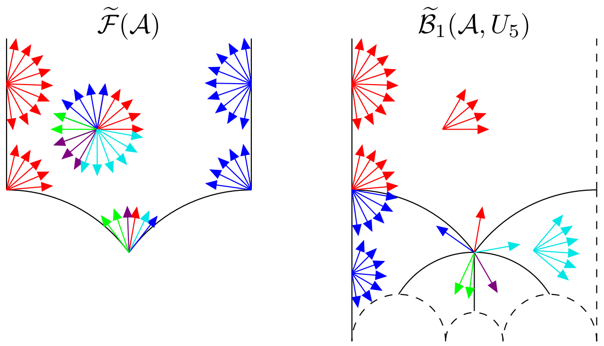


Figure 6.15: A partition of $\widetilde{\mathcal{F}}(\mathcal{A})$ and the cell $\widetilde{\mathcal{B}}_1(\mathcal{A}, U_5)$ in SH.

The unit tangent vectors in red belong to $\widetilde{\mathcal{A}}_{1,1}$, those in dark blue to $\widetilde{\mathcal{A}}_{1,2}$, those in green to $\widetilde{\mathcal{A}}_{1,3}$, those in violet to $\widetilde{\mathcal{A}}_{1,4}$, and those in light blue to $\widetilde{\mathcal{A}}_{1,4}$. The second figure in Fig. 6.15 shows the cell $\widetilde{\mathcal{B}}_1(\mathcal{A}, U_5)$ in SH.

Example 6.6.17. For the group Γ from Example 6.1.21 we choose as set of choices $\mathbb{S} = \{(\mathcal{A}_1, \mathrm{id}), (\mathcal{A}_2, S)\}$ (cf. Example 6.4.6). The cells in SH which arise from $(\mathcal{A}_1, \mathrm{id})$ are shown in Fig. 6.16. The cells in SH arising from $(\mathcal{A}_2, S)$ are indicated in Fig. 6.17.

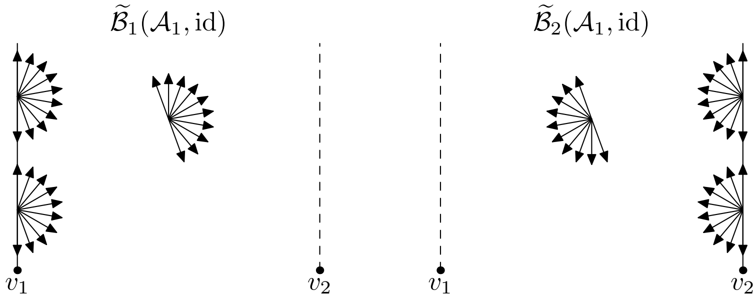


Figure 6.16: The cells in SH arising from $(\mathcal{A}_1, \mathrm{id})$.

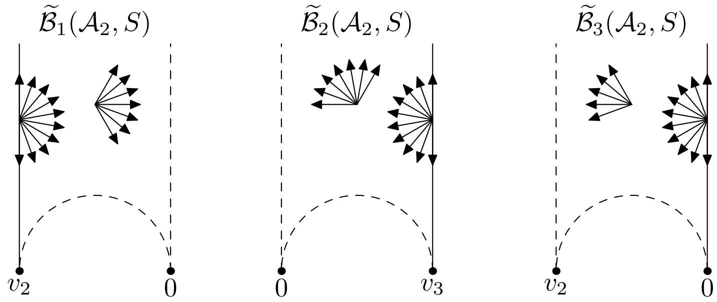


Figure 6.17: The cells in SH arising from $(\mathcal{A}_1, \mathrm{id})$.

Let $\mathbb{S}$ be a set of choices associated to $\mathbb{A}$.

Proposition 6.6.18. *The union* $\bigcup_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S}}} \widetilde{\mathcal{B}}$ *is disjoint and a fundamental set for* Γ *in* SH*.*

Proof. Constr. 6.6.15 picks a fundamental set $\widetilde{\mathcal{F}}$ for Γ in SH and chooses a family $\mathcal{P} := \{\widetilde{\mathcal{F}}_z(\mathcal{A}) \mid z \in \mathcal{F},\ \mathcal{A} \in \mathbb{A}\}$ of subsets of it. Since the union in (6.8) is disjoint, $\mathcal{P}$ is a partition of $\widetilde{\mathcal{F}}$. Recall the notation from Constr. 6.6.15. One considers the family

$$\mathcal{P}_1 := \left\{ \widetilde{\mathcal{F}}_z(\mathcal{A}_j) \,\middle|\, z \in \mathcal{F},\ (\mathcal{A}, h_{\mathcal{A}}) \in \mathbb{S},\ j = 1, \ldots, \mathrm{cyl}(\mathcal{A}) \right\}.$$

The elements of $\mathcal{P}_1$ are pairwise disjoint and each element of $\mathcal{P}$ is contained in $\mathcal{P}_1$. Hence, $\mathcal{P}_1$ is a partition of $\widetilde{\mathcal{F}}$. The next step is to partition each element of $\mathcal{P}_1$ into a finite number of subsets. Thus, $\widetilde{\mathcal{F}}$ is partitioned into some family $\mathcal{P}_2$ of subsets of $\widetilde{\mathcal{F}}$. Then each element W of $\mathcal{P}_2$ is translated by some element $g(W)$ in Γ to get the family $\mathcal{P}_3 := \{g(W)W \mid W \in \mathcal{P}_2\}$. Since $\widetilde{\mathcal{F}}$ is a fundamental set for Γ in SH, the elements of $\mathcal{P}_3$ are pairwise disjoint and $\bigcup \mathcal{P}_3$ is a fundamental set for Γ in SH. Now $\mathcal{P}_3$ is partitioned into certain subsets, say into the subsets $\mathcal{Q}_l$, $l \in L$. Each cell $\widetilde{\mathcal{B}}$ in SH is the union of the elements in some $\mathcal{Q}_{l(\widetilde{\mathcal{B}})}$ such that $l(\widetilde{\mathcal{B}}_1) \neq l(\widetilde{\mathcal{B}}_2)$ if $\widetilde{\mathcal{B}}_1 \neq \widetilde{\mathcal{B}}_2$. Therefore, the union $\bigcup \left\{ \widetilde{\mathcal{B}} \,\middle|\, \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S}} \right\}$ is disjoint and a fundamental set for Γ in SH. $\square$

For each $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S}}$ set $b(\widetilde{\mathcal{B}}) := \mathrm{pr}(\widetilde{\mathcal{B}}) \cap \partial\,\mathrm{pr}(\widetilde{\mathcal{B}})$ and let $\mathrm{CS}'(\widetilde{\mathcal{B}})$ be the set of unit tangent vectors in $\widetilde{\mathcal{B}}$ that are based on $b(\widetilde{\mathcal{B}})$ but do not point along $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}})$.

Example 6.6.19. Recall the congruence subgroup $\mathrm{P}\Gamma_0(5)$ and its cycles in $\mathbb{A} \times \mathrm{P}\Gamma_0(5)$ from Example 6.4.6. We choose $\mathbb{S} := \{(\mathcal{A}(v_4), h^{-1}), (\mathcal{A}(v_1), h_1)\}$ as set of choices associated to $\mathbb{A}$ and set

$$\widetilde{\mathcal{B}}_1 := \widetilde{\mathcal{B}}_1\big(\mathcal{A}(v_4), h^{-1}\big), \qquad \widetilde{\mathcal{B}}_4 := \widetilde{\mathcal{B}}_1\big(\mathcal{A}(v_1), h_1\big),$$
$$\widetilde{\mathcal{B}}_2 := \widetilde{\mathcal{B}}_2\big(\mathcal{A}(v_4), h^{-1}\big), \qquad \widetilde{\mathcal{B}}_5 := \widetilde{\mathcal{B}}_2\big(\mathcal{A}(v_1), h_1\big),$$
$$\widetilde{\mathcal{B}}_3 := \widetilde{\mathcal{B}}_3\big(\mathcal{A}(v_4), h^{-1}\big), \qquad \widetilde{\mathcal{B}}_6 := \widetilde{\mathcal{B}}_3\big(\mathcal{A}(v_1), h_1\big),$$

as well as

$$\mathrm{CS}'_j := \mathrm{CS}'(\widetilde{\mathcal{B}}_j)$$

for $j = 1, \ldots, 6$. Fig. 6.18 shows the sets CS'_j.

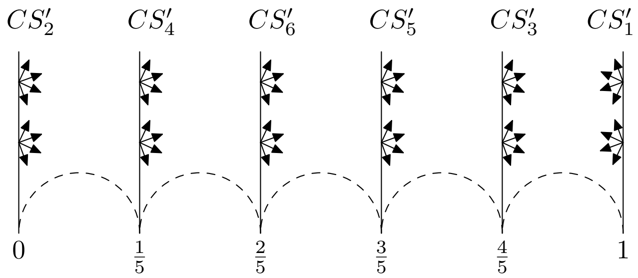


Figure 6.18: The sets CS'_j.

Lemma 6.6.20. *Let $\mathcal{A}_1, \mathcal{A}_2$ be two basal precells in H and let $g \in \Gamma$ such that $g \cdot \mathrm{vc}(\widetilde{\mathcal{A}_1}) \cap \mathrm{vc}(\widetilde{\mathcal{A}_2}) \neq \emptyset$. Suppose that $\mathcal{A}_1 \neq \mathcal{A}_2$ or $g \neq \mathrm{id}$. Then*

$$g \cdot \mathrm{vc}\left(\widetilde{\mathcal{A}_1}\right) \cap \mathrm{vc}\left(\widetilde{\mathcal{A}_2}\right) \subseteq g \cdot \mathrm{vb}\left(\widetilde{\mathcal{A}_1}\right) \cap \mathrm{vb}\left(\widetilde{\mathcal{A}_2}\right).$$

Moreover, suppose that $\mathcal{A}_1$ is cuspidal or non-cuspidal and that there is a unit tangent vector $w \in \mathrm{vc}(\widetilde{\mathcal{A}_1})$ pointing into a non-vertical side S_1 of $\mathcal{A}_1$ such that $gw \in \mathrm{vc}(\widetilde{\mathcal{A}_2})$. Then gw points into a non-vertical side S_2 of $\mathcal{A}_2$ and $gS_1 = S_2$.

Proof. We have $\mathrm{pr}(g \cdot \mathrm{vc}(\widetilde{\mathcal{A}_1})) = g\,\mathrm{pr}(\mathrm{vc}(\widetilde{\mathcal{A}_1})) = g\mathcal{A}_1$ and $\mathrm{pr}(g \cdot \mathrm{vb}(\widetilde{\mathcal{A}_1})) = g\partial\mathcal{A}_1$, and likewise for $\mathrm{pr}(\mathrm{vc}(\widetilde{\mathcal{A}_2})) = \mathcal{A}_2$ and $\mathrm{pr}(\mathrm{vb}(\widetilde{\mathcal{A}_2})) = \partial\mathcal{A}_2$. From $g \cdot \mathrm{vc}(\widetilde{\mathcal{A}_1}) \cap \mathrm{vc}(\widetilde{\mathcal{A}_2}) \neq \emptyset$ then follows that $g\mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$. By Prop. 6.2.26, either $g\mathcal{A}_1 = \mathcal{A}_2$ and $g \in \Gamma_\infty$ or $g\mathcal{A}_1 \cap \mathcal{A}_2 \subseteq g\partial\mathcal{A}_1 \cap \partial\mathcal{A}_2$. Assume for contradiction that $g\mathcal{A}_1 = \mathcal{A}_2$ with $g \in \Gamma_\infty$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are basal, Cor. 6.2.23 shows that $g = \mathrm{id}$ and $\mathcal{A}_1 = \mathcal{A}_2$. This contradicts the hypotheses of the lemma. Hence $g\mathcal{A}_1 \cap \mathcal{A}_2 \subseteq g\partial\mathcal{A}_1 \cap \partial\mathcal{A}_2$ and therefore $g \cdot \mathrm{vc}(\widetilde{\mathcal{A}_1}) \cap \mathrm{vc}(\widetilde{\mathcal{A}_2}) \subseteq g \cdot \mathrm{vb}(\widetilde{\mathcal{A}_1}) \cap \mathrm{vb}(\widetilde{\mathcal{A}_2})$.

Let $\mathcal{A}_1$ and w be as in the claim. Further let γ be the geodesic determined by w. Then $g\gamma$ is the geodesic determined by gw. By definition there exists $\varepsilon > 0$ such that $\gamma((0,\varepsilon)) \subseteq S_1$ and $g\gamma((0,\varepsilon)) \subseteq \mathcal{A}_2$. Then

$$\gamma\big((0,\varepsilon)\big) \subseteq S_1 \cap g^{-1} \subseteq \mathcal{A}_1 \cap g^{-1}\mathcal{A}_2.$$

Since the sets $\mathcal{A}_1$ and $g^{-1}\mathcal{A}_2$ intersect in more than one point and $\mathcal{A}_1 \neq g^{-1}\mathcal{A}_2$, Prop. 6.2.26 states that $\mathcal{A}_1 \cap g^{-1}\mathcal{A}_2$ is a common side of $\mathcal{A}_1$ and $g^{-1}\mathcal{A}_2$. Necessarily, this side is S_1. Prop. 6.2.26 shows further that gS_1 is a non-vertical side of $\mathcal{A}_2$. Thus, gw points along the non-vertical side gS_1 of $\mathcal{A}_2$. $\square$

Proposition 6.6.21. *Let $(\mathcal{A}, h_{\mathcal{A}}) \in \mathbb{S}$ and suppose that $\mathcal{A}$ is a non-cuspidal precell in H. Let $\big((\mathcal{A}_j, h_j)\big)_{j=1,\dots,k}$ be the cycle in $\mathbb{A}\times\Gamma$ determined by $(\mathcal{A}, h_{\mathcal{A}})$. For each $m = 1,\dots,\operatorname{cyl}(\mathcal{A})$ we have $b(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}})) = (h_m^{-1}\infty, \infty)$ and*

$$\operatorname{pr}\big(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}})\big) = \mathcal{B}(\mathcal{A}_m)^\circ \cup (h_m^{-1}\infty, \infty).$$

Moreover, $\mathrm{CS}'(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}}))$ is the set of unit tangent vectors based on $(h_m^{-1}\infty,\infty)$ that point into $\mathcal{B}(\mathcal{A}_m)^\circ$, and $\operatorname{pr}(\mathrm{CS}'(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}}))) = b(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}}))$.

Proof. We use the notation from Constr. 6.6.15. Let $j \in \{1,\dots,\operatorname{cyl}(\mathcal{A})\}$ and $z \in \mathcal{A}_j$. At first we show that $\widetilde{\mathcal{F}}_z(\mathcal{A}_j) \neq \emptyset$. For each choice of $\widetilde{\mathcal{F}}$ we have $\widetilde{\mathcal{A}_j} \subseteq \widetilde{\mathcal{F}} \cap \operatorname{vc}(\widetilde{\mathcal{A}_j})$. Remark 6.6.3 states that $\operatorname{pr}(\widetilde{\mathcal{A}_j}) = \mathcal{A}_j$. Hence $\widetilde{E}_z(\mathcal{A}_j) \cap \widetilde{\mathcal{A}_j} \neq \emptyset$. More precisely, if $\big(\widetilde{\mathcal{A}_j}\big)_z$ denotes the set of unit tangent vectors based on z that point into $\mathcal{A}_j^\circ$, then $\big(\widetilde{\mathcal{A}_j}\big)_z = \widetilde{E}_z(\mathcal{A}_j) \cap \widetilde{\mathcal{A}_j}$. The set $\big(\widetilde{\mathcal{A}_j}\big)_z$ is non-empty, since $\mathcal{A}_j$ is convex with non-empty interior. Let $k \in J$ such that $\mathcal{A}_k \neq \mathcal{A}_j$. Then

$$\widetilde{E}_z(\mathcal{A}_k) \cap \widetilde{E}_z(\mathcal{A}_j) \subseteq \operatorname{vc}\big(\widetilde{\mathcal{A}_k}\big) \cap \operatorname{vc}\big(\widetilde{\mathcal{A}_j}\big) \subseteq \operatorname{vb}\big(\widetilde{\mathcal{A}_k}\big) \cap \operatorname{vb}\big(\widetilde{\mathcal{A}_j}\big),$$

where the last inclusion follows from Lemma 6.6.20. Since $\widetilde{\mathcal{A}_j} \cap \operatorname{vb}(\widetilde{\mathcal{A}_j}) = \emptyset$ by Lemma 6.6.6, it follows that

$$\big(\widetilde{\mathcal{A}_j}\big) \cap \widetilde{E}_z(\mathcal{A}_k) = \widetilde{\mathcal{A}_j} \cap \widetilde{E}_z(\mathcal{A}_j) \cap \widetilde{E}_z(\mathcal{A}_k) \subseteq \widetilde{\mathcal{A}_j} \cap \operatorname{vb}\big(\widetilde{\mathcal{A}_j}\big) = \emptyset.$$

Hence

$$\big(\widetilde{\mathcal{A}_j}\big)_z \subseteq \widetilde{\mathcal{F}}_z(\mathcal{A}_j). \tag{6.9}$$

Let $j \in \{1,\dots,\operatorname{cyl}(\mathcal{A})\}$ set $\widetilde{\mathcal{B}}_j := \widetilde{\mathcal{B}}_j(\mathcal{A}, h_{\mathcal{A}})$ and

$$T_j := \mathcal{A}_j^\circ \cup (h_{j-1}s_{j-1}, g_j v] \cup [g_j v, s_j).$$

Let $m \in \{1,\dots,k\}$. Then

$$\begin{aligned}
\operatorname{pr}\big(\widetilde{\mathcal{A}}_{j,m}\big) &= \bigcup_{z\in\mathcal{A}_j} \operatorname{pr}\big(W_{j,z}^{(m)}\big) \\
&= \bigcup_{z\in T_j} \operatorname{pr}\big(W_{j,z}^{(m)}\big) \cup \bigcup_{z\in[s_j,\infty)} \operatorname{pr}\big(W_{j,z}^{(m)}\big) \cup \bigcup_{z\in[h_{j-1}s_{j-1},\infty)} \operatorname{pr}\big(W_{j,z}^{(m)}\big) \\
&= \begin{cases} T_j & \text{for } m \notin \{1,2\}, \\ T_j \cup [s_j,\infty) & \text{for } m = 1, \\ T_j \cup [h_{j-1}s_{j-1},\infty) & \text{for } m = 2. \end{cases}
\end{aligned}$$

Note that necessarily $k \geq 3$. Then

$$\begin{aligned}
\widetilde{\mathcal{B}}_j &= \bigcup_{l=1}^{k} g_j g_l^{-1} \widetilde{\mathcal{A}}_{l,l-j+1} \\
&= \bigcup_{l=1}^{j-1} g_j g_l^{-1} \widetilde{\mathcal{A}}_{l,l-j+1} \cup g_j g_j^{-1} \widetilde{\mathcal{A}}_{j,1} \cup g_j g_{j+1}^{-1} \widetilde{\mathcal{A}}_{j+1,2} \cup \bigcup_{l=j+2}^{k} g_j g_l^{-1} \widetilde{\mathcal{A}}_{l,l-j+1}.
\end{aligned}$$

Since $l-j+1 \not\equiv 1,2 \mod k$ for $l \in \{1,\ldots,j-1\} \cup \{j+2,\ldots,k\}$, it follows that

$$\begin{aligned}
\mathrm{pr}\left(\widetilde{\mathcal{B}}_j\right) &= \bigcup_{l=1}^{j-1} g_j g_l^{-1} \,\mathrm{pr}\left(\widetilde{\mathcal{A}}_{l,l-j+1}\right) \cup \mathrm{pr}\left(\widetilde{\mathcal{A}}_{j,1}\right) \cup h_j^{-1}\,\mathrm{pr}\left(\widetilde{\mathcal{A}}_{j+1,2}\right) \cup \bigcup_{l=j+2}^{k} g_j g_l^{-1}\,\mathrm{pr}\left(\widetilde{\mathcal{A}}_{l,l-j+1}\right) \\
&= \bigcup_{l=1}^{j-1} g_j g_l^{-1} T_l \cup T_j \cup [s_j,\infty) \cup h_j^{-1} T_{j+1} \cup h_j^{-1}[h_j s_j,\infty) \cup \bigcup_{l=j+2}^{k} g_j g_l^{-1} T_l \\
&= \bigcup_{l=1}^{k} g_j g_l^{-1} T_l \cup (h_j^{-1}\infty,\infty).
\end{aligned}$$

For the last equality we use that $\mathrm{pr}_\infty(s_j) = h_j^{-1}\infty$ by Lemma 6.2.8. Hence s_j is contained in the geodesic segment $\mathrm{pr}_\infty^{-1}(h_j^{-1}\infty) \cap H = (h_j^{-1}\infty,\infty)$, which shows that the union of the two geodesic segments $[s_j,\infty)$ and $[s_j,h_j^{-1}\infty)$ is indeed $(h_j^{-1}\infty,\infty)$. Prop. 6.4.11 implies that

$$\mathrm{pr}\left(\widetilde{\mathcal{B}}_j\right) = \mathcal{B}(\mathcal{A}_j)^\circ \cup (h_j^{-1}\infty,\infty).$$

This shows that $b(\widetilde{\mathcal{B}}_j) = (h_j^{-1}\infty,\infty)$. The set of unit tangent vectors in $\widetilde{\mathcal{B}}_j$ that are based on $b(\widetilde{\mathcal{B}}_j)$ is the disjoint union

$$\begin{aligned}
D'_j &:= \bigcup_{z\in[s_j,\infty)} W^{(1)}_{j,z} \cup h_j^{-1} \bigcup_{z\in[h_j s_j,\infty)} W^{(2)}_{j+1,z} \\
&= \bigcup_{z\in[s_j,\infty)} \widetilde{\mathcal{F}}_z(\mathcal{A}_j) \cup h_j^{-1} \bigcup_{z\in[h_j s_j,\infty)} \widetilde{\mathcal{F}}_z(\mathcal{A}_{j+1}).
\end{aligned}$$

To show that $\mathrm{CS}'(\widetilde{\mathcal{B}}_j)$ is the set of unit tangent vectors based on $b(\widetilde{\mathcal{B}}_j)$ that point into $\mathcal{B}(\mathcal{A}_j)^\circ$ we have to show that D'_j contains all unit tangent vectors based on $[s_j,\infty)$ that point into $\mathcal{A}_j^\circ$ and all unit tangent vectors based on $(h_j^{-1}\infty,s_j]$ that point into $h_j^{-1}\mathcal{A}_{j+1}^\circ$ and the unit tangent vector which is based at s_j and points into $[s_j,g_j v]$. If w is a unit tangent vector based on $[h_j s_j,\infty)$ that points into $\mathcal{A}_{j+1}^\circ$, then, clearly, $h_j^{-1}w$ is a unit tangent vector based on $[s_j,h_j^{-1}\infty)$ that points into $h_j^{-1}\mathcal{A}_{j+1}^\circ$. Hence, (6.9) shows that D'_j contains all unit tangent vectors of the first two kinds mentioned above. Let w be the unit tangent vector with $\mathrm{pr}(w) = s_j$ which points into $[s_j,g_j v]$.

Suppose first that $w \in \widetilde{\mathcal{F}}$. Then $w \in \mathrm{vc}(\widetilde{\mathcal{A}_j}) \cap \widetilde{\mathcal{F}}$ and therefore $w \in \widetilde{E}_{s_j}(\mathcal{A}_j)$. Let $k \in J$ with $\mathcal{A}_k \neq \mathcal{A}_j$. Assume for contradiction that $w \in \mathrm{vc}(\widetilde{\mathcal{A}_k})$. Lemma 6.6.20

implies that $[s_j, g_j v]$ is a non-vertical side of $\mathcal{A}_k$, which is a contradiction. Hence $w \notin \widetilde{E}_{s_j}(\mathcal{A}_k)$. Therefore, $w \in \widetilde{\mathcal{F}}_{s_j}(\mathcal{A}_j)$ and hence $w \in D'_j$.

Suppose now that $w \notin \widetilde{\mathcal{F}}$. Then there exists a unique $g \in \Gamma \smallsetminus \{\mathrm{id}\}$ such that $gw \in \widetilde{\mathcal{F}}$. Let $\mathcal{A}$ be a basal precell in H such that $gw \in \mathrm{vc}(\widetilde{\mathcal{A}}) \cap \widetilde{\mathcal{F}}$. Lemma 6.6.20 shows that $g[s_j, g_j v]$ is a non-vertical side S of $\mathcal{A}$. Thus, $g^{-1}\mathcal{A} \cap \mathcal{B}(\mathcal{A}_j)^\circ \neq \emptyset$. By Prop. 6.4.11(iv) there is a unique $l \in \{1, \ldots, k\}$ such that $g = g_l g_j^{-1}$ and $\mathcal{A} = \mathcal{A}_l$. Now $[s_j, g_j v]$ is mapped by h_j to the non-vertical side $[h_j s_j, g_{j+1} v]$ of $\mathcal{A}_{j+1}$. Thus, $g = h_j$ and $\mathcal{A} = \mathcal{A}_{j+1}$. Then $h_j w \in \mathrm{vc}(\widetilde{\mathcal{A}_{j+1}})$. As before we see that $w \in D'_j$. Moreover, $\mathrm{pr}(\mathrm{CS}'(\widetilde{\mathcal{B}}_j)) = b(\widetilde{\mathcal{B}}_j)$. □

Analogously to Prop. 6.6.21 one proves the following two propositions.

Proposition 6.6.22. *Let $(\mathcal{A}, h_{\mathcal{A}}) \in \mathbb{S}$ and suppose that $\mathcal{A}$ is a cuspidal precell in H. Let v be the vertex of $\mathcal{K}$ to which $\mathcal{A}$ is attached and let $\big((\mathcal{A}, g), (\mathcal{A}', g^{-1})\big)$ be the cycle in $\mathbb{A} \times \Gamma$ determined by $(\mathcal{A}, h_{\mathcal{A}})$. Then*

$$b\big(\widetilde{\mathcal{B}}_1(\mathcal{A}, h_{\mathcal{A}})\big) = (v, \infty),\ \ b\big(\widetilde{\mathcal{B}}_2(\mathcal{A}, h_{\mathcal{A}})\big) = (gv, \infty),\ \ b\big(\widetilde{\mathcal{B}}_3(\mathcal{A}, h_{\mathcal{A}})\big) = (g^{-1}\infty, \infty)$$

and

$$\begin{aligned} \mathrm{pr}\left(\widetilde{\mathcal{B}}_1(\mathcal{A}, h_{\mathcal{A}})\right) &= \mathcal{B}(\mathcal{A})^\circ \cup (v, \infty), \\ \mathrm{pr}\left(\widetilde{\mathcal{B}}_2(\mathcal{A}, h_{\mathcal{A}})\right) &= \mathcal{B}(\mathcal{A}')^\circ \cup (gv, \infty), \\ \mathrm{pr}\left(\widetilde{\mathcal{B}}_3(\mathcal{A}, h_{\mathcal{A}})\right) &= \mathcal{B}(\mathcal{A})^\circ \cup (g^{-1}\infty, \infty). \end{aligned}$$

Moreover, $\mathrm{CS}'(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}}))$ is the set of unit tangent vectors based on $b(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}}))$ that point into $\mathrm{pr}(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}}))^\circ$, and $\mathrm{pr}(\mathrm{CS}'(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}}))) = b(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}}))$ for $m = 1, 2, 3$.

Proposition 6.6.23. *Let $(\mathcal{A}, h_{\mathcal{A}}) \in \mathbb{S}$ and suppose that $\mathcal{A}$ is a strip precell. Let v_1, v_2 be the two (infinite) vertices of $\mathcal{K}$ to which $\mathcal{A}$ is attached and suppose that $v_1 < v_2$. For $m = 1, 2$ we have $b(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}})) = (v_m, \infty)$ and*

$$\mathrm{pr}\left(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}})\right) = \mathcal{B}(\mathcal{A})^\circ \cup (v_m, \infty) = \mathcal{A}^\circ \cup (v_m, \infty).$$

Moreover, $\mathrm{CS}'(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}}))$ is the set of unit tangent vectors based on (v_m, ∞) that point into $\mathcal{B}(\mathcal{A})^\circ$, and $\mathrm{pr}(\mathrm{CS}'(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}}))) = b(\widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}}))$.

Corollary 6.6.24. *Let $\widetilde{\mathcal{B}} \in \mathbb{B}_{\mathbb{S}}$. Then $\mathcal{B} := \mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$ is a cell in H and $b(\widetilde{\mathcal{B}})$ a side of $\mathcal{B}$. Moreover, $\mathrm{pr}(\widetilde{\mathcal{B}}) = \mathcal{B}^\circ \cup b(\widetilde{\mathcal{B}})$ and $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ = \mathcal{B}^\circ$.*

Proof. This follows directly from a combination of Prop. 6.6.21 with 6.4.11 resp. of Prop. 6.6.22 with 6.4.12 resp. of Prop. 6.6.23 with 6.4.13. □

The development of a symbolic dynamics for the geodesic flow on Y via the family $\widetilde{\mathbb{B}}_{\mathbb{S}}$ of cells in SH is based on the following properties of the cells $\widetilde{\mathcal{B}}$ in SH: It uses that $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$ is a convex polyhedron of which each side is a complete geodesic segment and that each side is the image under some element $g \in \Gamma$ of the complete geodesic segment $b(\widetilde{\mathcal{B}}')$ for some cell $\widetilde{\mathcal{B}}'$ in SH. It further uses that $\bigcup \widetilde{\mathbb{B}}_{\mathbb{S}}$ is a fundamental set for Γ in SH and that $\{g \cdot \mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}})) \mid g \in \Gamma,\ \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S}}\}$ is a tesselation of H. Moreover, one needs that $b(\widetilde{\mathcal{B}})$ is a vertical side of $\mathrm{pr}(\widetilde{\mathcal{B}})$ and that $\mathrm{CS}'(\widetilde{\mathcal{B}})$ is the set of unit tangent vectors based on $b(\widetilde{\mathcal{B}})$ that point into $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ$. It does not use that $\{\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}})) \mid \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S}}\}$ is the set of all cells in H nor does one need that for some cells $\widetilde{\mathcal{B}}_1, \widetilde{\mathcal{B}}_2 \in \widetilde{\mathbb{B}}_{\mathbb{S}}$ one has $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_1)) = \mathcal{B}(\mathrm{pr}(\widetilde{\mathcal{B}}_2))$. This means that one has the freedom to perform (horizontal) translations of single cells in SH by elements in Γ_∞. The following definition is motivated by this fact. We will see that in some situations the family of shifted cells in SH will induce a symbolic dynamics which has a generating function for the future part while the symbolic dynamics that is constructed from the original family of cells in SH has not.

Definition 6.6.25. Each map $\mathbb{T}\colon \widetilde{\mathbb{B}}_{\mathbb{S}} \to \Gamma_\infty$ ("translation") is called a *shift map* for $\widetilde{\mathbb{B}}_{\mathbb{S}}$. The family

$$\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} := \big\{ \mathbb{T}(\widetilde{\mathcal{B}})\widetilde{\mathcal{B}} \;\big|\; \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S}} \big\}$$

is called the *family of cells in SH associated to $\mathbb{A}$, $\mathbb{S}$ and $\mathbb{T}$.* Each element of $\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ is called a *shifted cell in SH.*

For each $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ define $b(\widetilde{\mathcal{B}}) := \mathrm{pr}(\widetilde{\mathcal{B}}) \cap \partial\,\mathrm{pr}(\widetilde{\mathcal{B}})$ and let $\mathrm{CS}'(\widetilde{\mathcal{B}})$ be the set of unit tangent vectors in $\widetilde{\mathcal{B}}$ that are based on $b(\widetilde{\mathcal{B}})$ but do not point along $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}})$.

Let $\mathbb{T}$ be a shift map for $\widetilde{\mathbb{B}}_{\mathbb{S}}$.

Remark 6.6.26. The results of Prop. 6.6.18 and 6.6.21-6.6.23 remain true for $\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ after the obvious changes. More precisely, the union $\bigcup \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ is disjoint and a fundamental set for Γ in SH, and if $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S}}$, then $\mathrm{pr}\big(\mathbb{T}(\widetilde{\mathcal{B}})\widetilde{\mathcal{B}}\big) = \mathbb{T}(\widetilde{\mathcal{B}})\,\mathrm{pr}(\widetilde{\mathcal{B}})$ and $b\big(\mathbb{T}(\widetilde{\mathcal{B}})\widetilde{\mathcal{B}}\big) = \mathbb{T}(\widetilde{\mathcal{B}})b(\widetilde{\mathcal{B}})$. Then $\mathrm{CS}'\big(\mathbb{T}(\widetilde{\mathcal{B}})\widetilde{\mathcal{B}}\big)$ is the set of unit tangent vectors based on $\mathbb{T}(\widetilde{\mathcal{B}})b(\widetilde{\mathcal{B}})$ that point into $\mathbb{T}(\widetilde{\mathcal{B}})\,\mathrm{pr}(\widetilde{\mathcal{B}})^\circ$.

6.7. Geometric symbolic dynamics

Let Γ be a geometrically finite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of which ∞ is a cuspidal point and which satisfies (A2). Suppose that the set of relevant isometric spheres is non-empty. Let $\mathbb{A}$ be a basal family of precells in H and denote the family of cells in H assigned to $\mathbb{A}$ by $\mathbb{B}$. Suppose that $\mathbb{S}$ is a set of choices associated to $\mathbb{A}$ and let $\widetilde{\mathbb{B}}_{\mathbb{S}}$ be the family of cells in SH associated to $\mathbb{A}$ and $\mathbb{S}$. Fix a shift map $\mathbb{T}$ for $\widetilde{\mathbb{B}}_{\mathbb{S}}$ and denote the family of cells in SH associated to $\mathbb{A}, \mathbb{S}$ and $\mathbb{T}$ by $\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. Recall the set $\mathrm{CS}'(\widetilde{\mathcal{B}})$ for $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ from Def. 6.6.25. We set

$$\mathrm{CS}'\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) := \bigcup_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} \mathrm{CS}'(\widetilde{\mathcal{B}}) \quad\text{and}\quad \widehat{\mathrm{CS}}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) := \pi\big(\,\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})\big).$$

In Sec. 6.7.1, we will use the results from Sec. 6.6 to show that $\widehat{\mathrm{CS}}$ satisfies (C1) and hence is a cross section for the geodesic flow on Y w. r. t. certain measures μ. It will turn out that the measures μ are characterized by the condition that NIC (see Remark 6.5.5) be a μ-null set. We start by showing that $\widehat{\mathrm{CS}} = \widehat{\mathrm{CS}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. This suffices to characterize the geodesics on Y which intersect $\widehat{\mathrm{CS}}$ at all. Then we investigate the location of the endpoints in $\partial_g H$ of the geodesics on H determined by the elements in $\mathrm{CS}'(\widetilde{\mathcal{B}})$, $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. This result and the fact that $\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is a set of representatives for $\widehat{\mathrm{CS}}$ allows us to provide a rather explicit description of the structure how $\mathrm{CS} = \pi^{-1}(\widehat{\mathrm{CS}})$ (see Sec. 6.5) decomposes into Γ-translates of the sets $\mathrm{CS}'(\widetilde{\mathcal{B}})$, $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. These investigations culminate, in Prop. 6.7.12, in the determination of the location of next and previous points of intersection of a geodesic γ on H and the set CS. The purpose of Prop. 6.7.12 is two-fold. At first we will use it to decide which geodesics on Y intersect $\widehat{\mathrm{CS}}$ infinitely often in future and past and for the determination of the maximal strong cross section contained in $\widehat{\mathrm{CS}}$. In Sec. 6.7.2, the results of Prop. 6.7.12 will allow to define a natural labeling of $\widehat{\mathrm{CS}}$ and a natural symbolic dynamics for the geodesic flow on Y.

6.7.1. Geometric cross section

Recall the set BS from Sec. 6.5.

Lemma 6.7.1. *Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. Then $\mathrm{pr}(\widetilde{\mathcal{B}})$ is a convex polyhedron and $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}})$ consists of complete geodesic segments. Moreover, we have that $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ \cap \mathrm{BS} = \emptyset$ and $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}}) \subseteq \mathrm{BS}$ and $\mathrm{pr}(\widetilde{\mathcal{B}}) \cap \mathrm{BS} = b(\widetilde{\mathcal{B}})$ and that $b(\widetilde{\mathcal{B}})$ is a connected component of BS.*

Proof. Let $\widetilde{\mathcal{B}}_1$ be the (unique) element in $\widetilde{\mathbb{B}}_{\mathbb{S}}$ such that $\widetilde{\mathcal{B}} = \mathbb{T}(\widetilde{\mathcal{B}}_1)\widetilde{\mathcal{B}}_1$. Cor. 6.6.24 states that $\mathcal{B}_1 := \mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_1))$ is a cell in H and that $b(\widetilde{\mathcal{B}}_1)$ is a side of $\mathcal{B}_1$. Moreover, $\mathrm{pr}(\widetilde{\mathcal{B}}_1) = \mathcal{B}_1^\circ \cup b(\widetilde{\mathcal{B}}_1)$ and hence $\mathrm{pr}(\widetilde{\mathcal{B}}_1)^\circ = \mathcal{B}_1^\circ$. Thus, $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}}_1) = \partial\mathcal{B}_1$ consists of complete geodesic segments, $\mathrm{pr}(\widetilde{\mathcal{B}}_1)^\circ \cap \mathrm{BS} = \emptyset$, $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}}_1) \subseteq \mathrm{BS}$ and $\mathrm{pr}(\widetilde{\mathcal{B}}_1) \cap \mathrm{BS} = b(\widetilde{\mathcal{B}}_1)$. Now the statements of the lemma follow from $\mathrm{pr}(\widetilde{\mathcal{B}}) = \mathbb{T}(\widetilde{\mathcal{B}}_1)\,\mathrm{pr}(\widetilde{\mathcal{B}}_1)$ and $b(\widetilde{\mathcal{B}}) = \mathbb{T}(\widetilde{\mathcal{B}}_1)b(\widetilde{\mathcal{B}}_1)$ and the Γ-invariance of BS. □

Proposition 6.7.2. *We have $\widehat{\mathrm{CS}} = \widehat{\mathrm{CS}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Moreover, the union*

$$\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) = \bigcup \left\{ \mathrm{CS}'(\widetilde{\mathcal{B}}) \;\middle|\; \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \right\}$$

is disjoint and $\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is a set of representatives for $\widehat{\mathrm{CS}}$.

Proof. We start by showing that $\widehat{\mathrm{CS}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) \subseteq \widehat{\mathrm{CS}}$. Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. Then there exists a (unique) $\widetilde{\mathcal{B}}_1 \in \widetilde{\mathbb{B}}_{\mathbb{S}}$ such that $\widetilde{\mathcal{B}} = \mathbb{T}(\widetilde{\mathcal{B}}_1)\widetilde{\mathcal{B}}_1$. Lemma 6.7.1 shows that $b(\widetilde{\mathcal{B}}_1)$ is a connected component of BS. The set $\mathrm{CS}'(\widetilde{\mathcal{B}}_1)$ consists of unit tangent vectors based on $b(\widetilde{\mathcal{B}}_1)$ which are not tangent to it. Therefore, $\mathrm{CS}'(\widetilde{\mathcal{B}}_1) \subseteq \mathrm{CS}$. Now $b(\widetilde{\mathcal{B}}) = \mathbb{T}(\widetilde{\mathcal{B}}_1)\widetilde{\mathcal{B}}_1$ and $\mathrm{CS}'(\widetilde{\mathcal{B}}) = \mathbb{T}(\widetilde{\mathcal{B}}_1)\,\mathrm{CS}'(\widetilde{\mathcal{B}}_1)$ with $\mathbb{T}(\widetilde{\mathcal{B}}_1) \in \Gamma$. Thus, we see that $\pi(b(\widetilde{\mathcal{B}})) \subseteq \pi(\mathrm{BS}) = \widehat{\mathrm{BS}}$ and $\pi(\mathrm{CS}'(\widetilde{\mathcal{B}})) \subseteq \pi(\mathrm{CS}) = \widehat{\mathrm{CS}}$. This shows that $\widehat{\mathrm{CS}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) \subseteq \widehat{\mathrm{CS}}$.

Conversely, let $\widehat{v} \in \widehat{\mathrm{CS}}$. We will show that there is a unique $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and a unique $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$ such that $\pi(v) = \widehat{v}$. Pick any $w \in \pi^{-1}(v)$. Remark 6.6.26

shows that the set $\mathcal{P} := \bigcup\{\widetilde{\mathcal{B}} \mid \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\}$ is a fundamental set for Γ in SH. Hence there exists a unique pair $(\widetilde{\mathcal{B}}, g) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma$ such that $v := gw \in \widetilde{\mathcal{B}}$. Note that $\pi^{-1}(\widehat{\mathrm{CS}}) = \mathrm{CS}$. Thus, $v \in \mathrm{CS}$ and hence $\mathrm{pr}(v) \in \mathrm{pr}(\widetilde{\mathcal{B}}) \cap \mathrm{BS}$. Lemma 6.7.1 shows that $\mathrm{pr}(v) \in b(\widetilde{\mathcal{B}})$. Therefore, $v \in \pi^{-1}(b(\widetilde{\mathcal{B}})) \cap \widetilde{\mathcal{B}}$. Since $v \in \mathrm{CS}$, it does not point along $b(\widetilde{\mathcal{B}})$. Hence v does not point along $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}})$, which shows that $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$. This proves that $\widehat{\mathrm{CS}} \subseteq \widehat{\mathrm{CS}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.

To see the uniqueness of $\widetilde{\mathcal{B}}$ and v suppose that $w_1 \in \pi^{-1}(\widehat{v})$. Let $(\widetilde{\mathcal{B}}_1, g_1) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma$ be the unique pair such that $g_1 w_1 \in \widetilde{\mathcal{B}}_1$. There exists a unique element $h \in \Gamma$ such that $hw = w_1$. Then $g_1 h g^{-1} v = g w_1$ and $v, g_1 h g^{-1} v \in \mathcal{P}$. Now $\mathcal{P}$ being a fundamental set shows that $g_1 h g^{-1} = \mathrm{id}$, which proves that $g_1 w_1 = g_1 h w = gw = v$ and $\widetilde{\mathcal{B}}_1 = \widetilde{\mathcal{B}}$. This completes the proof. □

Corollary 6.7.3. *Let $\widehat{\gamma}$ be a geodesic on Y which intersects $\widehat{\mathrm{CS}}$ in t. Then there is a unique geodesic γ on H which intersects $\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ in t such that $\pi(\gamma) = \widehat{\gamma}$.*

Definition 6.7.4. Let $\widehat{\gamma}$ be a geodesic on Y which intersects $\widehat{\mathrm{CS}}$ in $\widehat{\gamma}'(t_0)$. If

$$s := \min\left\{t > t_0 \;\middle|\; \widehat{\gamma}'(t) \in \widehat{\mathrm{CS}}\right\}$$

exists, we call s the *first return time* of $\widehat{\gamma}'(t_0)$ and $\widehat{\gamma}'(s)$ the *next point of intersection of $\widehat{\gamma}$ and* $\widehat{\mathrm{CS}}$. Let γ be a geodesic on H. If $\gamma'(t) \in \mathrm{CS}$, then we say that γ *intersects* CS *in* t. If there is a sequence $(t_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = \infty$ and $\gamma'(t_n) \in \mathrm{CS}$ for all $n \in \mathbb{N}$, then γ is said to *intersect* CS *infinitely often in future*. Analogously, if we find a sequence $(t_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = -\infty$ and $\gamma'(t_n) \in \mathrm{CS}$ for all $n \in \mathbb{N}$, then γ is said to *intersect* CS *infinitely often in past*. Suppose that γ intersects CS in t_0. If

$$s := \min\left\{t > t_0 \;\middle|\; \gamma'(t) \in \mathrm{CS}\right\}$$

exists, we call s the *first return time* of $\gamma'(t_0)$ and $\gamma'(s)$ the *next point of intersection of γ and* CS. Analogously, we define the *previous point of intersection* of $\widehat{\gamma}$ and $\widehat{\mathrm{CS}}$ resp. of γ and CS.

Remark 6.7.5. A geodesic $\widehat{\gamma}$ on Y intersects $\widehat{\mathrm{CS}}$ if and only if some (and hence any) representative of $\widehat{\gamma}$ on H intersects $\pi^{-1}(\widehat{\mathrm{CS}})$. Recall that $\mathrm{CS} = \pi^{-1}(\widehat{\mathrm{CS}})$, and that CS is the set of unit tangent vectors based on BS but which are not tangent to BS. Since BS is a totally geodesic submanifold of H (see Prop. 6.5.3), a geodesic γ on H intersects CS if and only if γ intersects BS transversely. Again because BS is totally geodesic, the geodesic γ intersects BS transversely if and only if γ intersects BS and is not contained in BS. Therefore, a geodesic $\widehat{\gamma}$ on Y intersects $\widehat{\mathrm{CS}}$ if and only if some (and thus any) representative γ of $\widehat{\gamma}$ on H intersects BS and $\gamma(\mathbb{R}) \not\subseteq \mathrm{BS}$.

A similar argument simplifies the search for previous and next points of intersection. To make this precise, suppose that $\widehat{\gamma}$ is a geodesic on Y which intersects $\widehat{\mathrm{CS}}$ in $\widehat{\gamma}'(t_0)$ and that γ is a representative of $\widehat{\gamma}$ on H. Then $\gamma'(t_0) \in \mathrm{CS}$. There is a next point of intersection of $\widehat{\gamma}$ and $\widehat{\mathrm{CS}}$ if and only if there is a next point of intersection of γ and CS. The hypothesis that $\gamma'(t_0) \in \mathrm{CS}$ implies that $\gamma(\mathbb{R})$ is not contained in BS. Hence each intersection of γ and BS is transversal. Then there is a next point of intersection of γ and CS if and only if $\gamma((t_0,\infty))$ intersects BS. Suppose that there is a next point of intersection. If $\gamma'(s)$ is the next point of intersection of γ and CS, then and only then $\widehat{\gamma}'(s)$ is the next point of intersection of $\widehat{\gamma}$ and $\widehat{\mathrm{CS}}$. In this case, $s = \min\{t > t_0 \mid \gamma(t) \in \mathrm{BS}\}$.

Likewise, there was a previous point of intersection of $\widehat{\gamma}$ and $\widehat{\mathrm{CS}}$ if and only if there was a previous point of intersection of γ and CS. Further, there was a previous point of intersection of γ and CS if and only if $\gamma((-\infty,t_0))$ intersects BS. If there was a previous point of intersection, then $\gamma'(s)$ is the previous point of intersection of γ and CS if and only if $\widehat{\gamma}'(s)$ was the previous point of intersection of $\widehat{\gamma}$ and $\widehat{\mathrm{CS}}$. Moreover, $s = \max\{t < t_0 \mid \gamma(t) \in \mathrm{BS}\}$.

Prop. 6.7.7 provides a characterization of the geodesics on Y which intersect $\widehat{\mathrm{CS}}$ at all. Its proof needs the following lemma.

Lemma 6.7.6. *Let U be a convex polyhedron in H and γ a geodesic on H.*

(i) *Suppose that $t \in \mathbb{R}$ such that $\gamma(t) \in \partial U$. If $\gamma((t,\infty)) \subseteq U$, then either there is a unique side S of U such that $\gamma((t,\infty)) \subseteq S$ or $\gamma((t,\infty)) \subseteq U^\circ$.*

(ii) *Suppose that $t_1, t_2, t_3 \in \mathbb{R}$ such that $t_1 < t_2 < t_3$ and $\gamma(t_1), \gamma(t_2), \gamma(t_3) \in \partial U$. Then there is a side S of U such that $S \subseteq \gamma(\mathbb{R})$.*

(iii) *If $\gamma(\pm\infty) \in \partial_g U$, then either $\gamma(\mathbb{R}) \subseteq U^\circ$ or $\gamma(\mathbb{R}) \subseteq \partial U$. If $\gamma(t) \in U$ and $\gamma(\infty) \in \partial_g U$, then either $\gamma((t,\infty)) \subseteq U^\circ$ or $\gamma([t,\infty)) \subseteq \partial U$.*

Proof. We will use the following specialization of [Rat06, Thm. 6.3.8]: Suppose that s is a non-trivial geodesic segment with endpoints a, b (possibly in $\partial_g H$) which is contained in U. If there is a side S of U such that $s \smallsetminus \{a,b\}$ intersects S, then $s \subseteq S$.

For (i) suppose that there exists $t_1 \in (t,\infty)$ such that $\gamma(t_1) \in \partial U$. If $\gamma(t_1)$ is an endpoint of some side of U, then there are two sides S_1, S_2 of U which have $\gamma(t_1)$ as an endpoint. Assume for contradiction that $S_1, S_2 \subseteq \gamma(\mathbb{R})$. Since $\gamma(t_1) \in S_1 \cap S_2$, the union $T := S_1 \cup S_2$ is a geodesic segment in ∂U and hence S is contained in a side of U. This contradicts to $\gamma(t_1)$ being an endpoint of the sides S_1 and S_2. Suppose that $S_1 \not\subseteq \gamma(\mathbb{R})$. Let $\langle S_1\rangle$ be the complete geodesic segment which contains S_1. Then $\langle S_1\rangle$ divides H into two closed halfplanes H_1 and H_2 (with $H_1 \cap H_2 = \langle S_1\rangle$) one of which contains $\gamma(t)$, say H_1. Now $\gamma(\mathbb{R})$ intersects $\langle S_1\rangle$ transversely in $\gamma(t_1)$. Since $t_1 > t$, the segment $\gamma((t_1,\infty))$ is contained in H_2. This contradicts to $\gamma((t,\infty)) \subseteq U$. Hence $\gamma(t_1)$ is not an endpoint of some side of U. Let S be the unique side of U with $\gamma(t_1) \in S$. Then $S \subseteq \gamma((t,\infty))$. The previous argument shows that $\gamma((t,\infty))$ does not contain an endpoint of S, hence $\gamma((t,\infty)) \subseteq S$. Finally, since S is closed, $\gamma([t,\infty)) \subseteq S$.

For (ii) let $s := [\gamma(t_1), \gamma(t_3)]$. Since $\gamma(t_1)$ and $\gamma(t_3)$ are in U, the convexity of U shows that $s \subseteq U$. Now $\gamma(t_2) \in (\gamma(t_1), \gamma(t_3)) \cap \partial U$. As in the proof of (i) it follows that $\gamma(t_2)$ is not an endpoint of some side of U. Let S be the unique side of U with $\gamma(t_2) \in S$. Then $s \subseteq S$. Since the geodesic segment S

contains (at least) two point of the complete geodesic segment $\gamma(\mathbb{R})$, it follows that $S \subseteq \gamma(\mathbb{R})$.

For (iii) it suffices to show that $\gamma(\mathbb{R}) \subseteq U$ resp. that $\gamma((t,\infty)) \subseteq U$. This follows from [Rat06, Thm. 6.4.2]. $\square$

Proposition 6.7.7. *Let $\widehat{\gamma}$ be a geodesic on Y. Then $\widehat{\gamma}$ intersects $\widehat{\mathrm{CS}}$ if and only if $\widehat{\gamma} \notin \mathrm{NC}$.*

Proof. Let $\mathbb{B}$ be the family of cells in H assigned to $\mathbb{A}$. Recall from Prop. 6.5.2 that $\mathrm{NC} = \mathrm{NC}(\mathbb{B})$. Suppose first that $\widehat{\gamma} \in \mathrm{NC}$. Then we find $\mathcal{B} \in \mathbb{B}$ and a representative γ of $\widehat{\gamma}$ on H such that $\gamma(\pm\infty) \in \mathrm{bd}(\mathcal{B})$. Since $\mathcal{B}$ is a convex polyhedron and $\gamma(\pm\infty) \in \partial_g \mathcal{B}$, Lemma 6.7.6(iii) states that either $\gamma(\mathbb{R}) \subseteq \mathcal{B}^\circ$ or $\gamma(\mathbb{R}) \subseteq \partial\mathcal{B}$. Cor. 6.4.18 shows that $B^\circ \cap \mathrm{BS} = \emptyset$ and $\partial\mathcal{B} \subseteq \mathrm{BS}$. Thus, either $\gamma(\mathbb{R})$ does not intersect BS or $\gamma(\mathbb{R}) \subseteq \mathrm{BS}$. Remark 6.7.5 shows that in both cases γ does not intersect CS, and therefore $\widehat{\gamma}$ does not intersect $\widehat{\mathrm{CS}}$.

Suppose now that $\widehat{\gamma}$ does not intersect $\widehat{\mathrm{CS}}$. Then each representative of $\widehat{\gamma}$ on H does not intersect CS. Let γ be any representative of $\widehat{\gamma}$ on H. We will show that there is a cell $\mathcal{B}$ in H and a to γ equivalent geodesic η such that $\eta(\pm\infty) \in \mathrm{bd}(\mathcal{B})$. Pick a unit tangent vector v to γ. Recall from Prop. 6.6.18 that $\bigcup\{\widetilde{\mathcal{B}} \mid \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S}}\}$ is a fundamental set for Γ in SH. Thus, there is a pair $(\widetilde{\mathcal{B}}, g) \in \widetilde{\mathbb{B}}_{\mathbb{S}} \times \Gamma$ such that $gv \in \widetilde{\mathcal{B}}$. Set $\eta := g\gamma$. Lemma 6.7.1 states that $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}})$ consists of complete geodesic segments and $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}}) \subseteq \mathrm{BS}$. By assumption, η does not intersect BS transversely, which implies that η does not intersect $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}})$ transversely. Because $\eta(\mathbb{R}) \cap \mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}})) \neq \emptyset$, it follows that $\eta(\mathbb{R}) \subseteq \mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$. Thus, $\eta(\pm\infty) \in \partial_g\,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$. By Cor. 6.6.24, $\mathcal{B} := \mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$ is a cell in H. Therefore $\eta(\pm\infty) \in \mathrm{bd}(\mathcal{B})$, which shows that $\widehat{\gamma} = \widehat{\eta} \in \mathrm{NC}(\mathcal{B}) \subseteq \mathrm{NC}$. $\square$

Suppose that we are given a geodesic $\widehat{\gamma}$ on Y which intersects $\widehat{\mathrm{CS}}$ in $\widehat{\gamma}'(t_0)$ and suppose that γ is the unique geodesic on H which intersects $\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ in $\gamma'(t_0)$ and which satisfies $\pi(\gamma) = \widehat{\gamma}$. Our next goal is to characterize when there is a next point of intersection of $\widehat{\gamma}$ and $\widehat{\mathrm{CS}}$ resp. of γ and CS, and, if there is one, where this point is located. Further we will do analogous investigations on the existence and location of previous points of intersections. To this end we need the following preparations.

Definition 6.7.8. Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and suppose that $b(\widetilde{\mathcal{B}})$ is the complete geodesic segment (a,∞) with $a \in \mathbb{R}$. We assign to $\widetilde{\mathcal{B}}$ two intervals $I(\widetilde{\mathcal{B}})$ and $J(\widetilde{\mathcal{B}})$ which are given as follows:

$$I(\widetilde{\mathcal{B}}) := \begin{cases} (a,\infty) & \text{if } \mathrm{pr}(\widetilde{\mathcal{B}}) \subseteq \{z \in H \mid \operatorname{Re} z \geq a\}, \\ (-\infty,a) & \text{if } \mathrm{pr}(\widetilde{\mathcal{B}}) \subseteq \{z \in H \mid \operatorname{Re} z \leq a\}, \end{cases}$$

and

$$J(\widetilde{\mathcal{B}}) := \begin{cases} (-\infty,a) & \text{if } \mathrm{pr}(\widetilde{\mathcal{B}}) \subseteq \{z \in H \mid \operatorname{Re} z \geq a\}, \\ (a,\infty) & \text{if } \mathrm{pr}(\widetilde{\mathcal{B}}) \subseteq \{z \in H \mid \operatorname{Re} z \leq a\}. \end{cases}$$

Note that the combination of Remark 6.6.26 with Prop. 6.4.11(i) and 6.6.21 resp. with Prop. 6.4.12(i) and 6.6.22 resp. with Remark 6.2.10 and Prop. 6.6.23 shows that indeed each $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ gets assigned a pair $\big(I(\widetilde{\mathcal{B}}), J(\widetilde{\mathcal{B}})\big)$ of intervals.

Lemma 6.7.9. *Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. For each $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$ let γ_v denote the geodesic on H determined by v. If $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$, then $(\gamma_v(\infty), \gamma_v(-\infty)) \in I(\widetilde{\mathcal{B}}) \times J(\widetilde{\mathcal{B}})$. Conversely, if $(x,y) \in I(\widetilde{\mathcal{B}}) \times J(\widetilde{\mathcal{B}})$, then there exists a unique element v in $\mathrm{CS}'(\widetilde{\mathcal{B}})$ such that $(\gamma_v(\infty), \gamma_v(-\infty)) = (x,y)$.*

Proof. Let $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$. By Prop. 6.6.21 resp. 6.6.22 resp. 6.6.23 (recall Remark 6.6.26), the unit tangent vector v points into $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ$ and $\mathrm{pr}(v) \in b(\widetilde{\mathcal{B}})$. By definition we find $\varepsilon > 0$ such that $\gamma_v((0,\varepsilon)) \subseteq \mathrm{pr}(\widetilde{\mathcal{B}})^\circ$. Then $\gamma_v(\mathbb{R})$ intersects $b(\widetilde{\mathcal{B}})$ in $\gamma_v(0) = \mathrm{pr}(v)$. From $\gamma_v(\varepsilon/2) \in \mathrm{pr}(\widetilde{\mathcal{B}})^\circ$ and hence $\gamma_v(\varepsilon/2) \notin b(\widetilde{\mathcal{B}})$, it follows that $\gamma_v(\mathbb{R}) \neq b(\widetilde{\mathcal{B}})$. Since $\gamma_v(\mathbb{R})$ and $b(\widetilde{\mathcal{B}})$ are both complete geodesic segments, this shows that $\mathrm{pr}(v)$ is the only intersection point of $\gamma_v(\mathbb{R})$ and $b(\widetilde{\mathcal{B}})$. Now $b(\widetilde{\mathcal{B}})$ divides H into two closed half-spaces H_1 and H_2 (with $H_1 \cap H_2 = b(\widetilde{\mathcal{B}})$) one of which contains $\mathrm{pr}(\widetilde{\mathcal{B}})$, say $\mathrm{pr}(\widetilde{\mathcal{B}}) \subseteq H_1$. Then $\gamma_v((0,\infty)) \subseteq H_1$ and $\gamma_v((-\infty,0)) \subseteq H_2$. The definition of $I(\widetilde{\mathcal{B}})$ and $J(\widetilde{\mathcal{B}})$ shows that $\big(\gamma_v(\infty), \gamma_v(-\infty)\big) \in I(\widetilde{\mathcal{B}}) \times J(\widetilde{\mathcal{B}})$.

Conversely, let $(x,y) \in I(\widetilde{\mathcal{B}}) \times J(\widetilde{\mathcal{B}})$. Suppose that $b(\widetilde{\mathcal{B}})$ is the geodesic segment (a,∞) and suppose w.l.o.g. that $I(\widetilde{\mathcal{B}})$ is the interval (a,∞) and $J(\widetilde{\mathcal{B}})$ the interval $(-\infty,a)$. Let c denote the complete geodesic segment $[x,y]$. Since $x > a > y$, the geodesic segment c intersects $b(\widetilde{\mathcal{B}})$ in a (unique) point z. There are exactly two unit tangent vectors w_j, $j = 1,2$, to c at z. For $j \in \{1,2\}$ let γ_{w_j} denote the geodesic on H determined by w_j. Then $\gamma_{w_j}(\mathbb{R}) = c$ and

$$\big(\gamma_{w_1}(\infty), \gamma_{w_1}(-\infty)\big) = \big(\gamma_{w_2}(\infty), \gamma_{w_2}(-\infty)\big)$$

with

$$\big(\gamma_{w_1}(\infty), \gamma_{w_1}(-\infty)\big) = (x,y) \quad \text{or} \quad \big(\gamma_{w_1}(\infty), \gamma_{w_1}(-\infty)\big) = (y,x).$$

W.l.o.g. suppose that $\big(\gamma_{w_1}(\infty), \gamma_{w_1}(-\infty)\big) = (x,y)$ and set $v := w_1$. We will show that v points into $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ$. The set $b(\widetilde{\mathcal{B}})$ is a side of $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$ and, since $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$ is a convex polyhedron with non-empty interior, $b(\widetilde{\mathcal{B}})$ is a side of $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ$, hence $b(\widetilde{\mathcal{B}}) \subseteq \partial\, \mathrm{pr}(\widetilde{\mathcal{B}})^\circ$. Since z is not an endpoint of $b(\widetilde{\mathcal{B}})$, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(z) \cap \mathrm{pr}\left(\widetilde{\mathcal{B}}\right)^\circ = B_\varepsilon(z) \cap \{z \in H \mid \mathrm{Re}\, z > a\}.$$

Now $\gamma_v((0,\infty)) \subseteq \{z \in H \mid \mathrm{Re}\, z > a\}$ with $\gamma_v(0) = z$. Hence there is $\delta > 0$ such that

$$\gamma_v\big((0,\delta)\big) \subseteq B_\varepsilon(z) \cap \{z \in H \mid \mathrm{Re}\, z > a\}.$$

Thus $\gamma_v((0,\delta)) \subseteq \mathrm{pr}(\widetilde{\mathcal{B}})^\circ$, which means that v point into $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ$. Then Prop. 6.6.21 resp. 6.6.22 resp. 6.6.23 states that $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$. This completes the proof. $\square$

Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and $g \in \Gamma$. Suppose that $I(\widetilde{\mathcal{B}}) = (a, \infty)$. Then

$$gI(\widetilde{\mathcal{B}}) = \begin{cases} (ga, g\infty) & \text{if } ga < g\infty, \\ (ga, \infty] \cup (-\infty, g\infty) & \text{if } g\infty < ga, \end{cases}$$

and

$$gJ(\widetilde{\mathcal{B}}) = \begin{cases} (ga, \infty] \cup (-\infty, g\infty) & \text{if } ga < g\infty, \\ (ga, g\infty) & \text{if } g\infty < ga. \end{cases}$$

Here, the interval $(b, \infty]$ denotes the union of the interval (b, ∞) with the point $\infty \in \partial_g H$. Hence, the set $I := (b, \infty] \cup (-\infty, c)$ is connected as a subset of $\partial_g H$. The interpretation of I is more eluminating in the ball model: Via the Cayley transform $\mathcal{C}$ the set $\partial_g H$ is homeomorphic to the unit sphere S^1. Let $b' := \mathcal{C}(b)$, $c' := \mathcal{C}(c)$ and $I' := \mathcal{C}(I)$. Then I' is the connected component of $S^1 \smallsetminus \{b', c'\}$ which contains $\mathcal{C}(\infty)$.

Suppose now that $I(\widetilde{\mathcal{B}}) = (-\infty, a)$. Then

$$gI(\widetilde{\mathcal{B}}) = \begin{cases} (-\infty, ga) \cup (g(-\infty), \infty] & \text{if } ga < g(-\infty), \\ (g(-\infty), ga) & \text{if } g(-\infty) < ga, \end{cases}$$

and

$$gJ(\widetilde{\mathcal{B}}) = \begin{cases} (g(-\infty), ga) & \text{if } ga < g(-\infty), \\ (-\infty, ga) \cup (g(-\infty), \infty] & \text{if } g(-\infty) < ga. \end{cases}$$

Note that for $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ we have

$$g(-\infty) = \lim_{t \searrow -\infty} \frac{\alpha t + \beta}{\gamma t + \delta} = \lim_{s \nearrow 0} \frac{\alpha + \beta s}{\gamma + \delta s} = \lim_{s \searrow 0} \frac{\alpha + \beta s}{\gamma + \delta s} = g\infty.$$

In particular, $\mathrm{id}(-\infty) = \infty$.

Let $a, b \in \overline{\mathbb{R}}$. For abbreviation we set $(a,b)_+ := (\min(a,b), \max(a,b))$ and $(a,b)_- := (\max(a,b), \infty] \cup (-\infty, \min(a,b))$.

Proposition 6.7.10. *Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and suppose that S is a side of $\mathrm{pr}(\widetilde{\mathcal{B}})$. Then there exist exactly two pairs $(\widetilde{\mathcal{B}}_1, g_1), (\widetilde{\mathcal{B}}_2, g_2) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma$ such that $S = g_j b(\widetilde{\mathcal{B}}_j)$. Moreover, $g_1 \operatorname{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_1)) = \operatorname{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$ and $g_2 \operatorname{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_2)) \cap \operatorname{cl}(\mathrm{pr}(\widetilde{\mathcal{B}})) = S$ or vice versa. The union $g_1 \mathrm{CS}'(\widetilde{\mathcal{B}}_1) \cup g_2 \mathrm{CS}'(\widetilde{\mathcal{B}}_2)$ is disjoint and equals the set of all unit tangent vectors in CS that are based on S. Let $a, b \in \partial_g H$ be the endpoints of S. Then $g_1 I(\widetilde{\mathcal{B}}_1) \times g_1 J(\widetilde{\mathcal{B}}_1) = (a,b)_+ \times (a,b)_-$ and $g_2 I(\widetilde{\mathcal{B}}_2) \times g_2 J(\widetilde{\mathcal{B}}_2) = (a,b)_- \times (a,b)_+$ or vice versa.*

Proof. Let D' denote the set of unit tangent vectors in CS that are based on S. By Lemma 6.7.1, S is a connected component of BS. Hence D' is the set of unit tangent vectors based on S but not tangent to S. The complete geodesic segment S divides H into two open half-spaces H_1, H_2 such that H is the disjoint union $H_1 \cup S \cup H_2$. Moreover, $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ$ is contained in H_1 or H_2, say $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ \subseteq H_1$. Then D' decomposes into the disjoint union $D'_1 \cup D'_2$ where D'_j denotes the non-empty set of elements in D' that point into H_j. For $j = 1, 2$ pick $v_j \in D'_j$. Since $\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is a set of representatives for $\widehat{\mathrm{CS}} = \pi(\mathrm{CS})$ (see Prop. 6.7.2), there exists a uniquely determined pair $(\widetilde{\mathcal{B}}_j, g_j) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma$ such that $v_j \in g_j \,\mathrm{CS}'(\widetilde{\mathcal{B}}_j)$. We will show that $S = g_j b(\widetilde{\mathcal{B}}_j)$. Assume for contradiction that $S \neq g_j b(\widetilde{\mathcal{B}}_j)$. Since S and $g_j b(\widetilde{\mathcal{B}}_j)$ are complete geodesic segments, the intersection of S and $g_j b(\widetilde{\mathcal{B}}_j)$ in $\mathrm{pr}(v_j)$ is transversal. Recall that $S \subseteq \partial\,\mathrm{pr}(\widetilde{\mathcal{B}})$ and $b(\widetilde{\mathcal{B}}_j) \subseteq \partial\,\mathrm{pr}(\widetilde{\mathcal{B}}_j)$ and that $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}}')^\circ = \partial\,\mathrm{pr}(\widetilde{\mathcal{B}}')$ for each $\widetilde{\mathcal{B}}' \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(\mathrm{pr}(v_j)) \cap \mathrm{pr}(\widetilde{\mathcal{B}})^\circ = B_\varepsilon(\mathrm{pr}(v_j)) \cap H_1$ and

$$B_\varepsilon\big(\mathrm{pr}(v_j)\big) \cap g_j \,\mathrm{pr}\,\big(\widetilde{\mathcal{B}}_j\big)^\circ \cap H_1 \neq \emptyset.$$

Hence $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ \cap g_j \,\mathrm{pr}(\widetilde{\mathcal{B}}_j)^\circ \neq \emptyset$. Prop. 6.4.15 resp. 6.4.16 resp. 6.4.17 in combination with Remark 6.6.26 shows that $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}})) = g_j \,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_j))$. But then $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}}) = g_j \partial\,\mathrm{pr}(\widetilde{\mathcal{B}}_j)$, which implies that $S = g_j b(\widetilde{\mathcal{B}}_j)$. This is a contradiction to the assumption that $S \neq g_j b(\widetilde{\mathcal{B}}_j)$. Therefore $S = g_j b(\widetilde{\mathcal{B}}_j)$. Then Lemma 6.7.9 implies that $g_j I(\widetilde{\mathcal{B}}_j) \times g_j J(\widetilde{\mathcal{B}}_j)$ equals $(a,b)_+ \times (a,b)_-$ or $(a,b)_- \times (a,b)_+$. On the other hand

$$\partial_g H_1 \times \partial_g H_2 = \big\{\big(\gamma_v(\infty), \gamma_v(-\infty)\big) \bigm| v \in D'_1\big\} = \big\{\big(\gamma_v(-\infty), \gamma_v(\infty)\big) \bigm| v \in D'_2\big\}$$

equals $(a,b)_+ \times (a,b)_-$ or $(a,b)_- \times (a,b)_+$. Therefore, again by Lemma 6.7.9, $g_j \,\mathrm{CS}'(\widetilde{\mathcal{B}}_j) = D'_j$. This shows that the union $g_1 \,\mathrm{CS}'(\widetilde{\mathcal{B}}_1) \cup g_2 \,\mathrm{CS}'(\widetilde{\mathcal{B}}_2)$ is disjoint and equals D'.

We have $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}})) \subseteq \overline{H}_1$ and $g_1 \,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_1)) \subseteq \overline{H}_1$ with $S \subseteq \partial\,\mathrm{pr}(\widetilde{\mathcal{B}}) \cap g_1 \partial\,\mathrm{pr}(\widetilde{\mathcal{B}}_1)$. Let $z \in S$. Then there exists $\varepsilon > 0$ such that

$$B_\varepsilon(z) \cap \mathrm{pr}\,\big(\widetilde{\mathcal{B}}\big)^\circ = B_\varepsilon(z) \cap H_1 = B_\varepsilon(z) \cap g_1 \,\mathrm{pr}\,\big(\widetilde{\mathcal{B}}_1\big)^\circ.$$

Hence $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ \cap g_1 \,\mathrm{pr}(\widetilde{\mathcal{B}}_1)^\circ \neq \emptyset$. As above we find that $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}})) = g_1 \,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_1))$. Finally, $g_2 \,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_2)) \subseteq \overline{H}_2$ with

$$S \subseteq g_2 \,\mathrm{cl}\,\big(\mathrm{pr}\,\big(\widetilde{\mathcal{B}}_2\big)\big) \cap \overline{H}_1 \subseteq \overline{H}_2 \cap \overline{H}_1 = S.$$

Hence $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}})) \cap g_2 \,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_2)) = S$. ◻

Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and suppose that S is a side of $\mathrm{pr}(\widetilde{\mathcal{B}})$. Let $(\widetilde{\mathcal{B}}_1, g_1), (\widetilde{\mathcal{B}}_2, g_2)$ be the two elements in $\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma$ such that $S = g_j b(\widetilde{\mathcal{B}}_j)$ and $g_1 \,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_1)) = \mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$ and $g_2 \,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_2)) \cap \mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}})) = S$. Then we define

$$p\big(\widetilde{\mathcal{B}}, S\big) := \big(\widetilde{\mathcal{B}}_1, g_1\big) \quad \text{and} \quad n\big(\widetilde{\mathcal{B}}, S\big) := \big(\widetilde{\mathcal{B}}_2, g_2\big).$$

Remark 6.7.11. Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and suppose that S is a side of $\mathrm{pr}(\widetilde{\mathcal{B}})$. We will show how one effectively finds the elements $p(\widetilde{\mathcal{B}}, S)$ and $n(\widetilde{\mathcal{B}}, S)$. Let

$$\big(\widetilde{\mathcal{B}}_1, k_1\big) := p\big(\widetilde{\mathcal{B}}, S\big) \quad \text{and} \quad \big(\widetilde{\mathcal{B}}_2, k_2\big) := n\big(\widetilde{\mathcal{B}}, S\big).$$

Suppose that $\widetilde{\mathcal{B}}'$ is the (unique) element in $\widetilde{\mathbb{B}}_{\mathbb{S}}$ such that $\mathbb{T}(\widetilde{\mathcal{B}}')\widetilde{\mathcal{B}}' = \widetilde{\mathcal{B}}$ and suppose further that $\widetilde{\mathcal{B}}'_j \in \widetilde{\mathbb{B}}_{\mathbb{S}}$ such that $\mathbb{T}(\widetilde{\mathcal{B}}'_j)\widetilde{\mathcal{B}}'_j = \widetilde{\mathcal{B}}_j$ for $j=1,2$. Set $S' := \mathbb{T}(\widetilde{\mathcal{B}}')^{-1}S$. Then S' is a side of $\operatorname{pr}(\widetilde{\mathcal{B}}')$. For $j=1,2$ we have

$$S' = \mathbb{T}\big(\widetilde{\mathcal{B}}'\big)^{-1}S = \mathbb{T}\big(\widetilde{\mathcal{B}}'\big)^{-1}k_j b\big(\widetilde{\mathcal{B}}_j\big) = \mathbb{T}\big(\widetilde{\mathcal{B}}'\big)^{-1}k_j\mathbb{T}\big(\widetilde{\mathcal{B}}'_j\big)b\big(\widetilde{\mathcal{B}}'_j\big)$$

and

$$k_j \operatorname{cl}\big(\operatorname{pr}\big(\widetilde{\mathcal{B}}_j\big)\big) = k_j\mathbb{T}\big(\widetilde{\mathcal{B}}'_j\big)\operatorname{cl}\big(\operatorname{pr}\big(\widetilde{\mathcal{B}}'_j\big)\big).$$

Moreover, $\operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}})) = \mathbb{T}(\widetilde{\mathcal{B}}')\operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}'))$. Then $k_1\operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}_1)) = \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}))$ is equivalent to

$$\mathbb{T}\big(\widetilde{\mathcal{B}}'\big)^{-1}k_1\mathbb{T}\big(\widetilde{\mathcal{B}}'_1\big)\operatorname{cl}\big(\operatorname{pr}\big(\widetilde{\mathcal{B}}'_1\big)\big) = \operatorname{cl}\big(\operatorname{pr}\big(\widetilde{\mathcal{B}}'\big)\big),$$

and $k_2\operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}_2)) \cap \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}})) = S$ is equivalent to

$$\mathbb{T}\big(\widetilde{\mathcal{B}}'\big)^{-1}k_2\mathbb{T}\big(\widetilde{\mathcal{B}}'_2\big)\operatorname{cl}\big(\operatorname{pr}\big(\widetilde{\mathcal{B}}'_2\big)\big) \cap \operatorname{cl}\big(\operatorname{pr}\big(\widetilde{\mathcal{B}}'\big)\big) = S'.$$

Therefore, $(\widetilde{\mathcal{B}}_1, k_1) = p(\widetilde{\mathcal{B}}, S)$ if and only if $(\widetilde{\mathcal{B}}'_1, \mathbb{T}(\widetilde{\mathcal{B}}')^{-1}k_1\mathbb{T}(\widetilde{\mathcal{B}}'_1)) = p(\widetilde{\mathcal{B}}', S')$, and $(\widetilde{\mathcal{B}}_2, k_2) = n(\widetilde{\mathcal{B}}, S)$ if and only if $(\widetilde{\mathcal{B}}'_2, \mathbb{T}(\widetilde{\mathcal{B}}')^{-1}k_2\mathbb{T}(\widetilde{\mathcal{B}}'_2)) = n(\widetilde{\mathcal{B}}', S')$.

By Cor. 6.6.24, the sets $\mathcal{B}' := \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}'))$ and $\mathcal{B}'_j := \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}'_j))$ are $\mathbb{A}$-cells in H. Suppose first that $\mathcal{B}'$ arises from the non-cuspidal basal precell $\mathcal{A}'$ in H. Then there is a unique element $(\mathcal{A}, h_{\mathcal{A}}) \in \mathbb{S}$ such that for some $h \in \Gamma$ the pair $(\mathcal{A}', h)$ is contained in the cycle in $\mathbb{A} \times \Gamma$ determined by $(\mathcal{A}, h_{\mathcal{A}})$. Necessarily, $\mathcal{A}$ is non-cuspidal. Let $\big((\mathcal{A}_j, h_j)\big)_{j=1,\ldots,k}$ be the cycle in $\mathbb{A} \times \Gamma$ determined by $(\mathcal{A}, h_{\mathcal{A}})$. Then $\mathcal{A}' = \mathcal{A}_m$ for some $m \in \{1, \ldots, \operatorname{cyl}(\mathcal{A})\}$ and hence $\mathcal{B}' = \mathcal{B}(\mathcal{A}_m)$ and $\widetilde{\mathcal{B}}' = \widetilde{\mathcal{B}}_m(\mathcal{A}, h_{\mathcal{A}})$. For $j = 1, \ldots, k$ set $g_1 := \mathrm{id}$ and $g_{j+1} := h_j g_j$. Prop. 6.4.11(iii) states that $\mathcal{B}(\mathcal{A}_m) = g_m\mathcal{B}(\mathcal{A})$ and Prop. 6.4.11(i) shows that S' is the geodesic segment $[g_m g_j^{-1}\infty, g_m g_{j+1}^{-1}\infty]$ for some $j \in \{1, \ldots, k\}$. Then $g_j g_m^{-1}S' = [\infty, h_j^{-1}\infty]$. Let $n \in \{1, \ldots, \operatorname{cyl}(\mathcal{A})\}$ such that $n \equiv j \mod \operatorname{cyl}(\mathcal{A})$. Then $h_n = h_j$ by Lemma 6.6.12. Prop. 6.6.21 shows that $b(\widetilde{\mathcal{B}}_n(\mathcal{A}, h_{\mathcal{A}})) = [\infty, h_n^{-1}\infty] = g_j g_m^{-1}S'$. We claim that $(\widetilde{\mathcal{B}}_j(\mathcal{A}, h_{\mathcal{A}}), g_m g_j^{-1}) = p(\widetilde{\mathcal{B}}', S')$. For this is remains to show that $g_m g_j^{-1}\operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}_j(\mathcal{A}, h_{\mathcal{A}}))) = \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}'))$. Prop. 6.6.21 shows that $\operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}_j(\mathcal{A}, h_{\mathcal{A}}))) = \mathcal{B}(\mathcal{A}_n)$ and Lemma 6.6.12 implies that $\mathcal{B}(\mathcal{A}_n) = \mathcal{B}(\mathcal{A}_j)$. Let v be the vertex of $\mathcal{K}$ to which $\mathcal{A}$ is attached. Then $g_j v \in \mathcal{B}(\mathcal{A}_j)^\circ$ and $g_m g_j^{-1}g_j v = g_m v \in \mathcal{B}(\mathcal{A}_m)^\circ$. Therefore $g_m g_j^{-1}\mathcal{B}(\mathcal{A}_j)^\circ \cap \mathcal{B}(\mathcal{A}_m)^\circ \neq \emptyset$. From Prop. 6.4.15 it follows that $g_m g_j^{-1}\mathcal{B}(\mathcal{A}_j) = \mathcal{B}(\mathcal{A}_m)$. Recall that $\mathcal{B}(\mathcal{A}_m) = \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}'))$. Hence $(\widetilde{\mathcal{B}}_j(\mathcal{A}, h_{\mathcal{A}}), g_m g_j^{-1}) = p(\widetilde{\mathcal{B}}', S')$ and

$$\Big(\mathbb{T}\big(\widetilde{\mathcal{B}}_j(\mathcal{A}, h_{\mathcal{A}})\big)\widetilde{\mathcal{B}}_j\big(\mathcal{A}, h_{\mathcal{A}}\big), \mathbb{T}\big(\widetilde{\mathcal{B}}'\big)g_m g_j^{-1}\mathbb{T}\big(\widetilde{\mathcal{B}}_j(\mathcal{A}, h_{\mathcal{A}})\big)^{-1}\Big) = p\big(\widetilde{\mathcal{B}}, S\big).$$

Analogously one proceeds if $\mathcal{A}'$ is cuspidal or a strip precell.

Now we show how one determines $(\widetilde{\mathcal{B}}_2, k_2)$. Suppose again that $\mathcal{B}'$ arises from the non-cuspidal basal precell $\mathcal{A}'$ in H. We use the notation from the determination of $p(\widetilde{\mathcal{B}}', S')$. By Cor. 6.2.23 there is a unique pair $(\widehat{\mathcal{A}}, s) \in \mathbb{A} \times \mathbb{Z}$ such that $b(\widetilde{\mathcal{B}}_n(\mathcal{A}, h_{\mathcal{A}})) \cap t_\lambda^s\widehat{\mathcal{A}} \neq \emptyset$ and $t_\lambda^s\widehat{\mathcal{A}} \neq \mathcal{A}_n$. Then $t_\lambda^{-s}g_j g_m^{-1}S'$ is a side of the cell $\mathcal{B}(\widehat{\mathcal{A}})$ in H. As before, we determine $(\widetilde{\mathcal{B}}_3, k_3) \in \widetilde{\mathbb{B}}_{\mathbb{S}} \times \Gamma$ such that $k_3 b(\widetilde{\mathcal{B}}_3) = t_\lambda^{-s}g_j g_m^{-1}S'$

and $k_3\,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_3)) = \mathcal{B}(\widehat{\mathcal{A}})$. Recall that $g_j g_m^{-1} S'$ is a side of $\mathcal{B}(\mathcal{A}_j) = \mathcal{B}(\mathcal{A}_n)$. We have

$$
\begin{aligned}
g_m g_j^{-1} t_\lambda^s k_3\,\mathrm{cl}\left(\mathrm{pr}\left(\widetilde{\mathcal{B}}_3\right)\right) \cap \mathrm{cl}\left(\mathrm{pr}\left(\widetilde{\mathcal{B}}'\right)\right) &= g_m g_j^{-1} t_\lambda^s \mathcal{B}(\widehat{\mathcal{A}}) \cap \mathcal{B}(\mathcal{A}_m) \\
&= g_m g_j^{-1} t_\lambda^s \mathcal{B}(\widehat{\mathcal{A}}) \cap g_m g_j^{-1} \mathcal{B}(\mathcal{A}_j) \\
&= g_m g_j^{-1} \left(t_\lambda^s \mathcal{B}(\widehat{\mathcal{A}}) \cap \mathcal{B}(\mathcal{A}_j)\right) \\
&= g_m g_j^{-1} \left(t_\lambda^s \mathcal{B}(\widehat{\mathcal{A}}) \cap \mathcal{B}(\mathcal{A}_n)\right) \\
&= g_m g_j^{-1} g_j g_m^{-1} S' \\
&= S'.
\end{aligned}
$$

Thus $n(\widetilde{\mathcal{B}}', S') = (\widetilde{\mathcal{B}}_3, g_m g_j^{-1} t_\lambda^s k_3)$ and

$$n(\widetilde{\mathcal{B}}, S) = \left(\mathbb{T}(\widetilde{\mathcal{B}}_3)\widetilde{\mathcal{B}}_3, \mathbb{T}(\widetilde{\mathcal{B}}')g_m g_j^{-1} t_\lambda^s k_3 \mathbb{T}(\widetilde{\mathcal{B}}_3)^{-1}\right).$$

If $\mathcal{B}'$ arises from a cuspidal or strip precell in H, then the construction of $n(\widetilde{\mathcal{B}}, S)$ is analogous.

Proposition 6.7.12. *Let $\widehat{\gamma}$ be a geodesic on Y and suppose that $\widehat{\gamma}$ intersects $\widehat{\mathrm{CS}}$ in $\widehat{\gamma}'(t_0)$. Let γ be the unique geodesic on H such that $\gamma'(t_0) \in \mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and $\pi(\gamma'(t_0)) = \widehat{\gamma}'(t_0)$. Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ be the unique shifted cell in SH such that $\gamma'(t_0) \in \mathrm{CS}'(\widetilde{\mathcal{B}})$.*

(i) *There is a next point of intersection of γ and CS if and only if $\gamma(\infty)$ does not belong to $\partial_g \mathrm{pr}(\widetilde{\mathcal{B}})$.*

(ii) *Suppose that $\gamma(\infty) \notin \partial_g \mathrm{pr}(\widetilde{\mathcal{B}})$. Then there is a unique side S of $\mathrm{pr}(\widetilde{\mathcal{B}})$ intersected by $\gamma((t_0, \infty))$. Suppose that $(\widetilde{\mathcal{B}}_1, g) = n(\widetilde{\mathcal{B}}, S)$. The next point of intersection is on $g\,\mathrm{CS}'(\widetilde{\mathcal{B}}_1)$.*

(iii) *Let $(\widetilde{\mathcal{B}}', h) = n(\widetilde{\mathcal{B}}, b(\widetilde{\mathcal{B}}))$. Then there was a previous point of intersection of γ and CS if and only if $\gamma(-\infty) \notin h\partial_g \mathrm{pr}(\widetilde{\mathcal{B}}')$.*

(iv) *Suppose that $\gamma(-\infty) \notin h\partial_g \mathrm{pr}(\widetilde{\mathcal{B}}')$. Then there is a unique side S of $h\,\mathrm{pr}(\widetilde{\mathcal{B}}')$ intersected by $\gamma((-\infty, t_0))$. Let $(\widetilde{\mathcal{B}}_2, h^{-1}k) = p(\widetilde{\mathcal{B}}', h^{-1}S)$. Then the previous point of intersection was on $k\,\mathrm{CS}'(\widetilde{\mathcal{B}}_2)$.*

Proof. We start by proving (i). Recall from Remark 6.7.5 that there is a next point of intersection of γ and CS if and only if $\gamma((t_0, \infty))$ intersects BS. Since $\gamma'(t_0) \in \mathrm{CS}'(\widetilde{\mathcal{B}})$, Prop. 6.6.21 resp. 6.6.22 resp. 6.6.23 in combination with Remark 6.6.26 shows that $\gamma'(t_0)$ points into $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ$. Lemma 6.7.1 states that $\mathrm{pr}(\widetilde{\mathcal{B}})^\circ \cap \mathrm{BS} = \emptyset$ and $\partial\,\mathrm{pr}(\widetilde{\mathcal{B}}) \subseteq \mathrm{BS}$. Hence $\gamma((t_0, \infty))$ does not intersect BS if and only if $\gamma((t_0, \infty)) \subseteq \mathrm{pr}(\widetilde{\mathcal{B}})^\circ$. In this case, $\gamma(\infty) \in \mathrm{cl}_{\overline{H}^g}(\mathrm{pr}(\widetilde{\mathcal{B}})) \cap \partial_g H = \partial_g \mathrm{pr}(\widetilde{\mathcal{B}})$. Conversely, if $\gamma(\infty) \in \partial_g \mathrm{pr}(\widetilde{\mathcal{B}})$, then Lemma 6.7.6(iii) states that $\gamma((t_0, \infty)) \subseteq \mathrm{pr}(\widetilde{\mathcal{B}})^\circ$ or $\gamma((t_0, \infty)) \subseteq \partial\,\mathrm{pr}(\widetilde{\mathcal{B}})$. In the latter case, Lemma 6.7.1 shows that $\gamma((t_0, \infty)) \subseteq \mathrm{BS}$. Hence, if $\gamma(\infty) \in \partial_g \mathrm{pr}(\widetilde{\mathcal{B}})$, then $\gamma((t_0, \infty)) \subseteq \mathrm{pr}(\widetilde{\mathcal{B}})^\circ$.

Suppose now that $\gamma(\infty) \notin \partial_g \operatorname{pr}(\widetilde{\mathcal{B}})$. The previous argument shows that $\gamma((t_0,\infty))$ intersects $\partial \operatorname{pr}(\widetilde{\mathcal{B}})$, say $\gamma(t_1) \in \partial \operatorname{pr}(\widetilde{\mathcal{B}})$ with $t_1 \in (t_0,\infty)$. If there was an element $t_2 \in (t_0,\infty) \smallsetminus \{t_1\}$ with $\gamma(t_2) \in \partial \operatorname{pr}(\widetilde{\mathcal{B}})$, then Lemma 6.7.6(ii) would imply that there is a side S of $\operatorname{pr}(\widetilde{\mathcal{B}})$ such that $\gamma(\mathbb{R}) = S$, where the equality follows from the fact that S is a complete geodesic segment (see Lemma 6.7.1). But then, by Lemma 6.7.1, $\gamma(\mathbb{R}) \subseteq \mathrm{BS}$, which contradicts to $\gamma'(t_0) \in \mathrm{CS}$. Thus, $\gamma(t_1)$ is the only intersection point of $\partial \operatorname{pr}(\widetilde{\mathcal{B}})$ and $\gamma((t_0,\infty))$. Since $\gamma((t_0,t_1)) \subseteq \operatorname{pr}(\widetilde{\mathcal{B}})^\circ$, $\gamma'(t_0)$ is the next point of intersection of γ and CS. Moreover, $\gamma'(t_1)$ points out of $\operatorname{pr}(\widetilde{\mathcal{B}})$, since otherwise $\gamma((t_1,\infty))$ would intersect $\partial \operatorname{pr}(\widetilde{\mathcal{B}})$ which would lead to a contradiction as before. Prop. 6.7.2 states that there is a unique pair $(\widetilde{\mathcal{B}}_1, g) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma$ such that $\gamma'(t_1) \in g\,\mathrm{CS}'(\widetilde{\mathcal{B}}_1)$. Then $\gamma'(t_1)$ points into $g \operatorname{pr}(\widetilde{\mathcal{B}}_1)^\circ$. Let S be the side of $\operatorname{pr}(\widetilde{\mathcal{B}})$ with $\gamma(t_1) \in S$. By Prop. 6.7.10, either $g \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}_1)) = \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}))$ or $g \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}_1)) \cap \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}})) = S$. In the first case, $\gamma'(t_1)$ points into $\operatorname{pr}(\widetilde{\mathcal{B}})^\circ$, which is a contradiction. Therefore $g \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}_1)) \cap \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}})) = S$, which shows that $(\widetilde{\mathcal{B}}_1, g) = n(\widetilde{\mathcal{B}}, S)$. This completes the proof of (ii).

Let $(\widetilde{\mathcal{B}}', h) = n(\widetilde{\mathcal{B}}, b(\widetilde{\mathcal{B}}))$. Since $\gamma(t_0) \in b(\widetilde{\mathcal{B}})$ and $\gamma'(t_0) \in \mathrm{CS}'(\widetilde{\mathcal{B}})$, Prop. 6.7.10 implies that $\gamma(t_0) \in hb(\widetilde{\mathcal{B}}')$ and $\gamma'(t_0) \notin h\,\mathrm{CS}'(\widetilde{\mathcal{B}}')$. Since $\gamma(\mathbb{R}) \not\subseteq h\partial \operatorname{pr}(\widetilde{\mathcal{B}}')$, the unit tangent vector $\gamma'(t_0)$ points out of $\operatorname{pr}(\widetilde{\mathcal{B}}')$. Because the intersection of $\gamma(\mathbb{R})$ and $hb(\widetilde{\mathcal{B}}')$ is transversal and $\operatorname{pr}(\widetilde{\mathcal{B}}')$ is a convex polyhedron with non-empty interior, $\gamma((t_0 - \varepsilon, t_0)) \cap h \operatorname{pr}(\widetilde{\mathcal{B}})^\circ \neq \emptyset$ for each $\varepsilon > 0$. As before we find that there was a previous point of intersection of γ and CS if and only if $\gamma((-\infty,t_0))$ intersects $\partial \operatorname{pr}(\widetilde{\mathcal{B}}')$ and that this is the case if and only if $\gamma(-\infty) \notin h\partial_g \operatorname{pr}(\widetilde{\mathcal{B}}')$.

Suppose that $\gamma(-\infty) \notin h\partial_g \operatorname{pr}(\widetilde{\mathcal{B}}')$. As before, there is a unique $t_{-1} \in (-\infty, t_0)$ such that $\gamma(t_{-1}) \in h\partial_g \operatorname{pr}(\widetilde{\mathcal{B}}')$. Let S be the side of $h \operatorname{pr}(\widetilde{\mathcal{B}}')$ with $\gamma(t_{-1}) \in S$. Necessarily, $\gamma((t_{-1}, t_0) \subseteq h \operatorname{pr}(\widetilde{\mathcal{B}}')^\circ$, which shows that $\gamma'(t_{-1})$ points into $h \operatorname{pr}(\widetilde{\mathcal{B}}')^\circ$ and that $\gamma'(t_{-1})$ is the previous point of intersection. Let $(\widetilde{\mathcal{B}}_2, k) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma$ be the unique pair such that $\gamma'(t_{-1}) \in k\,\mathrm{CS}'(\widetilde{\mathcal{B}}_2)$ (see Prop. 6.7.2). By Prop. 6.7.10, either $k \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}_2)) = h \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}'))$ or $k \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}_2)) \cap h \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}')) = S$. In the latter case, $\gamma'(t_{-1})$ points out of $h \operatorname{pr}(\widetilde{\mathcal{B}}')^\circ$ which is a contradiction. Hence $h^{-1}k \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}_2) = \operatorname{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}'))$, which shows that $(\widetilde{\mathcal{B}}_2, h^{-1}k) = p(\widetilde{\mathcal{B}}', h^{-1}S)$. ☐

Corollary 6.7.13. *Let $\widehat{\gamma}$ be a geodesic on Y and suppose that $\widehat{\gamma}$ does not intersect $\widehat{\mathrm{CS}}$ infinitely often in future. If $\widehat{\gamma}$ intersects $\widehat{\mathrm{CS}}$ at all, then there exists $t \in \mathbb{R}$ such that $\widehat{\gamma}'(t) \in \widehat{\mathrm{CS}}$ and $\widehat{\gamma}((t,\infty)) \cap \widehat{\mathrm{BS}} = \emptyset$. Analogously, suppose that $\widehat{\eta}$ is a geodesic on Y which does not intersect $\widehat{\mathrm{CS}}$ infinitely often in past. If $\widehat{\eta}$ intersects $\widehat{\mathrm{CS}}$ at all, then there exists $t \in \mathbb{R}$ such that $\widehat{\eta}'(t) \in \widehat{\mathrm{CS}}$ and $\widehat{\eta}((-\infty,t)) \cap \widehat{\mathrm{BS}} = \emptyset$.*

Proof. Since $\widehat{\gamma}$ does not intersect $\widehat{\mathrm{CS}}$ infinitely often in future, we find $s \in \mathbb{R}$ such that $\widehat{\gamma}'((s,\infty)) \cap \widehat{\mathrm{CS}} = \emptyset$. Suppose that $\widehat{\gamma}$ intersects $\widehat{\mathrm{CS}}$. Remark 6.7.5 shows that then $\widehat{\gamma}'((s,\infty)) \cap \widehat{\mathrm{CS}} = \emptyset$ is equivalent to $\widehat{\gamma}((s,\infty)) \cap \widehat{\mathrm{BS}} = \emptyset$. Pick $r \in (s,\infty)$ and let γ be any representative of $\widehat{\gamma}$ on H. Then $\gamma(r) \notin \mathrm{BS}$. Hence there is a pair $(B,g) \in \mathbb{B} \times \Gamma$ such that $\gamma(r) \in gB^\circ$. Since $g\partial B \subseteq \mathrm{BS}$ by the definition of BS, we have $\gamma((s,\infty)) \subseteq gB^\circ$. Since $\widehat{\gamma}$ intersects $\widehat{\mathrm{CS}}$, $\gamma(\mathbb{R})$ intersects $g\partial B$ transversely. Because gB is convex, this intersection is unique, say $\{\gamma(t)\} = \gamma(\mathbb{R}) \cap g\partial B$. Then $\gamma((t,\infty)) \subseteq gB^\circ$. Hence $\gamma'(t) \in \mathrm{CS}$. Thus $\widehat{\gamma}'(t) \in \widehat{\mathrm{CS}}$ and $\widehat{\gamma}((t,\infty)) \cap \widehat{\mathrm{BS}} = \emptyset$.

The proof of the claims on $\widehat{\eta}$ is analogous. $\square$

Example 6.7.14. For the Hecke triangle group G_5 with $\mathbb{A} = \{\mathcal{A}\}$, $\mathbb{S} = \{(\mathcal{A}, U_5)\}$ (see Example 6.6.16) and $\mathbb{T} \equiv \mathrm{id}$, Fig. 6.19 shows the translates of $\mathrm{CS}' := \mathrm{CS}'(\widetilde{\mathcal{B}})$ which are necessary to determine the location of next and previous points of intersection.

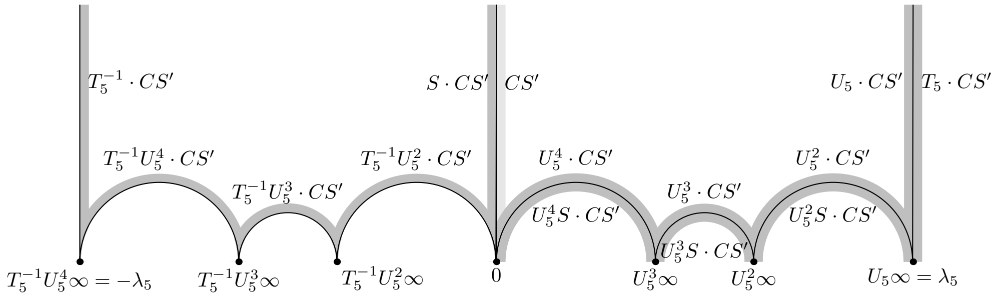


Figure 6.19: The shaded parts are translates of CS' (in unit tangent bundle) as indicated.

Example 6.7.15. Recall the setting of Example 6.6.19. We consider the two shift maps $\mathbb{T}_1 \equiv \mathrm{id}$, and

$$\mathbb{T}_2(\widetilde{\mathcal{B}}_1) := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbb{T}_2(\widetilde{\mathcal{B}}_j) := \mathrm{id} \quad \text{for } j = 2, \ldots, 6.$$

For simplicity set $\widetilde{\mathcal{B}}_{-1} := \mathbb{T}_2(\widetilde{\mathcal{B}}_1)\widetilde{\mathcal{B}}_1$ and $\mathrm{CS}'_{-1} := \mathbb{T}_2(\widetilde{\mathcal{B}}_1)\,\mathrm{CS}'_1$. Further we set

$$g_1 := \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}, \quad g_2 := \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}, \quad g_3 := \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}, \quad g_4 := \begin{pmatrix} 4 & -1 \\ 5 & -1 \end{pmatrix},$$
$$g_5 := \begin{pmatrix} 4 & -5 \\ 5 & -6 \end{pmatrix}, \quad g_6 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_7 := \begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix}.$$

Fig. 6.20 shows the translates of the sets CS'_j which are necessary to determine the location of the next point of intersection if the shift map is $\mathbb{T}_1$, and Fig. 6.21 those if $\mathbb{T}_2$ is the chosen shift map.

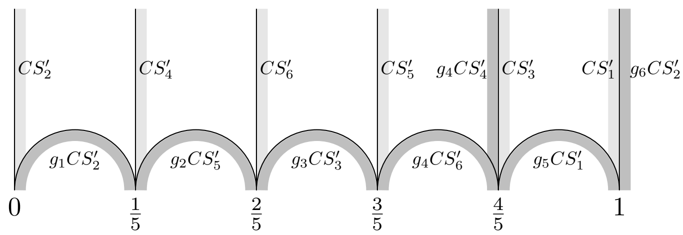


Figure 6.20: The translates of CS' relevant for determination of the location next point of intersection for the shift map $\mathbb{T}_1$.

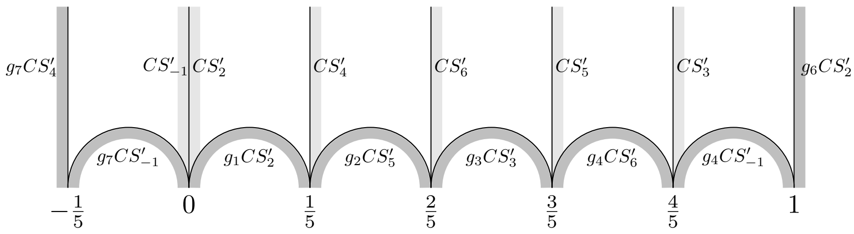


Figure 6.21: The translates of CS' relevant for determination of the location next point of intersection for the shift map $\mathbb{T}_2$.

Recall the set bd from Sec. 6.5.

Proposition 6.7.16. *Let $\widehat{\gamma}$ be a geodesic on Y.*

(i) *$\widehat{\gamma}$ intersects $\widehat{\mathrm{CS}}$ infinitely often in future if and only if $\widehat{\gamma}(\infty) \notin \pi(\mathrm{bd})$.*

(ii) *$\widehat{\gamma}$ intersects $\widehat{\mathrm{CS}}$ infinitely often in past if and only if $\widehat{\gamma}(-\infty) \notin \pi(\mathrm{bd})$.*

Proof. We will only show (i). The proof of (ii) is analogous.

Suppose first that $\widehat{\gamma}$ does not intersect $\widehat{\mathrm{CS}}$ infinitely often in future. If $\widehat{\gamma}$ does not intersect $\widehat{\mathrm{CS}}$ at all, then Prop. 6.7.7 states that $\widehat{\gamma} \in \mathrm{NC}$. Recall from Prop. 6.5.2 that $\mathrm{NC} = \mathrm{NC}(\mathbb{B})$. Hence there is $\mathcal{B} \in \mathbb{B}$ and a representative γ of $\widehat{\gamma}$ on H such that $\gamma(\pm\infty) \in \mathrm{bd}(\mathcal{B})$. Thus $\widehat{\gamma} \in \pi(\mathrm{bd}(\mathcal{B})) \subseteq \pi(\mathrm{bd})$. Suppose now that $\widehat{\gamma}$ intersects $\widehat{\mathrm{CS}}$. Cor. 6.7.13 shows that there is $t \in \mathbb{R}$ such that $\widehat{\gamma}'(t) \in \widehat{\mathrm{CS}}$ and $\widehat{\gamma}((t,\infty)) \cap \widehat{\mathrm{BS}} = \emptyset$. Let γ be the representative of $\widehat{\gamma}$ on H such that $\gamma'(t) \in \mathrm{CS}'(\mathbb{B}_{\mathbb{S},\mathbb{T}})$. Let $\widetilde{\mathcal{B}} \in \mathbb{B}_{\mathbb{S},\mathbb{T}}$ be the unique shifted cell in SH such that $\gamma'(t) \in \mathrm{CS}'(\widetilde{\mathcal{B}})$. From $\widehat{\gamma}((t,\infty)) \cap \widehat{\mathrm{BS}} = \emptyset$ it follows that $\gamma((t,\infty)) \cap \mathrm{BS} = \emptyset$. Since $\partial \operatorname{pr}(\widetilde{\mathcal{B}}) \subseteq \mathrm{BS}$ by Lemma 6.7.1, $\gamma((t,\infty)) \subseteq \operatorname{pr}(\widetilde{\mathcal{B}})^\circ$. Hence $\gamma(\infty) \in \partial_g \operatorname{pr}(\widetilde{\mathcal{B}})$. Let $\widetilde{\mathcal{B}}' \in \widetilde{\mathbb{B}}_{\mathbb{S}}$ such that $\mathbb{T}(\widetilde{\mathcal{B}}')\widetilde{\mathcal{B}}' = \widetilde{\mathcal{B}}$. Cor. 6.6.24 shows that $\mathcal{B}' := \mathrm{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}')) \in \mathbb{B}$. Hence

$$\partial_g \operatorname{pr}(\widetilde{\mathcal{B}}) = \partial_g \mathrm{cl}(\operatorname{pr}(\widetilde{\mathcal{B}})) = \mathbb{T}(\widetilde{\mathcal{B}}')\partial_g \mathcal{B}' = \mathbb{T}(\widetilde{\mathcal{B}}')\,\mathrm{bd}(\mathcal{B}') \subseteq \mathrm{bd}(\mathbb{B}).$$

Recall from Prop. 6.5.2 that $\mathrm{bd} = \mathrm{bd}(\mathbb{B})$. Therefore $\gamma(\infty) \in \mathrm{bd}$ and $\widehat{\gamma}(\infty) \in \pi(\mathrm{bd})$.

Suppose now that $\widehat{\gamma}(\infty) \in \pi(\mathrm{bd})$. We will show that $\widehat{\gamma}$ does not intersect $\widehat{\mathrm{CS}}$ infinitely often in future. Suppose first that $\widehat{\gamma}(\infty) = \pi(\infty)$. Choose a representative γ of $\widehat{\gamma}$ on H such that $\gamma(\infty) = \infty$. Lemma 6.1.33 shows that $\gamma(\mathbb{R}) \cap \mathcal{K} \neq \emptyset$. Pick $z \in \gamma(\mathbb{R}) \cap \mathcal{K}$, say $\gamma(t) = z$. By Cor. 6.2.23 we find a (not necessarily unique) pair $(\mathcal{A}, m) \in \mathbb{A} \times \mathbb{Z}$ such that $t_\lambda^m z \in \mathcal{A}$. The geodesic $\eta := t_\lambda^m \gamma$ is a representative of $\widehat{\gamma}$ on H with $\eta(\infty)\infty \in \partial_g \mathcal{A}$ and $\eta(t) \in \mathcal{A}$. Since $\mathcal{A}$ is convex, the geodesic segment $\eta([t,\infty))$ is contained in $\mathcal{A}$ and therefore in $\mathcal{B}(\mathcal{A})$ with $\eta(\infty) \in \partial_g \mathcal{B}(\mathcal{A})$. Because $\mathcal{B}(\mathcal{A})$ is convex, Lemma 6.7.6(iii) states that either $\eta([t,\infty)) \subseteq \mathcal{B}(\mathcal{A})^\circ$ or $\eta([t,\infty)) \subseteq \partial \mathcal{B}(\mathcal{A})$. Since $\partial \mathcal{B}(\mathcal{A})$ consists of complete geodesic segments, Lemma 6.7.6 implies that either $\eta(\mathbb{R}) \subseteq \mathcal{B}(\mathcal{A})^\circ$

or $\eta(\mathbb{R}) \subseteq \partial\mathcal{B}(\mathcal{A})$ or $\eta(\mathbb{R})$ intersects $\partial\mathcal{B}(\mathcal{A})$ in a unique point which is not an endpoint of some side. In the first two cases, $\eta(-\infty) \in \partial_g\mathcal{B}(\mathcal{A})$ and therefore $\widehat{\gamma} = \widehat{\eta} \in \mathrm{NC}(\mathcal{B}(\mathcal{A}))$. Prop. 6.7.7 shows that $\widehat{\gamma}$ does not intersect $\widehat{\mathrm{CS}}$. In the latter case, there is a unique side S of $\mathcal{B}(\mathcal{A})$ intersected by $\eta(\mathbb{R})$ and this intersection is transversal. Suppose that $\{\eta(s)\} = S \cap \eta(\mathbb{R})$ and let $v := \eta'(s)$. Since $\eta((s,\infty)) \subseteq \mathcal{B}(\mathcal{A})^\circ$, the unit tangent vector v points into $\mathcal{B}(\mathcal{A})^\circ$. Note that $v \in \mathrm{CS}$. By Prop. 6.7.10, there exists a (unique) pair $(\widetilde{\mathcal{B}}, g) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma$ such that $v \in g\,\mathrm{CS}'(\widetilde{\mathcal{B}})$. Moreover, $g\,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}})) = \mathcal{B}(\mathcal{A})$. Then $\alpha := g^{-1}\eta$ is a representative of $\widehat{\gamma}$ on H such that $\alpha'(s) = g^{-1}v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$ and $\alpha(\infty) \in \partial_g \,\mathrm{pr}(\widetilde{\mathcal{B}})$. Prop. 6.7.12(i) shows that there is no next point of intersection of α and CS. Hence $\widehat{\gamma}$ does not intersect $\widehat{\mathrm{CS}}$ infinitely often in future.

Suppose now that $\widehat{\gamma}(\infty) \notin \pi(\infty)$. We find a representative γ of $\widehat{\gamma}$ on H and a cell $\mathcal{B} \in \mathbb{B}$ in H such that $\gamma(\infty) \in \partial_g\mathcal{B} \cap \mathbb{R}$. Assume for contradiction that γ intersects CS infinitely often in future. Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathbb{R}$ such that $\gamma'(t_n) \in \mathrm{CS}$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n = \infty$. For $n \in \mathbb{N}$ let S_n be the connected component of BS such that $\gamma(t_n) \in S_n$. Note that S_n is a complete geodesic segment. We will show that there exists $n_0 \in \mathbb{N}$ such that both endpoints of S_{n_0} are in $\mathbb{R}$. Assume for contradiction that each S_n is vertical, hence $S_n = [a_n, \infty]$ with $a_n \in \mathbb{R}$. Then either $a_1 < a_2 < \ldots$ or $a_1 > a_2 > \ldots$. Theorem 6.2.20 shows that $\mathbb{A}$ is finite. Therefore $\mathbb{B}$ is so by Cor. 6.4.14. Recall that each S_n is a vertical side of some Γ_∞-translate of some element in $\mathbb{B}$. Hence there is $r > 0$ such that $|a_{n+1} - a_n| \geq r$ for each $n \in \mathbb{N}$. W.l.o.g. suppose that $a_1 < a_2 < \ldots$. Then $\lim_{n\to\infty} a_n = \infty$. For each $n \in \mathbb{N}$, $\gamma(\infty)$ is contained in the interval (a_n, ∞). Hence $\gamma(\infty) \in \bigcap_{n\in\mathbb{N}} (a_n, \infty) = \emptyset$. This is a contradiction. Therefore we find $k \in \mathbb{N}$ such that $S_k = [a_k, b_k]$ with $a_k, b_k \in \mathbb{R}$. W.l.o.g. $a_k < b_k$. Let $(\widetilde{\mathcal{B}}, g) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma$ such that $\gamma'(t_k) \in g\,\mathrm{CS}'(\widetilde{\mathcal{B}})$. Prop. 6.7.10 states that $gb(\widetilde{\mathcal{B}}) = S_k$ and $\gamma(\infty) \in (a_k, b_k)_+$ or $\gamma(\infty) \in (a_k, b_k)_-$. In each case $a_k < \gamma(\infty) < b_k$. Lemma 6.7.6(iii) shows that the complete geodesic segment $S := [\gamma(\infty), \infty]$ is contained in $\mathcal{B}$. It divides H into the two open half-spaces

$$H_1 := \{z \in H \mid \mathrm{Re}\, z < \gamma(\infty)\} \quad \text{and} \quad H_2 := \{z \in H \mid \mathrm{Re}\, z > \gamma(\infty)\}$$

such that H is the disjoint union $H_1 \cup S \cup H_2$. Neither a_n nor b_n is an endpoint of S but $(a_n, b_n) \in \partial_g H_1 \times \partial_g H_2$ or $(a_n, b_n) \in \partial_g H_2 \times \partial_g H_1$. In each case, S_n intersects S transversely. Then S_n intersects $\mathcal{B}^\circ$. Since S_n is the side of some Γ-translate of some cell in H, this is a contradiction to Cor. 6.4.18. This shows that γ does not intersect CS infinitely often in future and hence $\widehat{\gamma}$ does not intersect $\widehat{\mathrm{CS}}$ infinitely often in future. This completes the proof of (i). ☐

Recall the set NIC from Remark 6.5.5.

Theorem 6.7.17. *Let μ be a measure on the space of geodesics on Y. Then $\widehat{\mathrm{CS}}$ is a cross section w. r. t. μ for the geodesic flow on Y if and only if $\mu(\mathrm{NIC}) = 0$.*

Proof. Prop. 6.5.4 shows that $\widehat{\mathrm{CS}}$ satisfies (C2). Let $\widehat{\gamma}$ be a geodesic on Y. Then Prop. 6.7.16 implies that $\widehat{\gamma}$ intersects $\widehat{\mathrm{CS}}$ infinitely often in past and future if and only if $\widehat{\gamma} \notin \mathrm{NIC}$. This completes the proof. ☐

Let $\mathcal{E}$ denote the set of unit tangent vectors to the geodesics in NIC and set $\widehat{\mathrm{CS}}_{\mathrm{st}} := \widehat{\mathrm{CS}} \smallsetminus \mathcal{E}$.

Corollary 6.7.18. *Let μ be a measure on the space of geodesics on Y such that $\mu(\mathrm{NIC}) = 0$. Then $\widehat{\mathrm{CS}}_{st}$ is the maximal strong cross section w. r. t. μ contained in $\widehat{\mathrm{CS}}$.*

6.7.2. Geometric coding sequences and geometric symbolic dynamics

A *label* of a unit tangent vector in $\widehat{\mathrm{CS}}$ or CS is a symbol which is assigned to this vector. The *labeling* of $\widehat{\mathrm{CS}}$ resp. CS is the assigment of the labels to its elements. The set of labels is commonly called the *alphabet* of the arising symbolic dynamics.

We establish a labeling of $\widehat{\mathrm{CS}}$ and CS in the following way: Let $v \in \mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and suppose that $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ is the unique shifted cell in SH such that $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$. Let γ be the geodesic on H determined by v.

Suppose first that $\gamma(\infty) \notin \partial_g \operatorname{pr}(\widetilde{\mathcal{B}})$. Prop. 6.7.12(ii) states that there is a unique side S of $\operatorname{pr}(\widetilde{\mathcal{B}})$ intersected by $\gamma((0,\infty))$ and that the next point of intersection of γ and CS is on $g\,\mathrm{CS}'(\widetilde{\mathcal{B}}_1)$ if $(\widetilde{\mathcal{B}}_1, g) = n(\widetilde{\mathcal{B}}, S)$. We assign to v the label $(\widetilde{\mathcal{B}}_1, g)$.

Suppose now that $\gamma(\infty) \in \partial_g \operatorname{pr}(\widetilde{\mathcal{B}})$. Prop. 6.7.12(i) shows that there is no next point of intersection of γ and CS. Let ε be an abstract symbol which is not contained in Γ. Then we label v by ε (“end” or “empty word”).

Let $\widehat{v} \in \widehat{\mathrm{CS}}$. By Prop. 6.7.2 there is a unique $v \in \mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ such that $\pi(v) = \widehat{v}$. We endow $\widehat{v}$ and each element in $\pi^{-1}(\widehat{v})$ with the labels of v.

The following proposition is the key result for the determination of the set of labels.

Proposition 6.7.19. *Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and suppose that S_j, $j = 1, \ldots, k$, are the sides of $\operatorname{pr}(\widetilde{\mathcal{B}})$. For $j = 1, \ldots, k$ set $(\widetilde{\mathcal{B}}_j, g_j) := n(\widetilde{\mathcal{B}}, S_j)$. Let $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$ and suppose that γ is the geodesic determined by v. Then $\gamma(\infty) \in g_j I(\widetilde{\mathcal{B}}_j)$ if and only if $\gamma((0,\infty))$ intersects S_j. Moreover, if $S_j \neq b(\widetilde{\mathcal{B}})$, then $g_j I(\widetilde{\mathcal{B}}_j) \subseteq I(\widetilde{\mathcal{B}})$. If $S_j = b(\widetilde{\mathcal{B}})$, then $g_j I(\widetilde{\mathcal{B}}_j) = J(\widetilde{\mathcal{B}})$.*

Proof. Suppose that $\gamma((0,\infty))$ intersects S_j for some $j \in \{1, \ldots, k\}$, say in $\gamma(t_0)$. Assume for contradiction that $\gamma(\infty) \in \partial_g \operatorname{pr}(\widetilde{\mathcal{B}})$. Since $\mathrm{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}))$ is a convex polyhedron (see Lemma 6.7.1) and $\gamma(0) \in \mathrm{cl}(\operatorname{pr}(\widetilde{\mathcal{B}}))$, Lemma 6.7.6(iii) shows that $\gamma((0,\infty)) \subseteq \operatorname{pr}(\widetilde{\mathcal{B}})^\circ$ or $\gamma([0,\infty)) \subseteq \partial \operatorname{pr}(\widetilde{\mathcal{B}})$. From $\gamma(t_0) \in \partial \operatorname{pr}(\widetilde{\mathcal{B}})$ it follows that $\gamma([0,\infty)) \subseteq \partial \operatorname{pr}(\widetilde{\mathcal{B}})$. On the other hand $\gamma'(0) \in \mathrm{CS}'(\widetilde{\mathcal{B}})$, hence there is $\varepsilon > 0$ such that $\gamma((0,\varepsilon)) \subseteq \operatorname{pr}(\widetilde{\mathcal{B}})^\circ$. This is a contradiction. Therefore $\gamma(\infty) \neq \partial_g \operatorname{pr}(\widetilde{\mathcal{B}})$. Then Prop. 6.7.12(ii) shows that S_j is the only side of $\operatorname{pr}(\widetilde{\mathcal{B}})$ intersected by $\gamma((0,\infty))$ and that $\gamma'(t_0) \in g_j\,\mathrm{CS}'(\widetilde{\mathcal{B}}_j)$. Let η be the geodesic determined by $\gamma'(t_0)$. Note that $(\gamma(\infty), \gamma(-\infty)) = (\eta(\infty), \eta(-\infty))$. By Lemma 6.7.9, we have that $\gamma(\infty) = \eta(\infty) \in g_j I(\widetilde{\mathcal{B}}_j)$.

Conversely suppose that $\gamma(\infty) \in g_j I(\widetilde{\mathcal{B}}_j)$. The complete geodesic segment S_j divides H into two open convex half-spaces H_1, H_2 such that H is the disjoint union $H_1 \cup S \cup H_2$. Then $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_j)) \subseteq \overline{H}_k$ for some $k \in \{1,2\}$. W.l.o.g. $k=1$. Prop. 6.7.10 shows that $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}})) \subseteq \overline{H}_2$. From $\gamma(0) \in H_2$ and $\gamma(\infty) \in g_j I(\widetilde{\mathcal{B}}_j) = \mathrm{int}_g(\partial_g H_1)$ it follows that $\gamma((0,\infty)) \cap S_j \neq \emptyset$.

Finally suppose that $S_j \neq b(\widetilde{\mathcal{B}})$. Let $x \in g_j I(\widetilde{\mathcal{B}}_j)$ and choose $y \in g_j J(\widetilde{\mathcal{B}}_j)$. Lemma 6.7.9 shows that there is $w \in g_j\,\mathrm{CS}'(\widetilde{\mathcal{B}}_j)$ such that the geodesic α on H determined by w satisfies $\alpha(\infty) = x$. We will show that $\alpha(\infty) \in I(\widetilde{\mathcal{B}})$. By definition, $\mathrm{pr}(w) \in S_j$. Prop. 6.7.10 implies that w points out of $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$. Hence there is $\varepsilon > 0$ such that $\alpha((0,\varepsilon)) \subseteq \complement\,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$. At first we will show that $\alpha((0,\infty)) \subseteq \complement\,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$. Assume for contradiction that we find $r \in [\varepsilon,\infty)$ such that $\alpha(r) \in \mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$. Let H_1 and H_2 be the open convex half-spaces of H such that H is the disjoint union $H_1 \cup S \cup H_2$ and suppose w.l.o.g. that $g_j\,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}_j)) \subseteq \overline{H}_1$. Then $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}})) \subseteq \overline{H}_2$. Note that there is $\delta \in (0,\varepsilon)$ such that $\alpha((0,\delta]) \subseteq g_j\,\mathrm{pr}(\widetilde{\mathcal{B}}_j)^\circ$. Then $\alpha(r) \in \overline{H}_2$ and $\alpha(\delta) \in H_1$. Hence

$$\alpha([\delta,r]) \cap \partial H_1 = \alpha([\delta,r]) \cap S_j \neq \emptyset.$$

Then the complete geodesic segments $\alpha(\mathbb{R})$ and S_j have two points in common, which means that they are equal. This contradicts to $\alpha(\delta) \in H_1$.

Now let K_1, K_2 be the open convex half-spaces of H such that H is the disjoint union $K_1 \cup b(\widetilde{\mathcal{B}}) \cup K_2$ and suppose that $\mathrm{pr}(\widetilde{\mathcal{B}}) \subseteq \overline{K}_1$. Prop. 6.7.10 implies that $\mathrm{int}_g(\partial_g K_1) = I(\widetilde{\mathcal{B}})$. Assume for contradiction that $\alpha(\infty) \notin \mathrm{int}_g(\partial_g K_1)$. Then $\alpha(\infty) \in \partial_g K_2$. If $\alpha(\infty)$ is an endpoint of $b(\widetilde{\mathcal{B}})$, then $\alpha(\infty) \in \partial_g\,\mathrm{pr}(\widetilde{\mathcal{B}})$. Since $\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$ is a convex polyhedron, $\partial_g\,\mathrm{pr}(\widetilde{\mathcal{B}}) = \partial_g\,\mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$ and $\alpha(0) \in \mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$, Lemma 6.7.6(iii) shows that $\alpha((0,\infty)) \subseteq \mathrm{cl}(\mathrm{pr}(\widetilde{\mathcal{B}}))$. This is a contradiction. Therefore $\alpha(\infty) \in \mathrm{int}_g(\partial_g K_2)$. Then $\alpha(0,\infty) \cap b(\widetilde{\mathcal{B}}) \neq \emptyset$, say $\alpha(t_2) \in b(\widetilde{\mathcal{B}})$. Since $\mathrm{pr}(\widetilde{\mathcal{B}})$ is a convex polyhedron with non-empty interior, for each $z \in b(\widetilde{\mathcal{B}})$ there exists $\varepsilon > 0$ such that $B_\varepsilon(z) \cap \overline{K}_1 = B_\varepsilon(z) \cap \mathrm{pr}(\widetilde{\mathcal{B}})$. Consider $z = \alpha(t_2)$. Then there exists $s \in (0,\infty)$ such that $\alpha(s) \in \mathrm{pr}(\widetilde{\mathcal{B}})$, which again is a contradiction. Thus $\alpha(\infty) \in \mathrm{int}_g(\partial_g K_1) = I(\widetilde{\mathcal{B}})$.

Finally suppose that $S_j = b(\widetilde{\mathcal{B}})$. Since $p(\widetilde{\mathcal{B}}, b(\widetilde{\mathcal{B}})) = (\widetilde{\mathcal{B}}, \mathrm{id})$, Prop. 6.7.10 implies that $I(\widetilde{\mathcal{B}}) \times g_j I(\widetilde{\mathcal{B}}_j) = I(\widetilde{\mathcal{B}}) \times J(\widetilde{\mathcal{B}})$. Hence $g_j I(\widetilde{\mathcal{B}}_j) = J(\widetilde{\mathcal{B}})$. □

For $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ let $\mathrm{Sides}(\widetilde{\mathcal{B}})$ denote the set of sides of $\mathrm{pr}(\widetilde{\mathcal{B}})$.

Corollary 6.7.20. *Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. For each $S \in \mathrm{Sides}(\widetilde{\mathcal{B}})$ set $(\widetilde{\mathcal{B}}_S, g_S) := n(\widetilde{\mathcal{B}}, S)$. Then $I(\widetilde{\mathcal{B}})$ is the disjoint union*

$$I(\widetilde{\mathcal{B}}) = \Big(I(\widetilde{\mathcal{B}}) \cap \partial_g\,\mathrm{pr}(\widetilde{\mathcal{B}})\Big) \cup \bigcup_{S \in \mathrm{Sides}(\widetilde{\mathcal{B}}) \smallsetminus b(\widetilde{\mathcal{B}})} g_S I(\widetilde{\mathcal{B}}_S).$$

Proof. Let $K(\widetilde{\mathcal{B}})$ denote the set on the right hand side of the claimed equality. Prop. 6.7.19 shows that $K(\widetilde{\mathcal{B}}) \subseteq I(\widetilde{\mathcal{B}})$. For the converse inclusion let $x \in I(\widetilde{\mathcal{B}})$. Pick any $y \in J(\widetilde{\mathcal{B}})$. By Lemma 6.7.9 there is a unique element $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$ such that $\big(\gamma_v(\infty), \gamma_v(-\infty)\big) = (x, y)$. Prop. 6.7.12 shows that either $\gamma_v(\infty) \in \partial_g \operatorname{pr}(\widetilde{\mathcal{B}})$ or there is a unique side $S \in \operatorname{Sides}(\widetilde{\mathcal{B}})$ such that $\gamma_v((0,\infty))$ intersects $g_S \operatorname{CS}'(\widetilde{\mathcal{B}}_S)$. In the latter case, Lemma 6.7.9 shows that $\gamma_v(\infty) \in g_S I(\widetilde{\mathcal{B}}_S)$ and, since $I(\widetilde{\mathcal{B}}) \cap J(\widetilde{\mathcal{B}}) = \emptyset$, Prop. 6.7.19 implies that $S \neq b(\widetilde{\mathcal{B}})$. Therefore $I(\widetilde{\mathcal{B}}) \subseteq K(\widetilde{\mathcal{B}})$. The dichotomy between "$x = \gamma_v(\infty) \in \partial_g \operatorname{pr}(\widetilde{\mathcal{B}})$" and "$\gamma_v((0,\infty))$ intersects a side S of $\operatorname{pr}(\widetilde{\mathcal{B}})$" and the uniqueness of S shows that the union on the right hand side is indeed disjoint. □

Let Σ be the set of labels.

Corollary 6.7.21. *The set Σ of labels is given by*

$$\Sigma = \{\varepsilon\} \cup \\ \cup \left\{ (\widetilde{\mathcal{B}}, g) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma \;\middle|\; \exists\, \widetilde{\mathcal{B}}' \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\ \exists\, S \in \operatorname{Sides}(\widetilde{\mathcal{B}}') \smallsetminus b(\widetilde{\mathcal{B}}') \colon (\widetilde{\mathcal{B}}, g) = n(\widetilde{\mathcal{B}}', S) \right\}.$$

Moreover, Σ is finite.

Proof. Note that for each $\widetilde{\mathcal{B}}' \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ we have $\partial_g \operatorname{pr}(\widetilde{\mathcal{B}}') \cap I(\widetilde{\mathcal{B}}') \neq \emptyset$. Thus, ε is a label. Then the claimed equality follows immediately from Cor. 6.7.20. Since $\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ is finite and each shifted cell in SH has only finitely many sides, Σ is finite. □

Example 6.7.22. For the Hecke triangle group G_n let $\mathbb{A} = \{\mathcal{A}\}$, $\mathbb{S} = \{(\mathcal{A}, U_n)\}$ and $\mathbb{T} \equiv \mathrm{id}$. Then $\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} = \{\widetilde{\mathcal{B}}_1(\mathcal{A}, U_n)\}$. Set $\widetilde{\mathcal{B}} := \widetilde{\mathcal{B}}_1(\mathcal{A}, U_n)$. Then the set of labels is (cf. Example 6.7.14)

$$\Sigma = \left\{ \varepsilon, (\widetilde{\mathcal{B}}, U_n^k S) \;\middle|\; k \in \{1, \ldots, n-1\} \right\}.$$

Example 6.7.23. Recall Example 6.7.15. If the shift map is $\mathbb{T}_1$, then the set of labels is

$$\Sigma = \{\varepsilon, (\widetilde{\mathcal{B}}_2, g_1), (\widetilde{\mathcal{B}}_4, \mathrm{id}), (\widetilde{\mathcal{B}}_5, g_2), (\widetilde{\mathcal{B}}_6, \mathrm{id}), (\widetilde{\mathcal{B}}_3, g_3), (\widetilde{\mathcal{B}}_5, \mathrm{id}), (\widetilde{\mathcal{B}}_6, g_4), (\widetilde{\mathcal{B}}_3, \mathrm{id}), \\ (\widetilde{\mathcal{B}}_1, g_5), (\widetilde{\mathcal{B}}_2, g_6), (\widetilde{\mathcal{B}}_4, g_4)\}.$$

With the shift map $\mathbb{T}_2$, the set of labels is

$$\Sigma = \{\varepsilon, (\widetilde{\mathcal{B}}_4, g_7), (\widetilde{\mathcal{B}}_{-1}, g_7), (\widetilde{\mathcal{B}}_2, g_1), (\widetilde{\mathcal{B}}_4, \mathrm{id}), (\widetilde{\mathcal{B}}_5, g_2), (\widetilde{\mathcal{B}}_6, \mathrm{id}), (\widetilde{\mathcal{B}}_3, g_3), \\ (\widetilde{\mathcal{B}}_5, \mathrm{id}), (\widetilde{\mathcal{B}}_6, g_4), (\widetilde{\mathcal{B}}_3, \mathrm{id}), (\widetilde{\mathcal{B}}_{-1}, g_4), (\widetilde{\mathcal{B}}_2, g_6)\}.$$

Definition and Remark 6.7.24. Let $v \in \operatorname{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and suppose that γ is the geodesic on H determined by v. Prop. 6.7.12 implies that there is a unique sequence $(t_n)_{n\in J}$ in $\mathbb{R}$ which satisfies the following properties:

(i) $J = \mathbb{Z} \cap (a,b)$ for some interval (a,b) with $a, b \in \mathbb{Z} \cup \{\pm\infty\}$ and $0 \in (a,b)$,

(ii) the sequence $(t_n)_{n\in J}$ is increasing,

(iii) $t_0 = 0$,

(iv) for each $n \in J$ we have $\gamma'(t_n) \in \mathrm{CS}$ and $\gamma'((t_n, t_{n+1})) \cap \mathrm{CS} = \emptyset$ and $\gamma'((t_{n-1}, t_n)) \cap \mathrm{CS} = \emptyset$ where we set $t_b := \infty$ if $b < \infty$ and $t_a := -\infty$ if $a > -\infty$.

The sequence $(t_n)_{n\in J}$ is said to be the *sequence of intersection times of* v *(w. r. t.* CS*)*.

Let $\widehat{v} \in \widehat{\mathrm{CS}}$ and set $v := \left(\pi|_{\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})}\right)^{-1}(\widehat{v})$. Then the *sequence of intersection times (w. r. t.* CS*) of* $\widehat{v}$ and *of* each $w \in \pi^{-1}(\widehat{v})$ is defined to be the sequence of intersection times of v.

Now we define the geometric coding sequences.

Definition 6.7.25. For each $s \in \Sigma$ we set

$$\widehat{\mathrm{CS}}_s := \left\{\widehat{v} \in \widehat{\mathrm{CS}} \;\middle|\; \widehat{v} \text{ is labeled with } s\right\}$$

and

$$\mathrm{CS}_s := \pi^{-1}\big(\widehat{\mathrm{CS}}_s\big) = \left\{v \in \mathrm{CS} \;\middle|\; v \text{ is labeled with } s\right\}.$$

Let $\widehat{v} \in \widehat{\mathrm{CS}}$ and let $(t_n)_{n\in J}$ be the sequence of intersection times of $\widehat{v}$. Suppose that $\widehat{\gamma}$ is the geodesic on Y determined by $\widehat{v}$. The *geometric coding sequence* of $\widehat{v}$ is the sequence $(a_n)_{n\in J}$ in Σ defined by

$$a_n := s \quad \text{if and only if} \quad \widehat{\gamma}'(t_n) \in \widehat{\mathrm{CS}}_s$$

for each $n \in J$.

Let $v \in \mathrm{CS}$. The *geometric coding sequence* of v is defined to be the geometric coding sequence of $\pi(v)$.

Proposition 6.7.26. *Let $v \in \mathrm{CS}'$. Suppose that $(t_n)_{n\in J}$ is the sequence of intersection times of v and that $(a_n)_{n\in J}$ is the geometric coding sequence of v. Let γ be the geodesic on H determined by v. Suppose that $J = \mathbb{Z} \cap (a,b)$ with $a, b \in \mathbb{Z} \cup \{\pm\infty\}$.*

(i) *If $b = \infty$, then $a_n \in \Sigma \smallsetminus \{\varepsilon\}$ for each $n \in J$.*

(ii) *If $b < \infty$, then $a_n \in \Sigma \smallsetminus \{\varepsilon\}$ for each $n \in (a, b-2] \cap \mathbb{Z}$ and $a_{b-1} = \varepsilon$.*

(iii) *Suppose that $a_n = (\widetilde{\mathcal{B}}_n, h_n)$ for $n \in (a, b-1) \cap \mathbb{Z}$ and set*

$$\begin{aligned} g_0 &:= h_0 && \text{if } b \geq 2, \\ g_{n+1} &:= g_n h_{n+1} && \text{for } n \in [0, b-2) \cap \mathbb{Z}, \\ g_{-1} &:= \mathrm{id}, && \\ g_{-(n+1)} &:= g_{-n} h_{-n}^{-1} && \text{for } n \in [1, -(a+1)) \cap \mathbb{Z}. \end{aligned}$$

Then $\gamma'(t_{n+1}) \in g_n \, \mathrm{CS}'(\widetilde{\mathcal{B}}_n)$ for each $n \in (a, b-1) \cap \mathbb{Z}$.

Proof. We start with some preliminary considerations which will prove (i) and (ii) and simplify the argumentation for (iii). Let $n \in J$ and consider $w := \gamma'(t_n)$. The definition of geometric coding sequences shows that $\gamma'(t_n) \in \mathrm{CS}_{a_n}$. Since CS is the disjoint union $\bigcup_{k\in\Gamma} k\,\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ (see Prop. 6.7.2), there is a unique $k \in \Gamma$ such that $k^{-1}w \in \mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. The label of $k^{-1}w$ is a_n. Let η be the geodesic on H determined by $k^{-1}w$. Note that $\eta(t) := k^{-1}\gamma(t+t_n)$ for each $t \in \mathbb{R}$. The definition of labels shows that $a_n = \varepsilon$ if and only if there is no next point of intersection of η and CS. In this case $\gamma'((t_n,\infty)) \cap \mathrm{CS} = \emptyset$ and hence $b = n+1$. This shows (i) and (ii). Suppose now that $a_n = (\widetilde{\mathcal{B}}, g)$. Then there is a next point of intersection of η and CS, say $\eta'(s)$, and this is on $g\,\mathrm{CS}'(\widetilde{\mathcal{B}})$. Then $k^{-1}\gamma'(s+t_n) \in g\,\mathrm{CS}'(\widetilde{\mathcal{B}})$ and $k^{-1}\gamma'((t_n, s+t_n)) \cap \mathrm{CS} = \emptyset$. Hence $t_{n+1} = s + t_n$ and $\gamma'(t_{n+1}) \in kg\,\mathrm{CS}'(\widetilde{\mathcal{B}})$.

Now we show (iii). Suppose that $b \geq 2$. Then $v = \gamma'(t_0)$ is labeled with $(\widetilde{\mathcal{B}}_0, h_0)$. Hence for the next point of intersection $\gamma'(t_1)$ of γ and CS we have

$$\gamma'(t_1) \in h_0\,\mathrm{CS}'\big(\widetilde{\mathcal{B}}_0\big) = g_0\,\mathrm{CS}'\big(\widetilde{\mathcal{B}}_0\big).$$

Suppose that we have already shown that

$$\gamma'(t_{n+1}) \in g_n\,\mathrm{CS}'\big(\widetilde{\mathcal{B}}_n\big)$$

for some $n \in [0, b-1) \cap \mathbb{Z}$ and that $b \geq n+3$. By (i) resp. (ii), $\gamma'((t_{n+1},\infty)) \cap \mathrm{CS} \neq \emptyset$ and hence $\gamma'(t_{n+1})$ is labeled with $(\widetilde{\mathcal{B}}_{n+1}, h_{n+1})$. Our preliminary considerations show that

$$\gamma'(t_{n+2}) \in g_n h_{n+1}\,\mathrm{CS}'\big(\widetilde{\mathcal{B}}_{n+1}\big) = g_{n+1}\,\mathrm{CS}'\big(\widetilde{\mathcal{B}}_{n+1}\big).$$

Therefore $\gamma'(t_{n+1}) \in g_n\,\mathrm{CS}'(\widetilde{\mathcal{B}}_n)$ for each $n \in [0, b-1) \cap \mathbb{Z}$.

Suppose that $a \leq -2$. The element $\gamma'(t_{-1})$ is labeled with $(\widetilde{\mathcal{B}}_{-1}, h_{-1})$. Since $\gamma'(t_{-1}) \in k\,\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ for some $k \in \Gamma$, our preliminary considerations show that $\gamma'(t_0) \in kh_{-1}\,\mathrm{CS}'(\widetilde{\mathcal{B}}_{-1})$. Because $\gamma'(t_0) = v \in \mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$, Prop. 6.7.2 implies that $k = h_{-1}^{-1}$ and

$$\gamma'(t_0) \in \mathrm{CS}'\big(\widetilde{\mathcal{B}}_{-1}\big) = g_{-1}\,\mathrm{CS}'\big(\widetilde{\mathcal{B}}_{-1}\big) \text{ and } \gamma'(t_{-1}) \in h_{-1}^{-1}\,\mathrm{CS}'\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) = g_{-2}\,\mathrm{CS}'\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big).$$

Suppose that we have already shown that

$$\gamma'(t_{-(n-1)}) \in g_{-n}\,\mathrm{CS}'\big(\widetilde{\mathcal{B}}_{-n}\big) \quad \text{and} \quad \gamma'(t_{-n}) \in g_{-(n+1)}\,\mathrm{CS}'\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big)$$

for some $n \in [1, -a) \cap \mathbb{Z}$ and suppose that $a \leq -n-2$. Then $\gamma'(t_{-n-1})$ exists and is labeled with $(\widetilde{\mathcal{B}}_{-n-1}, h_{-n-1})$. Since $\gamma'(t_{-n-1}) \in h\,\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ for some $h \in \Gamma$, we know that $\gamma'(t_{-n}) \in hh_{-n-1}\,\mathrm{CS}'(\widetilde{\mathcal{B}}_{-n-1})$. Therefore

$$\gamma'(t_{-n}) \in hh_{-(n+1)}\,\mathrm{CS}'\big(\widetilde{\mathcal{B}}_{-(n+1)}\big) \cap g_{-(n+1)}\,\mathrm{CS}'\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big).$$

Prop. 6.7.2 implies that $hh_{-(n+1)} = g_{-(n+1)}$ and $\gamma'(t_{-n}) \in g_{-(n+1)}\,\mathrm{CS}'(\widetilde{\mathcal{B}}_{-(n+1)})$ and

$$\gamma'(t_{-(n+1)}) \in g_{-(n+1)}h_{-(n+1)}^{-1}\,\mathrm{CS}'\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) = g_{-(n+2)}\,\mathrm{CS}'\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big).$$

Therefore $\gamma'(t_{n+1}) \in g_n\,\mathrm{CS}'(\widetilde{\mathcal{B}}_n)$ for each $n \in (a, -1] \cap \mathbb{Z}$. This completes the proof. $\square$

Let Λ denote the set of geometric coding sequences and let Λ_σ be the subset of Λ which contains the geometric coding sequences $(a_n)_{n\in(a,b)\cap\mathbb{Z}}$ with $a,b\in\mathbb{Z}\cup\{\pm\infty\}$ for which $b\geq 2$. Let Σ^{all} denote the set of all finite and one- or two-sided infinite sequences in Σ. The left shift $\sigma\colon\Sigma^{\mathrm{all}}\to\Sigma^{\mathrm{all}}$,

$$\sigma\big((a_n)_{n\in J}\big)_k := a_{k+1} \quad \text{for all } k\in J$$

induces a partially defined map $\sigma\colon\Lambda\to\Lambda$ resp. a map $\sigma\colon\Lambda_\sigma\to\Lambda$. Suppose that $\mathrm{Seq}\colon\widehat{\mathrm{CS}}\to\Lambda$ is the map which assigns to $\widehat{v}\in\widehat{\mathrm{CS}}$ the geometric coding sequence of $\widehat{v}$. Recall the first return map R from Sec. 5.

Proposition 6.7.27. *The diagram*

$$\begin{tikzcd} \widehat{\mathrm{CS}} \arrow[r, "R"] \arrow[d, "\mathrm{Seq}"'] & \widehat{\mathrm{CS}} \arrow[d, "\mathrm{Seq}"] \\ \Lambda \arrow[r, "\sigma"] & \Lambda \end{tikzcd}$$

commutes. In particular, for $\widehat{v}\in\widehat{\mathrm{CS}}$, the element $R(\widehat{v})$ is defined if and only if $\mathrm{Seq}(\widehat{v})\in\Lambda_\sigma$.

Proof. This follows immediately from the definition of geometric coding sequences, Prop. 6.7.12 and 6.7.26. □

Set $\mathrm{CS}_{\mathrm{st}} := \pi^{-1}(\widehat{\mathrm{CS}}_{\mathrm{st}})$ and $\mathrm{CS}'_{\mathrm{st}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) := \mathrm{CS}_{\mathrm{st}}\cap\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and let Λ_{st} denote the set of two-sided infinite geometric coding sequences.

Remark 6.7.28. The set of geometric coding sequences of elements in $\mathrm{CS}_{\mathrm{st}}$ (or only $\mathrm{CS}'_{\mathrm{st}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$) is Λ_{st}. Moreover, $\Lambda_{\mathrm{st}}\subseteq\Lambda_\sigma$.

In the following we will show that $(\Lambda_{\mathrm{st}},\sigma)$ is a symbolic dynamics for the geodesic flow on $\widehat{\Phi}$.

Lemma 6.7.29. *Suppose that $x,y\in\partial_g H\setminus\mathrm{bd}$, $x<y$. Then there exists a connected component $S=[a,b]$ of* BS *with $a,b\in\mathbb{R}$, $a<b$, such that $x<a<y<b$.*

Proof. Consider the point $z:=y+i\frac{y-x}{2}$. By Cor. 6.4.18 we find a pair $(\mathcal{B},g)$ in $\mathbb{B}\times\Gamma$ such that $z\in g\mathcal{B}$. Recall that each side of $g\mathcal{B}$ is a complete geodesic segment. We claim that there is a non-vertical side S of $g\mathcal{B}$ which intersects $(y,z]$. Assume for contradiction that no side of $g\mathcal{B}$ intersects the geodesic segment $(y,z]$. Then $\mathcal{B}$ arises from a strip precell in H and the two sides of $g\mathcal{B}$ are vertical. Then $g\mathcal{B}=\mathrm{pr}_\infty^{-1}([c,d])\cap H$ for some $c,d\in\mathbb{R}$, $c<d$. From $z\in g\mathcal{B}$ it follows that $y=\mathrm{pr}_\infty(z)\in\partial_g g\mathcal{B}=[c,d]$. This is a contradiction to $y\notin\mathrm{bd}$. Hence we find a side S of $g\mathcal{B}$ such that $S\cap(y,z]\neq\emptyset$. Assume for contradiction that S is vertical. Then $S=(y,\infty)$ and $y\in\partial_g g\mathcal{B}$, which again is a contradiction. Thus, S is non-vertical. Suppose that S is the complete geodesic segment $[a,b]$ with $a,b\in\mathbb{R}$, $a<b$. Since S is a (Euclidean) half-circle centered at some $r\in\mathbb{R}$ and S intersects $\big(y,y+i\frac{y-x}{2}\big)$, we know that $x<a<y<b$. □

For the proof of the following proposition recall that each connected component of BS is a complete geodesic segment and that it is of the form $\mathrm{pr}(g\,\mathrm{CS}'(\widetilde{\mathcal{B}}))$ for some pair $(\widetilde{\mathcal{B}}, g) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma$. Conversely, for each pair $(\widetilde{\mathcal{B}}, g) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$, the set $\mathrm{pr}(g\,\mathrm{CS}'(\widetilde{\mathcal{B}}))$ is a connected component of BS.

Proposition 6.7.30. *Let $v, w \in \mathrm{CS}'_{st}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. If the geometric coding sequences of v and w are equal, then $v = w$.*

Proof. Let $\big((\widetilde{\mathcal{B}}_j, h_j)\big)_{j\in\mathbb{Z}}$ be the geometric coding sequence of v and let $\big((\widetilde{\mathcal{B}}'_j, k_j)\big)_{j\in\mathbb{Z}}$ be that of w. Suppose that $v \neq w$. Suppose first that

$$\big(\gamma_v(\infty), \gamma_v(-\infty)\big) = \big(\gamma_w(\infty), \gamma_w(-\infty)\big).$$

Prop. 6.7.26 shows that $v \in \mathrm{CS}'(\widetilde{\mathcal{B}}_{-1})$ and $w \in \mathrm{CS}'(\widetilde{\mathcal{B}}'_{-1})$. Lemma 6.7.9 implies that $\widetilde{\mathcal{B}}_{-1} \neq \widetilde{\mathcal{B}}'_{-1}$, which shows that the geometric coding sequences of v and w are different.

Suppose now that

$$\big(\gamma_v(\infty), \gamma_v(-\infty)\big) \neq \big(\gamma_w(\infty), \gamma_w(-\infty)\big).$$

Assume for contradiction that $\big((\widetilde{\mathcal{B}}_j, h_j)\big)_{j\in\mathbb{Z}} = \big((\widetilde{\mathcal{B}}'_j, k_j)\big)_{j\in\mathbb{Z}}$. Let $(t_n)_{n\in\mathbb{Z}}$ be the sequence of intersection times of v and $(s_n)_{n\in\mathbb{Z}}$ be that of w. Prop 6.7.26(iii) implies that for each $n \in \mathbb{Z}$, the elements $\mathrm{pr}(\gamma'_v(t_n))$ and $\mathrm{pr}(\gamma'_w(s_n))$ are on the same connected component of BS. For each connected component S of BS let $H_{1,S}, H_{2,S}$ denote the open convex half spaces such that H is the disjoint union $H = H_{1,S} \cup S \cup H_{2,S}$.

Suppose first that $\gamma_v(\infty) \neq \gamma_w(\infty)$. Prop. 6.7.16 shows that $\gamma_v(\infty), \gamma_w(\infty) \notin \mathrm{bd}$. By Lemma 6.7.29 we find a connected component S of BS such that $\gamma_v(\infty) \in \partial_g H_{1,S} \smallsetminus \partial_g S$ and $\gamma_w(\infty) \in \partial_g H_{2,S} \smallsetminus \partial_g S$ (or vice versa). Since BS is a manifold, each connected component of BS other than S is either contained in $H_{1,S}$ or in $H_{2,S}$. In particular, we may assume that $\mathrm{pr}(v), \mathrm{pr}(w) \in H_{1,S}$. Then $\gamma_v([0,\infty)) \subseteq H_{1,S}$ and $\gamma_w((t,\infty)) \subseteq H_{2,S}$ for some $t > 0$. Hence there is $n \in \mathbb{N}$ such that $\mathrm{pr}(\gamma'_w(s_n)) \in H_{2,S}$, which implies that $\mathrm{pr}(\gamma'_v(t_n))$ and $\mathrm{pr}(\gamma'_w(s_n))$ are not on the same connected component of BS.

Suppose now that $\gamma_v(-\infty) \neq \gamma_w(-\infty)$ and let S be a connected component of BS auch that $\gamma_v(-\infty) \in \partial_g H_{1,S} \smallsetminus \partial_g S$ and $\gamma_w(-\infty) \in \partial_g H_{2,S} \smallsetminus \partial_g S$ (or vice versa). Again, we may assume that $\mathrm{pr}(v), \mathrm{pr}(w) \in H_{1,S}$. Then $\gamma_v((-\infty, 0]) \subseteq H_{1,S}$ and $\gamma_w(-\infty, s)) \subseteq H_{2,S}$ for some $s < 0$. Thus we find $n \in \mathbb{N}$ such that $\mathrm{pr}(\gamma'_w(s_{-n})) \in H_{2,S}$. Hence $\mathrm{pr}(\gamma'_v(t_{-n}))$ and $\mathrm{pr}(\gamma'_w(s_{-n}))$ are not on the same connected component of BS. In both cases we find a contradiction. Therefore the geometric coding sequences are not equal. $\square$

Corollary 6.7.31. *The map $\mathrm{Seq}|_{\widehat{\mathrm{CS}}_{st}} \colon \widehat{\mathrm{CS}}_{st} \to \Lambda_{st}$ is bijective.*

Remark 6.7.32. If there is more than one shifted cell in SH or if there is a strip precell in H, then the map $\mathrm{Seq}\colon \widehat{\mathrm{CS}} \smallsetminus \widehat{\mathrm{CS}}_{\mathrm{st}} \to \Lambda \smallsetminus \Lambda_{\mathrm{st}}$ is not injective. This is due to the decision to label each $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$, for each $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$, with the same label ε if $\gamma_v(\infty) \in \partial_g \operatorname{pr}(\widetilde{\mathcal{B}})$ without distinguishing between different points in $\partial_g \operatorname{pr}(\widetilde{\mathcal{B}})$ and without distinguishing between different shifted cells in SH.

Let $\mathrm{Cod} := \left(\mathrm{Seq}\,|_{\widehat{\mathrm{CS}}_{\mathrm{st}}}\right)^{-1} : \Lambda_{\mathrm{st}} \to \widehat{\mathrm{CS}}_{\mathrm{st}}$.

Corollary 6.7.33. *The diagram*

$$\begin{tikzcd} \widehat{\mathrm{CS}}_{st} \arrow[r, "R"] & \widehat{\mathrm{CS}}_{st} \\ \Lambda_{st} \arrow[u, "\mathrm{Cod}"] \arrow[r, "\sigma"] & \Lambda_{st} \arrow[u, "\mathrm{Cod}"'] \end{tikzcd}$$

commutes and (Λ_{st}, σ) *is a symbolic dynamics for the geodesic flow on* Y.

We end this section with the explanation of the acronyms NC and NIC (cf. Remark 6.5.5).

Remark 6.7.34. Let $\widehat{v}$ be a unit tangent vector in SH based on $\widehat{\mathrm{BS}}$ and let $\widehat{\gamma}$ be the geodesic determined by $\widehat{v}$. Then $\widehat{v}$ has no geometric coding sequence if and only if $\widehat{v} \notin \widehat{\mathrm{CS}}$. By Prop. 6.7.7 this is the case if and only if $\widehat{\gamma} \in \mathrm{NC}$. This is the reason why NC stands for “not coded”.

Suppose now that $\widehat{v} \in \widehat{\mathrm{CS}}$. Then the geometric coding sequence is not two-sided infinite if and only if $\widehat{\gamma}$ does not intersect $\widehat{\mathrm{CS}}$ infinitely often in past and future, which by Prop. 6.7.16 is equivalent to $\widehat{\gamma} \in \mathrm{NIC}$. This explains why NIC is for “not infinitely often coded”.

6.8. Reduction and arithmetic symbolic dynamics

Let Γ be a geometrically finite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of which ∞ is a cuspidal point and which satisfies (A2). Suppose that the set of relevant isometric spheres is non-empty. Fix a basal family $\mathbb{A}$ of precells in H and let $\mathbb{B}$ be the family of cells in H assigned to $\mathbb{A}$. Let $\mathbb{S}$ be a set of choices associated to $\mathbb{A}$ and suppose that $\widetilde{\mathbb{B}}_{\mathbb{S}}$ is the family of cells in SH associated to $\mathbb{A}$ and $\mathbb{S}$. Let $\mathbb{T}$ be a shift map for $\widetilde{\mathbb{B}}_{\mathbb{S}}$ and let $\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ denote the family of cells in SH associated to $\mathbb{A}$, $\mathbb{S}$ and $\mathbb{T}$.

Recall the geometric symbolic dynamics for the geodesic flow on Y which we constructed in Sec. 6.7 with respect to $\mathbb{A}$, $\mathbb{S}$ and $\mathbb{T}$. In particular, recall the set $\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ of representatives for the cross section $\widehat{\mathrm{CS}} = \widehat{\mathrm{CS}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$, its subsets $\mathrm{CS}'(\widetilde{\mathcal{B}})$ for $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$, and the definition of the labeling of $\widehat{\mathrm{CS}}$.

Let $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$ for some $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and consider the geodesic γ_v on H determined by v. Suppose that $(a_n)_{n\in J}$ is the geometric coding sequence of v. The combination of Prop. 6.7.19 and 6.7.12 allows to determine the label a_0 of v from the location of $\gamma_v(\infty)$, and then to iteratively reconstruct the complete future part $(a_n)_{n\in[0,\infty)\cap J}$ of the geometric coding sequence of v. Hence, if the unit tangent vector $v \in \mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is known, or more precisely, if the shifted cell $\widetilde{\mathcal{B}}$ in SH with $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$ is known, then one can reconstruct at least the future part of the geometric coding sequence of v. However, if γ_v intersects $\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ in more than one point, then one cannot derive the shifted cell $\widetilde{\mathcal{B}}$ in SH from the endpoints of γ_v and therefore one cannot construct a symbolic dynamics or discrete dynamical system on the boundary $\partial_g H$ of H which is conjugate to $(\widehat{\mathrm{CS}}, R)$ or $(\widehat{\mathrm{CS}}_{\mathrm{st}}, R)$. Recall from Sec. 5 that R denotes the first return map.

To overcome this problem, we restrict, in Sec. 6.8.1, our cross section $\widehat{\mathrm{CS}}$ to a subset $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ (resp. to $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ for the strong cross section $\widehat{\mathrm{CS}}_{\mathrm{st}}$). We will show that $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ are cross sections for the geodesic flow on Y w.r.t. to the same measure as $\widehat{\mathrm{CS}}$ and $\widehat{\mathrm{CS}}_{\mathrm{st}}$. More precisely, it will turn out that exactly those geodesics on Y which intersect $\widehat{\mathrm{CS}}$ at all also intersect $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ at all, and that exactly those which intersect $\widehat{\mathrm{CS}}$ infinitely often in future and past also intersect $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ infinitely often in future and past. Moreover, $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is the maximal strong cross section contained in $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. In contrast to $\widehat{\mathrm{CS}}$ and $\widehat{\mathrm{CS}}_{\mathrm{st}}$, the sets $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ do depend on the choice of the family $\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. Moreover, we will find discrete dynamical systems $(\widetilde{\mathrm{DS}}, \widetilde{F})$ and $(\widetilde{\mathrm{DS}}_{\mathrm{st}}, \widetilde{F}_{\mathrm{st}})$ with $\widetilde{\mathrm{DS}}_{\mathrm{st}} \subseteq \widetilde{\mathrm{DS}} \subseteq \mathbb{R} \times \mathbb{R}$ which are conjugate to $(\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}), R)$ resp. $(\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}), R)$.

In Sec. 6.8.2 we will introduce a natural labeling of $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and define for each $\widehat{v} \in \widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and each $v \in \mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ a so-called reduced coding sequence. The labeling is not only given by a geometric definition analogous to that for the geometric symbolic dynamics. It is also induced, in the way explained in Sec. 5, by a certain decomposition of the set $\widetilde{\mathrm{DS}}$ resp. $\widetilde{\mathrm{DS}}_{\mathrm{st}}$. Therefore, contrary to the geometric coding sequence, the reduced coding sequence of v can completely be reconstructed from the location of the endpoints of the geodesic γ_v. The labeling of $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ clearly restricts to a labeling of $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. In Sec. 6.8.3 we will provide a (constructive) proof that in certain situations there is a generating function for the future part of the symbolic dynamics arising from the labeling of $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.

All construction in this process are of geometrical nature and effectively doable in a finite number of steps. Moreover, the set of labels is finite.

6.8.1. Reduced cross section

The set $\{I(\widetilde{\mathcal{B}}) \mid \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\}$ decomposes into two sequences

$$\mathcal{I}_1\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) := \Big\{ I\big(\widetilde{\mathcal{B}}_{1,1}\big) \supseteq I\big(\widetilde{\mathcal{B}}_{1,2}\big) \supseteq \ldots \supseteq I\big(\widetilde{\mathcal{B}}_{1,k_1}\big) \Big\}$$

where $I(\widetilde{\mathcal{B}}_{1,j}) = (a_j, \infty)$ and $a_1 < a_2 < \ldots < a_{k_1}$, and

$$\mathcal{I}_2\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) := \Big\{ I\big(\widetilde{\mathcal{B}}_{2,1}\big) \supseteq I\big(\widetilde{\mathcal{B}}_{2,2}\big) \supseteq \ldots \supseteq I\big(\widetilde{\mathcal{B}}_{2,k_2}\big) \Big\}$$

where $I(\widetilde{\mathcal{B}}_{2,j}) = (-\infty, b_j)$ and $b_1 > b_2 > \ldots > b_{k_2}$.

Set $I(\widetilde{\mathcal{B}}_{1,k_1+1}) := \emptyset$ and

$$I_{\mathrm{red}}\big(\widetilde{\mathcal{B}}_{1,j}\big) := I\big(\widetilde{\mathcal{B}}_{1,j}\big) \setminus I\big(\widetilde{\mathcal{B}}_{1,j+1}\big) \qquad \text{for } j = 1, \ldots, k_1,$$

and set $I(\widetilde{\mathcal{B}}_{2,k_2+1}) := \emptyset$ and

$$I_{\mathrm{red}}\big(\widetilde{\mathcal{B}}_{2,j}\big) := I\big(\widetilde{\mathcal{B}}_{2,j}\big) \smallsetminus I\big(\widetilde{\mathcal{B}}_{2,j+1}\big) \qquad \text{for } j = 1, \ldots, k_2.$$

For each $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ set

$$\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}) := \Big\{ v \in \mathrm{CS}'(\widetilde{\mathcal{B}}) \;\Big|\; \big(\gamma_v(\infty), \gamma_v(-\infty)\big) \in I_{\mathrm{red}}(\widetilde{\mathcal{B}}) \times J(\widetilde{\mathcal{B}}) \Big\}$$

and

$$\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) := \bigcup_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} \mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}).$$

Define

$$\widehat{\mathrm{CS}}_{\mathrm{red}}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) := \pi\big(\,\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})\big) \quad \text{and} \quad \mathrm{CS}_{\mathrm{red}}\,\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) := \pi^{-1}\big(\widehat{\mathrm{CS}}_{\mathrm{red}}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big)\big).$$

Further set

$$\begin{aligned}
\mathrm{CS}'_{\mathrm{st,red}}(\widetilde{\mathcal{B}}) &:= \mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}) \cap \mathrm{CS}_{\mathrm{st}} \qquad \text{for each } \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}},\\
\mathrm{CS}'_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) &:= \mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) \cap \mathrm{CS}_{\mathrm{st}} = \bigcup_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} \mathrm{CS}'_{\mathrm{st,red}}(\widetilde{\mathcal{B}}),\\
\mathrm{CS}_{\mathrm{st,red}}\,\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) &:= \mathrm{CS}_{\mathrm{red}}\,\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) \cap \mathrm{CS}_{\mathrm{st}},\\
\text{and} \qquad \widehat{\mathrm{CS}}_{\mathrm{st,red}}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) &:= \widehat{\mathrm{CS}}_{\mathrm{red}}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) \cap \widehat{\mathrm{CS}}_{\mathrm{st}}.
\end{aligned}$$

Remark 6.8.1. Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. Note that the sets $I_{\mathrm{red}}(\widetilde{\mathcal{B}})$ and $\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}})$ not only depend on $\widetilde{\mathcal{B}}$ but also on the choice of the complete family $\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. The set $\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}})$ is identical to

$$\big\{ v \in \mathrm{CS}'(\widetilde{\mathcal{B}}) \;\big|\; \gamma_v\big((0,\infty)\big) \cap \mathrm{CS}'\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) = \emptyset \big\}.$$

In other words, if we say that $v \in \mathrm{CS}'(\widetilde{\mathcal{B}})$ has an *inner intersection* if and only if $\gamma_v((0,\infty)) \cap \mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) \neq \emptyset$, then $\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}})$ is the subset of $\mathrm{CS}'(\widetilde{\mathcal{B}})$ of all elements without inner intersection.

Remark 6.8.2. By Prop. 6.7.2, the union $\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) = \bigcup\{\mathrm{CS}'(\widetilde{\mathcal{B}}) \mid \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\}$ is disjoint and $\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is a set of representatives for $\widehat{\mathrm{CS}}$. Since $\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}})$ is a subset of $\mathrm{CS}'(\widetilde{\mathcal{B}})$ for each $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) = \pi(\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}))$, the set $\mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is a set of representatives for $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and $\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is the disjoint union $\bigcup_{g\in\Gamma} g\,\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Further, one easily sees that

$$\pi^{-1}\big(\widehat{\mathrm{CS}}_{\mathrm{st,red}}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big)\big) = \mathrm{CS}_{\mathrm{st,red}}\,\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big)$$

and

$$\pi\big(\,\mathrm{CS}'_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})\big) = \widehat{\mathrm{CS}}_{\mathrm{st,red}}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big).$$

Moreover, $\mathrm{CS}'_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is a set of representatives for $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.

Example 6.8.3. Recall Example 6.7.15. Suppose first that the shift map is $\mathbb{T}_1$. Then

$$I(\widetilde{\mathcal{B}}_1) = (-\infty, 1), \quad I(\widetilde{\mathcal{B}}_2) = (0, \infty), \quad I(\widetilde{\mathcal{B}}_4) = \left(\tfrac{1}{5}, \infty\right), \quad I(\widetilde{\mathcal{B}}_6) = \left(\tfrac{2}{5}, \infty\right),$$
$$I(\widetilde{\mathcal{B}}_5) = \left(\tfrac{3}{5}, \infty\right), \quad I(\widetilde{\mathcal{B}}_3) = \left(\tfrac{4}{5}, \infty\right).$$

Therefore we have

$$\begin{aligned} I_{\text{red}}(\widetilde{\mathcal{B}}_1) &= (-\infty, 1), & I_{\text{red}}(\widetilde{\mathcal{B}}_2) &= \left(0, \tfrac{1}{5}\right], & I_{\text{red}}(\widetilde{\mathcal{B}}_4) &= \left(\tfrac{1}{5}, \tfrac{2}{5}\right], \\ I_{\text{red}}(\widetilde{\mathcal{B}}_6) &= \left(\tfrac{2}{5}, \tfrac{3}{5}\right], & I_{\text{red}}(\widetilde{\mathcal{B}}_5) &= \left(\tfrac{3}{5}, \tfrac{4}{5}\right], & I_{\text{red}}(\widetilde{\mathcal{B}}_3) &= \left(\tfrac{4}{5}, \infty\right). \end{aligned}$$

If the shift map is $\mathbb{T}_2$, then we find

$$\begin{aligned} I_{\text{red}}(\widetilde{\mathcal{B}}_{-1}) &= (-\infty, 0), & I_{\text{red}}(\widetilde{\mathcal{B}}_2) &= \left(0, \tfrac{1}{5}\right], & I_{\text{red}}(\widetilde{\mathcal{B}}_4) &= \left(\tfrac{1}{5}, \tfrac{2}{5}\right], \\ I_{\text{red}}(\widetilde{\mathcal{B}}_6) &= \left(\tfrac{2}{5}, \tfrac{3}{5}\right], & I_{\text{red}}(\widetilde{\mathcal{B}}_5) &= \left(\tfrac{3}{5}, \tfrac{4}{5}\right], & I_{\text{red}}(\widetilde{\mathcal{B}}_3) &= \left(\tfrac{4}{5}, \infty\right). \end{aligned}$$

Note that with $\mathbb{T}_2$, the sets $I_{\text{red}}(\cdot)$ are pairwise disjoint, whereas with $\mathbb{T}_1$ they are not.

Lemma 6.8.4. *Let*

$$(x, y) \in \bigcup_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} I_{red}(\widetilde{\mathcal{B}}) \times J(\widetilde{\mathcal{B}}).$$

Then there is a unique $v \in \mathrm{CS}'_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ such that $(x, y) = \big(\gamma_v(\infty), \gamma_v(-\infty)\big)$. Conversely, if $v \in \mathrm{CS}'_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$, then

$$\big(\gamma_v(\infty), \gamma_v(-\infty)\big) \in \bigcup_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} I_{red}(\widetilde{\mathcal{B}}) \times J(\widetilde{\mathcal{B}}).$$

Proof. The combination of the definition of $\mathrm{CS}'_{\text{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and Lemma 6.7.9 shows that there is at least one $v \in \mathrm{CS}'_{\text{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ such that $\big(\gamma_v(\infty), \gamma_v(-\infty)\big) = (x, y)$ and that for each $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ there there is at most one such v. By construction,

$$\Big(I_{\text{red}}(\widetilde{\mathcal{B}}_a) \times J(\widetilde{\mathcal{B}}_a)\Big) \cap \Big(I_{\text{red}}(\widetilde{\mathcal{B}}_b) \times J(\widetilde{\mathcal{B}}_b)\Big) = \emptyset$$

for $\widetilde{\mathcal{B}}_a, \widetilde{\mathcal{B}}_b \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$, $\widetilde{\mathcal{B}}_a \neq \widetilde{\mathcal{B}}_b$. Hence there is a unique such $v \in \mathrm{CS}'_{\text{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. The remaining assertion is clear from the definition of the sets $\mathrm{CS}'_{\text{red}}(\widetilde{\mathcal{B}})$ for $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. □

Let $l_1(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ be the number of connected components of BS of the form (a, ∞) (geodesic segment) with $a_1 \le a \le a_{k_1}$ and let $l_2(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ be the number of connected components of BS of the form (a, ∞) with $b_{k_2} \le a \le b_1$. Define

$$l(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) := \max\Big\{l_1(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}), l_2(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})\Big\}.$$

Proposition 6.8.5. *Let $v \in \mathrm{CS}'(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and suppose that η is the geodesic on H determined by v. Let $(s_j)_{j\in(\alpha,\beta)\cap\mathbb{Z}}$ be the geometric coding sequence of v. Suppose that $s_j = (\widetilde{\mathcal{B}}_j, h_j)$ for $j = 0,\ldots,\beta-2$. For $j = 0,\ldots,\beta-2$ set $g_{-1} := \mathrm{id}$ and $g_j := g_{j-1}h_j$. If $\alpha = -1$, then let $\widetilde{\mathcal{B}}_{-1}$ be the shifted cell in SH such that $v \in \mathrm{CS}'(\widetilde{\mathcal{B}}_{-1})$. Then*

$$s_0 := \min\left\{t \geq 0 \;\middle|\; \eta'(t) \in \mathrm{CS}_{red}\left(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\right)\right\}$$

exists and

$$\eta'(s_0) \in \bigcup_{l=-1}^{\kappa} g_l\, \mathrm{CS}'_{red}(\widetilde{\mathcal{B}}_l)$$

where $\kappa := \min\{l(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) - 1, \beta - 2\}$. More precisely, $\eta'(s_0) \in g_l\, \mathrm{CS}'_{red}(\widetilde{\mathcal{B}}_l)$ for $l \in \{-1,\ldots,\kappa\}$ if and only if $\eta(\infty) \in g_l I_{red}(\widetilde{\mathcal{B}}_l)$ and $\eta(\infty) \notin g_k I_{red}(\widetilde{\mathcal{B}}_k)$ for $k = -1,\ldots,l-1$. Moreover, if $v \in \mathrm{CS}'_{st}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$, then $\eta'(s_0) \in \mathrm{CS}_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.

Proof. Let $(t_n)_{n\in(\alpha,\beta)\cap\mathbb{Z}}$ be the sequence of intersection times of v (w. r. t. CS). Prop. 6.7.26(iii) resp. the choice of $\widetilde{\mathcal{B}}_{-1}$ shows that $\eta'(t_n) \in g_{n-1}\, \mathrm{CS}'(\widetilde{\mathcal{B}}_{n-1})$ for each $n \in [0,\beta)\cap\mathbb{Z}$. Since $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) \subseteq \mathrm{CS}$, the minimum s_0 exists if and only if $\eta'(t_m) \in g_{m-1}\, \mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}_{m-1})$ for some $m \in [0,\beta)\cap\mathbb{Z}$. In this case, $s_0 = t_n$ and $\eta'(s_0) \in g_{n-1}\, \mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}_{n-1})$ where

$$n := \min\left\{m \in [0,\beta)\cap\mathbb{Z} \;\middle|\; \eta'(t_m) \in g_{m-1}\, \mathrm{CS}'_{\mathrm{red}}\big(\widetilde{\mathcal{B}}_{m-1}\big)\right\}.$$

Suppose that $s_0 = t_n$. Note that for each $m \in [0,\beta)\cap\mathbb{Z}$ we have that $\eta(-\infty)$ is in $J(\widetilde{\mathcal{B}}_{m-1})$. Then the definition of $\mathrm{CS}'_{\mathrm{red}}(\cdot)$ shows that

$$n = \min\left\{m \in [0,\beta)\cap\mathbb{Z} \;\middle|\; \eta(\infty) \in g_{m-1} I_{\mathrm{red}}\big(\widetilde{\mathcal{B}}_{m-1}\big)\right\}.$$

Hence it remains to show that the element s_0 exists and that $s_0 = t_n$ for some $n \in \{0,\ldots,\kappa+1\}$.

W.l.o.g. suppose that $I(\widetilde{\mathcal{B}}_{-1}) \in \mathcal{I}_1(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Then $I(\widetilde{\mathcal{B}}_{-1}) = (a,\infty)$ for some $a \in \mathbb{R}$. Let $c_1 < c_2 < \ldots < c_k$ be the increasing sequence in $\mathbb{R}$ such that $c_1 = a$ and $c_k = a_{k_1}$ and such that the set

$$\{(c_j,\infty) \mid j = 1,\ldots,k\}$$

of geodesic segments is the set of connected components of BS of the form (c,∞) with $a \leq c \leq a_{k_1}$. Then $k \leq l(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Let $\{(c_{j_i},\infty) \mid i = 1,\ldots,m\}$ be its subfamily (indexed by $\{1,\ldots,m\}$) of geodesic segments such that for each $i \in \{1,\ldots,m\}$ we have $(c_{j_i},\infty) = b(\widetilde{\mathcal{B}}'_i)$ for some $\widetilde{\mathcal{B}}'_i \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ such that $b(\widetilde{\mathcal{B}}'_i) \in \mathcal{I}_1(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and $c_{j_1} \leq c_{j_2} \leq \ldots \leq c_{j_m}$. The definition of $I_{\mathrm{red}}(\cdot)$ shows that $I(\widetilde{\mathcal{B}}_{-1})$ is the disjoint union $\bigcup_{i=1}^m I_{\mathrm{red}}(\widetilde{\mathcal{B}}'_i)$. Moreover, $J(\widetilde{\mathcal{B}}'_i) \supseteq J(\widetilde{\mathcal{B}}_{-1})$ for $i = 1,\ldots,m$. From Lemma 6.7.9 we know that

$$\big(\eta(\infty), \eta(-\infty)\big) \in I(\widetilde{\mathcal{B}}_{-1}) \times J(\widetilde{\mathcal{B}}_{-1}).$$

Hence there is a unique $i \in \{1, \ldots, m\}$ such that

$$\big(\eta(\infty), \eta(-\infty)\big) \in I_{\mathrm{red}}(\widetilde{\mathcal{B}}'_i) \times J(\widetilde{\mathcal{B}}'_i).$$

In turn, η intersects $\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}'_i)$. Now, if η does not intersect $\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}'_1) = \mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}_{-1})$, then $\eta([0,\infty))$ intersects (c_2, ∞) and we have $g_0 b(\widetilde{\mathcal{B}}_0) = (c_2, \infty)$. If η does not intersect $g_0\,\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}_0)$, then $\eta([0,\infty))$ intersects (c_3,∞) and hence $g_1 b(\widetilde{\mathcal{B}}_1) = (c_3, \infty)$ and so on. This shows that

$$\mathrm{CS}'(\widetilde{\mathcal{B}}'_i) = \mathrm{CS}'(\widetilde{\mathcal{B}}_{j_i-2}) = g_{j_i-2}\,\mathrm{CS}'(\widetilde{\mathcal{B}}_{j_i-2})$$

and hence $\eta'(t_{j_i-1}) \in g_{j_i-2}\,\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}_{j_i-2})$, where $j_i - 1 \leq k \leq l(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. If $\beta < \infty$, then $j_i - 1 \leq \beta - 1$. Thus, s_0 exists and $\eta'(s_0) \in \bigcup_{l=-1}^{\kappa} g_l\,\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}_l)$. Finally, if $v \in \mathrm{CS}'_{\mathrm{st}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$, then $\eta'(s_0) \in \mathrm{CS}_{\mathrm{st}}$ and hence $\eta'(s_0) \in \mathrm{CS}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. $\square$

Recall the shift map $\sigma\colon \Lambda \to \Lambda$ from Sec. 6.7.2.

Proposition 6.8.6. *Let $v \in \mathrm{CS}'_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and suppose that η is the geodesic on H determined by v. Let $\big(s_j\big)_{j\in(\alpha,\beta)\cap\mathbb{Z}}$ be the geometric coding sequence of v. Suppose that $s_j = (\widetilde{\mathcal{B}}_j, h_j)$ for $j = 0, \ldots, \beta - 2$.*

(i) *Set $g_0 := h_0$ and for $j = 0, \ldots, \beta - 3$ define $g_{j+1} := g_j h_{j+1}$. If there is a next point of intersection of η and $\mathrm{CS}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$, then this is on*

$$\bigcup_{l=0}^{\kappa} g_l\,\mathrm{CS}'_{red}(\widetilde{\mathcal{B}}_l)$$

where $\kappa := \min\{l(\mathbb{S},\mathbb{T}), \beta - 2\}$. In this case it is on $g_l\,\mathrm{CS}'_{red}(\widetilde{\mathcal{B}}_l)$ if and only if $\eta(\infty) \in g_l I_{red}(\widetilde{\mathcal{B}}_l)$ and $\eta(\infty) \notin g_k I_{red}(\widetilde{\mathcal{B}}_k)$ for $k = 0, \ldots, l-1$. If $\beta \geq 2$, then there is a next point of intersection of η and $\mathrm{CS}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.

(ii) *Suppose that $v \in \mathrm{CS}'_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Let $(t_n)_{n\in\mathbb{Z}}$ be the sequence of intersection times of v (w. r. t. CS). Then there was a previous point of intersection of η and $\mathrm{CS}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and this is contained in*

$$\big\{\eta'(t_{-n}) \;\big|\; n = 1, \ldots, l\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) + 1\big\}.$$

Proof. We will first prove (i). Suppose that $\beta \geq 2$. Then there is a next point of intersection of η and CS. Let t_0 be the first return time of v w. r. t. CS. Then $\eta'(t_0) \in g_0\,\mathrm{CS}'(\widetilde{\mathcal{B}}_0)$. Set $w := g_0^{-1}\eta'(t_0)$. The geometric coding sequence of w is given by $\sigma\big((s_j)_{j\in(\alpha,\beta)\cap\mathbb{Z}}\big)$. Let γ be the geodesic on H determined by w. Prop. 6.8.5 shows that there is a next point of intersection of γ and $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$, say $\gamma'(s_0)$, and that

$$\gamma'(s_0) \in \bigcup_{l=0}^{\kappa} g_0^{-1} g_l\,\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}_l).$$

Note that the condition that $\kappa \leq \beta - 2$ is caused by the length of the geometric coding sequence, not by any properties of $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Then $\eta'(s_0) = g_0\gamma'(s_0)$

is the next point of intersection of η and $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. The remaining part of (i) is shown by Prop. 6.8.5.

To prove (ii) consider $\eta'(t_{-(l(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})+1)})$. There exists $k \in \Gamma$ such that $u := k\eta'(t_{-(l(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})+1)}) \in \mathrm{CS}'_{\mathrm{st}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Let α be the geodesic on H determined by u. The first $l(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})+1$ intersections of $\alpha'([0,\infty))$ and CS are given by

$$k\eta'\big(t_{-(l(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})+1)}\big), k\eta'\big(t_{-l(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})}\big), \ldots, k\eta'\big(t_{-2}\big), k\eta'\big(t_{-1}\big)$$

(in this order). Prop. 6.8.5 implies that at least one of these is in $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Note that none of these elements equals kv. Hence there was a previous point of intersection of η and $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. □

Recall from Remark 6.5.5 that NIC denotes the set of geodesics on Y with at least one endpoint contained in $\pi(\mathrm{bd})$.

Corollary 6.8.7. *Let μ be a measure on the space of geodesics on Y. The sets $\widehat{\mathrm{CS}}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and $\widehat{\mathrm{CS}}_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ are cross sections w. r. t. μ if and only if* NIC *is a μ-null set. Moreover, $\widehat{\mathrm{CS}}_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is the maximal strong cross section contained in $\widehat{\mathrm{CS}}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.*

Proof. Let $v \in \mathrm{CS}'_{\mathrm{st}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and suppose that γ is the geodesic determined by v. Since $\gamma(\mathbb{R}) \cap \mathrm{CS} \subseteq \mathrm{CS}_{\mathrm{st}}$, each intersection of γ and $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is an intersection of γ and $\mathrm{CS}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Then Prop. 6.8.6 implies that each geodesic on Y which intersects $\widehat{\mathrm{CS}}$ infinitely often in future and past also intersects $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ infinitely often in future and past. Because

$$\mathrm{CS}_{\mathrm{st,red}}\left(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\right) \subseteq \mathrm{CS}_{\mathrm{red}}\left(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\right) \subseteq \mathrm{CS},$$

Theorem 6.7.17 shows that $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ are cross section w. r. t. μ if and only if $\mu(\mathrm{NIC}) = 0$.

Moreover, each geodesic on Y which does not intersect $\widehat{\mathrm{CS}}$ infinitely often in future or past cannot intersect $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ or $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ infinitely often in future or past. This and the previous observation imply that $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is indeed the maximal strong cross section contained in $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. □

6.8.2. Reduced coding sequences and arithmetic symbolic dynamics

Analogous to the labeling of CS in Sec. 6.7.2 we define a labeling of $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.

Let $v \in \mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and let γ denote the geodesic on H determined by v. Suppose first that $\gamma((0,\infty)) \cap \mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) \neq \emptyset$. Prop. 6.8.6 implies that there is a next point of intersection of γ and $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and that this is on $g\,\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}})$ for a (uniquely determined) pair $(\widetilde{\mathcal{B}}, g) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma$. We endow v with the label $(\widetilde{\mathcal{B}}, g)$.

Suppose now that $\gamma((0,\infty)) \cap \mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) = \emptyset$. Then there is no next point of intersection of γ and $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. We label v by ε.

Let $\widehat{v} \in \widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and let $v := \left(\pi|_{\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})}\right)^{-1}(\widehat{v})$. The label of $\widehat{v}$ and of each element in $\pi^{-1}(\widehat{v})$ is defined to be the label of v.

Suppose that Σ_{red} denotes the set of labels of $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.

Remark 6.8.8. Recall from Cor. 6.7.21 that Σ is finite. Then Prop. 6.8.6 implies that also Σ_{red} is finite. Moreover, Remark 6.7.11 shows that the elements of Σ can be effectively determined. From Prop. 6.8.6 then follows that also the elements of Σ_{red} can be effectively determined.

The following definition is analogous to the corresponding definitions in Sec. 6.7.2.

Definition 6.8.9. Let $v \in \mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and suppose that γ is the geodesic on H determined by v. Prop. 6.8.5 and 6.8.6 imply that there is a unique sequence $(t_n)_{n\in J}$ in $\mathbb{R}$ which satisfies the following properties:

(i) $J = \mathbb{Z} \cap (a,b)$ for some interval (a,b) with $a,b \in \mathbb{Z} \cup \{\pm\infty\}$ and $0 \in (a,b)$,

(ii) the sequence $(t_n)_{n\in J}$ is increasing,

(iii) $t_0 = 0$,

(iv) for each $n \in J$ we have $\gamma'(t_n) \in \mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and $\gamma'((t_n,t_{n+1})) \cap \mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) = \emptyset$ and $\gamma'((t_{n-1},t_n)) \cap \mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) = \emptyset$ where we set $t_b := \infty$ if $b < \infty$ and $t_a := -\infty$ if $a > -\infty$.

The sequence $(t_n)_{n\in J}$ is said to be the *sequence of intersection times of* v *w.r.t.* $\mathrm{CS}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.

Let $\widehat{v} \in \widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and set $v := \left(\pi|_{\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})}\right)^{-1}(\widehat{v})$. Then the *sequence of intersection times w.r.t.* $\mathrm{CS}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ *of* $\widehat{v}$ and *of* each $w \in \pi^{-1}(\widehat{v})$ is defined to be the sequence of intersection times of v w.r.t. $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.

For each $s \in \Sigma_{\mathrm{red}}$ set

$$\widehat{\mathrm{CS}}_{\mathrm{red},s}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) := \big\{\widehat{v} \in \widehat{\mathrm{CS}}_{\mathrm{red}}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big) \;\big|\; \widehat{v} \text{ is labeled with } s\big\}$$

and

$$\mathrm{CS}_{\mathrm{red},s}\left(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\right) := \pi^{-1}\big(\widehat{\mathrm{CS}}_{\mathrm{red}}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big)\big) = \big\{v \in \mathrm{CS}_{\mathrm{red}}\left(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\right) \;\big|\; v \text{ is labeled with } s\big\}.$$

Let $\widehat{v} \in \widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and let $(t_n)_{n\in J}$ be the sequence of intersection times of $\widehat{v}$ w.r.t. $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Suppose that $\widehat{\gamma}$ is the geodesic on Y determined by $\widehat{v}$. The *reduced coding sequence* of $\widehat{v}$ is the sequence $(a_n)_{n\in J}$ in Σ_{red} defined by

$$a_n := s \quad \text{if and only if} \quad \widehat{\gamma}'(t_n) \in \widehat{\mathrm{CS}}_{\mathrm{red},s}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big)$$

for each $n \in J$.

Let $w \in \mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. The *reduced coding sequence* of w is defined to be the reduced coding sequence of $\pi(w)$.

Let Λ_{red} denote the set of reduced coding sequences and let $\Lambda_{\mathrm{red},\sigma}$ be the subset of Λ_{red} which contains the reduced coding sequences $(a_n)_{n\in(a,b)\cap\mathbb{Z}}$ with $a, b \in \mathbb{Z}\cup\{\pm\infty\}$ for which $b \geq 2$. Further, let $\Lambda_{\mathrm{st,red}}$ denote the set of two-sided infinite reduced coding sequences. Let $\Sigma^{\mathrm{all}}_{\mathrm{red}}$ be the set of all finite and one- or two-sided infinite sequences in Σ_{red}. Finally, let $\mathrm{Seq}_{\mathrm{red}}\colon \widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) \to \Lambda_{\mathrm{red}}$ be the map which assigns to $\widehat{v} \in \widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ the reduced coding sequence of $\widehat{v}$.

The proofs of Prop. 6.8.10, 6.8.11 and 6.8.12 are analogous to those of the corresponding statements in Sec. 6.7.2.

Proposition 6.8.10. *Let $v \in \mathrm{CS}'_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Suppose that $(t_n)_{n\in J}$ is the sequence of intersection times of v and that $(a_n)_{n\in J}$ is the reduced coding sequence of v. Let γ be the geodesic on H determined by v. Suppose that $J = \mathbb{Z}\cap(a,b)$ with $a, b \in \mathbb{Z}\cup\{\pm\infty\}$.*

(i) *If $b = \infty$, then $a_n \in \Sigma_{red}\setminus\{\varepsilon\}$ for each $n \in J$.*

(ii) *If $b < \infty$, then $a_n \in \Sigma_{red}\setminus\{\varepsilon\}$ for each $n \in (a, b-2]\cap\mathbb{Z}$ and $a_{b-1} = \varepsilon$.*

(iii) *Suppose that $a_n = (\widetilde{\mathcal{B}}_n, h_n)$ for $n \in (a, b-1)\cap\mathbb{Z}$ and set*

$$\begin{aligned} g_0 &:= h_0 && \text{if } b \geq 2,\\ g_{n+1} &:= g_n h_{n+1} && \text{for } n \in [0, b-2)\cap\mathbb{Z},\\ g_{-1} &:= \mathrm{id}, && \\ g_{-(n+1)} &:= g_{-n}h_{-n}^{-1} && \text{for } n \in [1, -(a+1))\cap\mathbb{Z}. \end{aligned}$$

Then $\gamma'(t_{n+1}) \in g_n\,\mathrm{CS}'_{red}(\widetilde{\mathcal{B}}_n)$ for each $n \in (a, b-1)\cap\mathbb{Z}$.

Proposition 6.8.11. *Let $v, w \in \mathrm{CS}'_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. If the reduced coding sequences of v and w are equal, then $v = w$.*

Proposition 6.8.12.

(i) *The left shift $\sigma\colon \Sigma^{all}_{red} \to \Sigma^{all}_{red}$ induces a partially defined map $\sigma\colon \Lambda_{red} \to \Lambda_{red}$ resp. a map $\sigma\colon \Lambda_{red,\sigma} \to \Lambda_{red}$. Moreover, $\Lambda_{st,red} \subseteq \Lambda_{red,\sigma}$ and σ restricts to a map $\Lambda_{st,red} \to \Lambda_{st,red}$.*

(ii) *The map $\mathrm{Seq}_{red}|_{\widehat{\mathrm{CS}}_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})}\colon \widehat{\mathrm{CS}}_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) \to \Lambda_{st,red}$ is bijective.*

(iii) *Let $\mathrm{Cod}_{red} := \left(\mathrm{Seq}_{red}|_{\widehat{\mathrm{CS}}_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})}\right)^{-1}$. Then the diagrams*

$$\begin{tikzcd} \widehat{\mathrm{CS}}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) \arrow[r, "R"] \arrow[d, "\mathrm{Seq}_{red}"'] & \widehat{\mathrm{CS}}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) \arrow[d, "\mathrm{Seq}_{red}"] \\ \Lambda_{red} \arrow[r, "\sigma"] & \Lambda_{red} \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathrm{CS}}_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) \arrow[r, "R"] & \widehat{\mathrm{CS}}_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) \\ \Lambda_{st,red} \arrow[r, "\sigma"] \arrow[u, "\mathrm{Cod}_{red}"] & \Lambda_{st,red} \arrow[u, "\mathrm{Cod}_{red}"'] \end{tikzcd}$$

commute and $(\Lambda_{st,red}, \sigma)$ is a symbolic dynamics for the geodesic flow on Y.

We will now show that the reduced coding sequence of $\widehat{v} \in \widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ can be completely constructed from the knowledge of the pair $\tau(\widehat{v})$.

Definition 6.8.13. Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. Define

$$\Sigma_{\mathrm{red}}\big(\widetilde{\mathcal{B}}\big) := \big\{ s \in \Sigma_{\mathrm{red}} \,\big|\, \exists\, v \in \mathrm{CS}'_{\mathrm{red}}\big(\widetilde{\mathcal{B}}\big) \colon v \text{ is labeled with } s \big\}$$

and for $s \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}})$ set

$$D_s\big(\widetilde{\mathcal{B}}\big) := I_{\mathrm{red}}\big(\widetilde{\mathcal{B}}\big) \cap g I_{\mathrm{red}}\big(\widetilde{\mathcal{B}}'\big) \quad \text{if } s = \big(\widetilde{\mathcal{B}}', g\big)$$

and

$$D_\varepsilon\big(\widetilde{\mathcal{B}}\big) := I_{\mathrm{red}}\big(\widetilde{\mathcal{B}}\big) \setminus \bigcup \big\{ D_s\big(\widetilde{\mathcal{B}}\big) \,\big|\, s \in \Sigma_{\mathrm{red}}\big(\widetilde{\mathcal{B}}\big) \setminus \{\varepsilon\} \big\}.$$

Example 6.8.14. Recall the Example 6.7.15. Suppose first that the shift map is $\mathbb{T}_1$. Then we have

$$\begin{aligned}
\Sigma_{\mathrm{red}}\big(\widetilde{\mathcal{B}}_1\big) &= \big\{\varepsilon, \big(\widetilde{\mathcal{B}}_1, g_5\big), \big(\widetilde{\mathcal{B}}_4, g_4\big), \big(\widetilde{\mathcal{B}}_6, g_4\big), \big(\widetilde{\mathcal{B}}_5, g_4\big), \big(\widetilde{\mathcal{B}}_3, g_4\big)\big\},\\
\Sigma_{\mathrm{red}}\big(\widetilde{\mathcal{B}}_2\big) &= \big\{\varepsilon, \big(\widetilde{\mathcal{B}}_2, g_1\big), \big(\widetilde{\mathcal{B}}_4, g_1\big), \big(\widetilde{\mathcal{B}}_6, g_1\big), \big(\widetilde{\mathcal{B}}_5, g_1\big), \big(\widetilde{\mathcal{B}}_3, g_1\big)\big\},\\
\Sigma_{\mathrm{red}}\big(\widetilde{\mathcal{B}}_3\big) &= \big\{\varepsilon, \big(\widetilde{\mathcal{B}}_1, g_5\big), \big(\widetilde{\mathcal{B}}_2, g_6\big), \big(\widetilde{\mathcal{B}}_4, g_6\big), \big(\widetilde{\mathcal{B}}_6, g_6\big), \big(\widetilde{\mathcal{B}}_5, g_6\big), \big(\widetilde{\mathcal{B}}_3, g_6\big)\big\},\\
\Sigma_{\mathrm{red}}\big(\widetilde{\mathcal{B}}_4\big) &= \big\{\varepsilon, \big(\widetilde{\mathcal{B}}_5, g_2\big), \big(\widetilde{\mathcal{B}}_3, g_2\big)\big\},\\
\Sigma_{\mathrm{red}}\big(\widetilde{\mathcal{B}}_5\big) &= \big\{\varepsilon, \big(\widetilde{\mathcal{B}}_6, g_4\big), \big(\widetilde{\mathcal{B}}_5, g_4\big), \big(\widetilde{\mathcal{B}}_3, g_4\big)\big\},\\
\Sigma_{\mathrm{red}}\big(\widetilde{\mathcal{B}}_6\big) &= \big\{\varepsilon, \big(\widetilde{\mathcal{B}}_3, g_3\big)\big\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&D_{(\widetilde{\mathcal{B}}_1,g_5)}\big(\widetilde{\mathcal{B}}_1\big) = \big(\tfrac{4}{5}, 1\big), && D_{(\widetilde{\mathcal{B}}_4,g_4)}\big(\widetilde{\mathcal{B}}_1\big) = \big(-\infty, \tfrac{3}{5}\big], && D_{(\widetilde{\mathcal{B}}_6,g_4)}\big(\widetilde{\mathcal{B}}_1\big) = \big(\tfrac{3}{5}, \tfrac{7}{10}\big],\\
&D_{(\widetilde{\mathcal{B}}_5,g_4)}\big(\widetilde{\mathcal{B}}_1\big) = \big(\tfrac{7}{10}, \tfrac{11}{15}\big], && D_{(\widetilde{\mathcal{B}}_3,g_4)}\big(\widetilde{\mathcal{B}}_1\big) = \big(\tfrac{11}{15}, \tfrac{4}{5}\big), && D_\varepsilon\big(\widetilde{\mathcal{B}}_1\big) = \big\{\tfrac{4}{5}\big\},
\end{aligned}$$

and

$$\begin{aligned}
&D_{(\widetilde{\mathcal{B}}_2,g_1)}\big(\widetilde{\mathcal{B}}_2\big) = \big(0, \tfrac{1}{10}\big], && D_{(\widetilde{\mathcal{B}}_4,g_1)}\big(\widetilde{\mathcal{B}}_2\big) = \big(\tfrac{1}{10}, \tfrac{2}{15}\big], && D_{(\widetilde{\mathcal{B}}_6,g_1)}\big(\widetilde{\mathcal{B}}_2\big) = \big(\tfrac{2}{15}, \tfrac{3}{20}\big],\\
&D_{(\widetilde{\mathcal{B}}_5,g_1)}\big(\widetilde{\mathcal{B}}_2\big) = \big(\tfrac{3}{20}, \tfrac{4}{25}\big], && D_{(\widetilde{\mathcal{B}}_3,g_1)}\big(\widetilde{\mathcal{B}}_2\big) = \big(\tfrac{4}{25}, \tfrac{1}{5}\big), && D_\varepsilon\big(\widetilde{\mathcal{B}}_2\big) = \big\{\tfrac{1}{5}\big\},
\end{aligned}$$

and

$$\begin{aligned}
&D_{(\widetilde{\mathcal{B}}_1,g_5)}\big(\widetilde{\mathcal{B}}_3\big) = \big(\tfrac{4}{5}, 1\big), && D_{(\widetilde{\mathcal{B}}_2,g_6)}\big(\widetilde{\mathcal{B}}_3\big) = \big(1, \tfrac{6}{5}\big], && D_{(\widetilde{\mathcal{B}}_4,g_6)}\big(\widetilde{\mathcal{B}}_3\big) = \big(\tfrac{6}{5}, \tfrac{7}{5}\big],\\
&D_{(\widetilde{\mathcal{B}}_6,g_6)}\big(\widetilde{\mathcal{B}}_3\big) = \big(\tfrac{7}{5}, \tfrac{8}{5}\big], && D_{(\widetilde{\mathcal{B}}_5,g_6)}\big(\widetilde{\mathcal{B}}_3\big) = \big(\tfrac{8}{5}, \tfrac{9}{5}\big], && D_{(\widetilde{\mathcal{B}}_3,g_6)}\big(\widetilde{\mathcal{B}}_3\big) = \big(\tfrac{9}{5}, \infty\big),\\
&D_\varepsilon\big(\widetilde{\mathcal{B}}_3\big) = \{1\},
\end{aligned}$$

and

$$D_{(\widetilde{\mathcal{B}}_5,g_2)}\big(\widetilde{\mathcal{B}}_4\big) = \big(\tfrac{1}{5}, \tfrac{3}{10}\big], \quad D_{(\widetilde{\mathcal{B}}_3,g_2)}\big(\widetilde{\mathcal{B}}_4\big) = \big(\tfrac{3}{10}, \tfrac{2}{5}\big), \quad D_\varepsilon\big(\widetilde{\mathcal{B}}_4\big) = \big\{\tfrac{2}{5}\big\},$$

and

$$D_{(\widetilde{\mathcal{B}}_6,g_4)}(\widetilde{\mathcal{B}}_5) = \left(\tfrac{3}{5}, \tfrac{7}{10}\right], \quad D_{(\widetilde{\mathcal{B}}_5,g_4)}(\widetilde{\mathcal{B}}_5) = \left(\tfrac{7}{10}, \tfrac{11}{15}\right], \quad D_{(\widetilde{\mathcal{B}}_3,g_4)}(\widetilde{\mathcal{B}}_5) = \left(\tfrac{11}{15}, \tfrac{4}{5}\right),$$
$$D_\varepsilon(\widetilde{\mathcal{B}}_5) = \left\{\tfrac{4}{5}\right\},$$

and

$$D_{(\widetilde{\mathcal{B}}_3,g_3)}(\widetilde{\mathcal{B}}_6) = \left(\tfrac{2}{5}, \tfrac{3}{5}\right), \qquad D_\varepsilon(\widetilde{\mathcal{B}}_6) = \left\{\tfrac{3}{5}\right\}.$$

Suppose now that the shift map is $\mathbb{T}_2$. Then $\Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}}_2), \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}}_4), \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}}_5)$ and $\Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}}_6)$ are as for $\mathbb{T}_1$. The sets $D_*(\widetilde{\mathcal{B}}_2), D_*(\widetilde{\mathcal{B}}_4), D_*(\widetilde{\mathcal{B}}_5)$ and $D_*(\widetilde{\mathcal{B}}_6)$ remain unchanged as well. We have

$$\Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}}_{-1}) = \left\{\varepsilon, (\widetilde{\mathcal{B}}_{-1}, g_7), (\widetilde{\mathcal{B}}_4, g_7), (\widetilde{\mathcal{B}}_6, g_7), (\widetilde{\mathcal{B}}_5, g_7), (\widetilde{\mathcal{B}}_3, g_7)\right\}$$

and

$$\Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}}_3) = \left\{\varepsilon, (\widetilde{\mathcal{B}}_{-1}, g_4), (\widetilde{\mathcal{B}}_2, g_2), (\widetilde{\mathcal{B}}_4, g_6), (\widetilde{\mathcal{B}}_6, g_6), (\widetilde{\mathcal{B}}_5, g_6), (\widetilde{\mathcal{B}}_3, g_6)\right\}.$$

Therefore

$$\begin{aligned} D_{(\widetilde{\mathcal{B}}_{-1},g_7)}(\widetilde{\mathcal{B}}_{-1}) &= \left(-\tfrac{1}{5}, 0\right), & D_{(\widetilde{\mathcal{B}}_4,g_7)}(\widetilde{\mathcal{B}}_{-1}) &= \left(-\infty, -\tfrac{2}{5}\right), \\ D_{(\widetilde{\mathcal{B}}_6,g_7)}(\widetilde{\mathcal{B}}_{-1}) &= \left(-\tfrac{2}{5}, -\tfrac{3}{10}\right), & D_{(\widetilde{\mathcal{B}}_5,g_7)}(\widetilde{\mathcal{B}}_{-1}) &= \left(-\tfrac{3}{10}, -\tfrac{4}{15}\right), \\ D_{(\widetilde{\mathcal{B}}_3,g_7)}(\widetilde{\mathcal{B}}_{-1}) &= \left(-\tfrac{4}{15}, -\tfrac{1}{5}\right), & D_\varepsilon(\widetilde{\mathcal{B}}_{-1}) &= \left\{-\tfrac{1}{5}\right\}, \end{aligned}$$

and

$$\begin{aligned} D_{(\widetilde{\mathcal{B}}_{-1},g_4)}(\widetilde{\mathcal{B}}_3) &= \left(\tfrac{4}{5}, 1\right), & D_{(\widetilde{\mathcal{B}}_2,g_6)}(\widetilde{\mathcal{B}}_3) &= \left(1, \tfrac{6}{5}\right], \\ D_{(\widetilde{\mathcal{B}}_4,g_6)}(\widetilde{\mathcal{B}}_3) &= \left(\tfrac{6}{5}, \tfrac{7}{5}\right], & D_{(\widetilde{\mathcal{B}}_6,g_6)}(\widetilde{\mathcal{B}}_3) &= \left(\tfrac{7}{5}, \tfrac{8}{5}\right], \\ D_{(\widetilde{\mathcal{B}}_5,g_6)}(\widetilde{\mathcal{B}}_3) &= \left(\tfrac{8}{5}, \tfrac{9}{5}\right], & D_{(\widetilde{\mathcal{B}}_3,g_6)}(\widetilde{\mathcal{B}}_3) &= \left(\tfrac{9}{5}, \infty\right), \\ D_\varepsilon(\widetilde{\mathcal{B}}_3) &= \{1\}. \end{aligned}$$

The next corollary follows immediately from Prop. 6.8.6.

Corollary 6.8.15. *Let $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. Then $I_{red}(\widetilde{\mathcal{B}})$ decomposes into the disjoint union $\bigcup\{D_s(\widetilde{\mathcal{B}}) \mid s \in \Sigma_{red}(\widetilde{\mathcal{B}})\}$. Let $v \in \mathrm{CS}'_{red}(\widetilde{\mathcal{B}})$ and suppose that γ is the geodesic on H determined by v. Then v is labeled with s if and only if $\gamma(\infty)$ belongs to $D_s(\widetilde{\mathcal{B}})$.*

Our next goal is to find a discrete dynamical system on the geodesic boundary of H which is conjugate to $(\widehat{\mathrm{CS}}_{\mathrm{red}}(\mathbb{B}_{\mathbb{S},\mathbb{T}}), R)$. To that end we set

$$\widetilde{\mathrm{DS}} := \bigcup_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} I_{\mathrm{red}}(\widetilde{\mathcal{B}}) \times J(\widetilde{\mathcal{B}}).$$

For $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and $s \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}})$ we set

$$\widetilde{D}_s(\widetilde{\mathcal{B}}) := D_s(\widetilde{\mathcal{B}}) \times J(\widetilde{\mathcal{B}}).$$

We define the partial map $\widetilde{F}\colon \widetilde{\mathrm{DS}} \to \widetilde{\mathrm{DS}}$ by

$$\widetilde{F}|_{\widetilde{D}_s(\widetilde{\mathcal{B}})}(x,y) := (g^{-1}x, g^{-1}y)$$

if $s = (\widetilde{\mathcal{B}}', g) \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}})$ and $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$.

Recall the map

$$\tau\colon \begin{cases} \widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}) & \to & \partial_g H \times \partial_g H \\ \widehat{v} & \mapsto & \big(\gamma_v(\infty), \gamma_v(-\infty)\big) \end{cases}$$

where $v := \left(\pi|_{\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})}\right)^{-1}(\widehat{v})$ and γ_v is the geodesic on H determined by v.

Proposition 6.8.16. *The set $\widetilde{\mathrm{DS}}$ is the disjoint union*

$$\widetilde{\mathrm{DS}} = \bigcup \left\{ \widetilde{D}_s(\widetilde{\mathcal{B}}) \;\middle|\; \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}},\ s \in \Sigma_{red}(\widetilde{\mathcal{B}}) \right\}.$$

If $(x,y) \in \widetilde{\mathrm{DS}}$, then there is a unique element $v \in \mathrm{CS}'_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ such that

$$\big(\gamma_v(\infty), \gamma_v(-\infty)\big) = (x,y).$$

If and only if $(x,y) \in \widetilde{D}_s(\widetilde{\mathcal{B}})$, the element v is labeled with s. Moreover, the partial map $\widetilde{F}$ is well-defined and the discrete dynamical system $(\widetilde{\mathrm{DS}}, \widetilde{F})$ is conjugate to $(\widehat{\mathrm{CS}}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}), R)$ via τ.

Proof. The sets $I_{\mathrm{red}}(\widetilde{\mathcal{B}}) \times J(\widetilde{\mathcal{B}})$, $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$, are pairwise disjoint by construction. Cor. 6.8.15 states that for each $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ the set $I_{\mathrm{red}}(\widetilde{\mathcal{B}})$ is the disjoint union $\bigcup\{D_s(\widetilde{\mathcal{B}}) \mid s \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}})\}$. Therefore, the sets $\widetilde{D}_s(\widetilde{\mathcal{B}})$, $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$, $s \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}})$, are pairwise disjoint and

$$\widetilde{\mathrm{DS}} = \bigcup \left\{ \widetilde{D}_s(\widetilde{\mathcal{B}}) \;\middle|\; \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}},\ s \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}}) \right\}.$$

This implies that $\widetilde{F}$ is well-defined. Let $(x,y) \in \widetilde{\mathrm{DS}}$. By Lemma 6.8.4 there is a unique $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and a unique $v \in \mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}})$ such that $\big(\gamma_v(\infty), \gamma_v(-\infty)\big) = (x,y)$. Cor. 6.8.15 shows that v is labeled with $s \in \Sigma_{\mathrm{red}}$ if and only if $\gamma_v(\infty) \in D_s(\widetilde{\mathcal{B}})$, hence if $(x,y) \in \widetilde{D}_s(\widetilde{\mathcal{B}})$. It remains to show that $(\widetilde{\mathrm{DS}}, \widetilde{F})$ is conjugate to $(\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}), R)$ by τ. Lemma 6.8.4 shows that τ is a bijection between $\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and $\widetilde{\mathrm{DS}}$. Let $\widehat{v} \in \widehat{\mathrm{CS}}_{\mathrm{red}}$ and $v := \big(\pi|_{\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})}\big)^{-1}(\widehat{v})$. Suppose that $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ is the (unique) shifted cell in SH such that $v \in \mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}})$, and let $(s_j)_{j\in(\alpha,\beta)\cap\mathbb{Z}}$ be the reduced coding sequence of v. Recall that s_0 is the label of v and $\widehat{v}$. Cor. 6.8.15 shows that $\gamma_v(\infty) \in D_{s_0}(\widetilde{\mathcal{B}})$. The map R is defined for $\widehat{v}$ if and only if $s_0 \neq \varepsilon$. In precisely this case, $\widetilde{F}$ is defined for $\tau(\widehat{v})$.

Suppose that $s_0 \neq \varepsilon$, say $s_0 = (\widetilde{\mathcal{B}}', g)$. Then the next intersection of γ_v and $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is on $g\,\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}')$, say it is w. Then $R(\widehat{v}) = \pi(w) =: \widehat{w}$ and $\big(\pi|_{\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})}\big)^{-1}(\widehat{w}) = g^{-1}w$. Let η be the geodesic on H determined by $g^{-1}w$. We have to show that $\widetilde{F}(\tau(\widehat{v})) = \big(\eta(\infty), \eta(-\infty)\big)$. To that end note that $g\eta(\mathbb{R}) = \gamma_v(\mathbb{R})$ and hence $\big(\eta(\infty), \eta(-\infty)\big) = \big(g^{-1}\gamma_v(\infty), g^{-1}\gamma_v(-\infty)\big)$. Since $\tau(\widehat{v}) \in \widetilde{D}_{s_0}(\widetilde{\mathcal{B}})$, the definition of $\widetilde{F}$ shows that

$$\widetilde{F}(\tau(\widehat{v})) = \widetilde{F}\big((\gamma_v(\infty), \gamma_v(-\infty))\big) = \big(g^{-1}\gamma_v(\infty), g^{-1}\gamma_v(-\infty)\big) = \big(\eta(\infty), \eta(-\infty)\big).$$

Thus, $(\widetilde{\mathrm{DS}}, \widetilde{F})$ is conjugate to $(\widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}), R)$ by τ. $\square$

The following corollary proves that we can reconstruct the future part of the reduced coding sequence of $\widehat{v} \in \widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ from $\tau(\widehat{v})$.

Corollary 6.8.17. *Let $\widehat{v} \in \widehat{\mathrm{CS}}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and suppose that $(s_j)_{j\in J}$ is the reduced coding sequence of v. Then*

$$s_j = s \qquad \textit{if and only if} \qquad \widetilde{F}^j\big(\tau(\widehat{v})\big) \in \widetilde{D}_s\big(\widetilde{\mathcal{B}}\big) \textit{ for some } \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$$

for each $j \in J \cap \mathbb{N}_0$. For $j \in \mathbb{N}_0 \smallsetminus J$, the map $\widetilde{F}^j$ is not defined for $\tau(\widehat{v})$.

The next proposition shows that we can also reconstruct the past part of the reduced coding sequence of $\widehat{v} \in \widehat{\mathrm{CS}}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ from $\tau(\widehat{v})$. Its proof is constructive.

Proposition 6.8.18.

(i) *The elements of*

$$\Big\{ g^{-1}\widetilde{D}_{(\widetilde{\mathcal{B}},g)}(\widetilde{\mathcal{B}}') \;\Big|\; \widetilde{\mathcal{B}}' \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}},\ (\widetilde{\mathcal{B}}, g) \in \Sigma_{red}(\widetilde{\mathcal{B}}') \Big\}$$

are pairwise disjoint.

(ii) *Let $\widehat{v} \in \widehat{\mathrm{CS}}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and suppose that $(a_j)_{j\in J}$ is the reduced coding sequence of $\widehat{v}$. Then $-1 \in J$ if and only if*

$$\tau(\widehat{v}) \in \bigcup \Big\{ g^{-1}\widetilde{D}_{(\widetilde{\mathcal{B}},g)}(\widetilde{\mathcal{B}}') \;\Big|\; \widetilde{\mathcal{B}}' \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}},\ (\widetilde{\mathcal{B}}, g) \in \Sigma_{red}(\widetilde{\mathcal{B}}') \Big\}.$$

In this case,

$$a_{-1} = \big(\widetilde{\mathcal{B}}, g\big) \qquad \textit{if and only if} \qquad \tau(\widehat{v}) \in g^{-1}\widetilde{D}_{(\widetilde{\mathcal{B}},g)}\big(\widetilde{\mathcal{B}}'\big)$$

for some $\widetilde{\mathcal{B}}' \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and $(\widetilde{\mathcal{B}}, g) \in \Sigma_{red}(\widetilde{\mathcal{B}}')$.

Proof. We will prove (ii), which directly implies (i). To that end set

$$v := \Big(\pi|_{\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})}\Big)^{-1}(\widehat{v})$$

and suppose that $(t_j)_{j\in J}$ is the sequence of intersection times of v w. r. t. $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.

Suppose first that $v \in \mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}})$ and that $-1 \in J$. Then there exists a (unique) pair $(\widetilde{\mathcal{B}}', g) \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \times \Gamma$ such that $\gamma'_v(t_{-1}) \in g^{-1}\,\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}}')$. Since the unit tangent vector $\gamma'_v(t_0) = v$ is contained in $\mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathcal{B}})$, the element $\gamma'_v(t_{-1})$ is labeled with $(\widetilde{\mathcal{B}}, g)$. Hence $(\widetilde{\mathcal{B}}, g) \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}}')$. Then

$$\begin{aligned}
\tau(\widehat{v}) = \big(\gamma_v(\infty), \gamma_v(-\infty)\big) &\in \Big(g^{-1} I_{\mathrm{red}}(\widetilde{\mathcal{B}}') \times g^{-1} J(\widetilde{\mathcal{B}}')\Big) \cap \Big(I_{\mathrm{red}}(\widetilde{\mathcal{B}}) \times J(\widetilde{\mathcal{B}})\Big) \\
&= \Big(g^{-1} I_{\mathrm{red}}(\widetilde{\mathcal{B}}') \cap I_{\mathrm{red}}(\widetilde{\mathcal{B}})\Big) \times \Big(g^{-1} J(\widetilde{\mathcal{B}}') \cap J(\widetilde{\mathcal{B}})\Big) \\
&= g^{-1}\Big(\Big(I_{\mathrm{red}}(\widetilde{\mathcal{B}}') \cap g I_{\mathrm{red}}(\widetilde{\mathcal{B}})\Big) \times \Big(J(\widetilde{\mathcal{B}}) \cap g J(\widetilde{\mathcal{B}})\Big)\Big) \\
&\subseteq g^{-1} \widetilde{D}_{(\widetilde{\mathcal{B}}, g)}(\widetilde{\mathcal{B}}').
\end{aligned}$$

Conversely suppose that $\tau(\widehat{v}) \in g^{-1}\widetilde{D}_{(\widetilde{\mathcal{B}},g)}(\widetilde{\mathcal{B}}')$ for some $\widetilde{\mathcal{B}}' \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and some $(\widetilde{\mathcal{B}}, g) \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}}')$. Consider the geodesic $\eta := g\gamma_v$. Then

$$\big(\eta(\infty), \eta(-\infty)\big) = g\tau(\widehat{v}) \in \widetilde{D}_{(\widetilde{\mathcal{B}},g)}(\widetilde{\mathcal{B}}').$$

By Prop. 6.8.16, there is a unique $u \in \mathrm{CS}'_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ such that $\big(\gamma_u(\infty), \gamma_u(-\infty)\big) = \big(\eta(\infty), \eta(-\infty)\big)$. Moreover, u is labeled with $(\widetilde{\mathcal{B}}, g)$. Let $(s_k)_{k\in K}$ be the sequence of intersection times of u w.r.t. $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Then $1 \in K$ and, by Prop. 6.8.16,

$$\begin{aligned}
\tau\big(\pi(\gamma'_u(s_1))\big) &= \tau\big(R(\pi(u))\big) = \widetilde{F}\big(\tau(\pi(u))\big) = \widetilde{F}\big(\gamma_u(\infty), \gamma_u(-\infty)\big) \\
&= \big(g^{-1}\gamma_u(\infty), g^{-1}\gamma_u(-\infty)\big) = \big(\gamma_v(\infty), \gamma_v(-\infty)\big) = \tau(\widehat{v}).
\end{aligned}$$

This shows that $\gamma'_u(s_1) = gv = g\gamma'_v(t_0)$. Then

$$g^{-1}\gamma'_u(s_0) \in \gamma'_v\big((-\infty, 0)\big) \cap \mathrm{CS}_{\mathrm{red}}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}\big).$$

Hence, there was a previous point of intersection of γ_v and $\mathrm{CS}_{\mathrm{red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and this is $g^{-1}\gamma'_u(s_0)$. Recall that $g^{-1}\gamma'_u(s_0)$ is labeled with $(\widetilde{\mathcal{B}}, g)$. This completes the proof. □

Let $\widetilde{F}_{\mathrm{bk}} \colon \widetilde{\mathrm{DS}} \to \widetilde{\mathrm{DS}}$ be the partial map defined by

$$\widetilde{F}_{\mathrm{bk}}|_{g^{-1}\widetilde{D}_{(\widetilde{\mathcal{B}},g)}(\widetilde{\mathcal{B}}')}(x, y) := (gx, gy)$$

for $\widetilde{\mathcal{B}}' \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and $(\widetilde{\mathcal{B}}, g) \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}}')$.

Corollary 6.8.19.

(i) *The partial map $\widetilde{F}_{bk}$ is well-defined.*

(ii) *Let $\widehat{v} \in \widehat{\mathrm{CS}}_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ and suppose that $(s_j)_{j\in J}$ is the reduced coding sequence of $\widehat{v}$. For each $j \in J \cap (-\infty, -1]$ and each $(\widetilde{\mathcal{B}}, g) \in \Sigma_{red}$ we have*

$$s_j = (\widetilde{\mathcal{B}}, g) \qquad \text{if and only if} \qquad \widetilde{F}^j_{bk}\big(\tau(\widehat{v})\big) \in g^{-1}\widetilde{D}_{(\widetilde{\mathcal{B}},g)}(\widetilde{\mathcal{B}}')$$

for some $\widetilde{\mathcal{B}}' \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. For $j \in \mathbb{Z}_{<0} \setminus J$, the map $\widetilde{F}^j_{bk}$ is not defined for $\tau(\widehat{v})$.

We end this section with the statement of the discrete dynamical system which is conjugate to the strong reduced cross section $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.

The set of labels of $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is given by

$$\Sigma_{\mathrm{st,red}} := \Sigma_{\mathrm{red}} \setminus \{\varepsilon\}.$$

For each $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ set

$$\Sigma_{\mathrm{st,red}}\big(\widetilde{\mathcal{B}}\big) := \Sigma_{\mathrm{red}}\big(\widetilde{\mathcal{B}}\big) \setminus \{\varepsilon\}.$$

Recall the set bd from Sec. 6.5. For $s \in \Sigma_{\mathrm{st,red}}(\widetilde{\mathcal{B}})$ set

$$D_{\mathrm{st},s}\big(\widetilde{\mathcal{B}}\big) := D_s\big(\widetilde{\mathcal{B}}\big) \setminus \mathrm{bd}$$

and

$$\widetilde{D}_{\mathrm{st},s}\big(\widetilde{\mathcal{B}}\big) := D_{\mathrm{st},s}\big(\widetilde{\mathcal{B}}\big) \times \big(J\big(\widetilde{\mathcal{B}}\big) \setminus \mathrm{bd}\,\big).$$

Further let

$$\widetilde{\mathrm{DS}}_{\mathrm{st}} := \bigcup_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} \big(I_{\mathrm{red}}\big(\widetilde{\mathcal{B}}\big) \setminus \mathrm{bd}\,\big) \times \big(J\big(\widetilde{\mathcal{B}}\big) \setminus \mathrm{bd}\,\big)$$

and define the map $\widetilde{F}_{\mathrm{st}} \colon \widetilde{\mathrm{DS}}_{\mathrm{st}} \to \widetilde{\mathrm{DS}}_{\mathrm{st}}$ by

$$\widetilde{F}_{\mathrm{st}}|_{\widetilde{D}_{\mathrm{st},s}(\widetilde{\mathcal{B}})}(x,y) := (g^{-1}x, g^{-1}y)$$

if $s = (\widetilde{\mathcal{B}}', g) \in \Sigma_{\mathrm{st,red}}(\widetilde{\mathcal{B}})$ and $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. The map $\widetilde{F}_{\mathrm{st}}$ is the "restriction" of $\widetilde{F}$ to the strong reduced cross section $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. In particular, the following proposition is the "reduced" analogon of Prop. 6.8.16.

Proposition 6.8.20.

(i) *The set $\widetilde{\mathrm{DS}}_{st}$ is the disjoint union*

$$\widetilde{\mathrm{DS}}_{st} = \bigcup \left\{ \widetilde{\mathrm{D}}_{st,s}\big(\widetilde{\mathcal{B}}\big) \;\middle|\; \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}},\;\; s \in \Sigma_{st,red}\big(\widetilde{\mathcal{B}}\big) \right\}.$$

If $(x,y) \in \widetilde{\mathrm{DS}}_{st}$, then there is a unique element $v \in \mathrm{CS}'_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ such that $\big(\gamma_v(\infty), \gamma_v(-\infty)\big) = (x,y)$. If and only if $(x,y) \in \widetilde{D}_{st,s}(\widetilde{\mathcal{B}})$, the element v is labeled with s.

(ii) *The map $\widetilde{F}_{st}$ is well-defined and the discrete dynamical system $\big(\widetilde{\mathrm{DS}}_{st}, \widetilde{F}_{st}\big)$ is conjugate to $\big(\widehat{\mathrm{CS}}_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}), R\big)$ via τ.*

6.8.3. Generating function for the future part

Suppose that the sets $I_{\mathrm{red}}(\widetilde{\mathcal{B}})$, $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$, are pairwise disjoint. Set

$$\mathrm{DS} := \bigcup_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} I_{\mathrm{red}}\big(\widetilde{\mathcal{B}}\big)$$

and consider the partial map $F\colon \mathrm{DS} \to \mathrm{DS}$ given by

$$F|_{D_s(\widetilde{\mathcal{B}})} x := g^{-1} x$$

if $s = (\widetilde{\mathcal{B}}', g) \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}})$ and $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$.

Proposition 6.8.21.

(i) *The set* DS *is the disjoint union*

$$\mathrm{DS} = \bigcup \left\{ D_s(\widetilde{\mathcal{B}}) \;\middle|\; \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}},\ s \in \Sigma_{red}(\widetilde{\mathcal{B}}) \right\}.$$

If $x \in \mathrm{DS}$, then there is (a non-unique) $v \in \mathrm{CS}'_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ such that $\gamma_v(\infty) = x$. Suppose that $v \in \mathrm{CS}'_{red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ with $\gamma_v(\infty) = x$ and let $(a_n)_{n\in J}$ be the reduced coding sequence of v. Then $a_0 = s$ if and only if $x \in D_s(\widetilde{\mathcal{B}})$ for some $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$.

(ii) *The partial map F is well-defined.*

Proof. Suppose that $\widetilde{\mathcal{B}}_1, \widetilde{\mathcal{B}}_2 \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and $s_1 \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}}_1)$, $s_2 \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}}_2)$ such that $D_{s_1}(\widetilde{\mathcal{B}}_1) \cap D_{s_2}(\widetilde{\mathcal{B}}_2) \neq \emptyset$. Pick $x \in D_{s_1}(\widetilde{\mathcal{B}}_1) \cap D_{s_2}(\widetilde{\mathcal{B}}_2)$. Then $x \in I_{\mathrm{red}}(\widetilde{\mathcal{B}}_1) \cap I_{\mathrm{red}}(\widetilde{\mathcal{B}}_2)$, which implies that $\widetilde{\mathcal{B}}_1 = \widetilde{\mathcal{B}}_2$. Now Cor. 6.8.15 yields that $s_1 = s_2$. Therefore the union in (i) is disjoint and hence F is well-defined. Cor. 6.8.15 shows that the union equals DS.

Let $(x, y) \in \widetilde{\mathrm{DS}}$. Then $(x, y) \in \widetilde{D}_s(\widetilde{\mathcal{B}})$ if and only if $x \in D_s(\widetilde{\mathcal{B}})$. Prop. 6.8.16 implies the remaining statements of (i). $\square$

Prop. 6.8.21 shows that

$$\left(F, (\mathrm{DS}_s(\widetilde{\mathcal{B}}))_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}, s \in \Sigma_{\mathrm{red}}(\widetilde{\mathcal{B}})}\right)$$

is like a generating function for the future part of the symbolic dynamics $(\Lambda_{\mathrm{red}}, \sigma)$. In comparison with a real generating function, the map $i\colon \Lambda_{\mathrm{red}} \to \mathrm{DS}$ is missing. Indeed, if there are strip precells in H, then there is no unique choice for the map i. To overcome this problem, we restrict ourselves to the strong reduced cross section $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$.

Proposition 6.8.22. *Let $v, w \in \mathrm{CS}'_{st,red}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Suppose that $(a_n)_{n\in\mathbb{Z}}$ is the reduced coding sequence of v and $(b_n)_{n\in\mathbb{Z}}$ that of w. If $(a_n)_{n\in\mathbb{N}_0} = (b_n)_{n\in\mathbb{N}_0}$, then $\gamma_v(\infty) = \gamma_w(\infty)$.*

Proof. The proof of Prop. 6.7.30 shows the corresponding statement for geometric coding sequences. The proof of the present statement is analogous. $\square$

We set

$$\mathrm{DS}_{\mathrm{st}} := \bigcup_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} I_{\mathrm{red}}(\widetilde{\mathcal{B}}) \smallsetminus \mathrm{bd}$$

and define the map $F_{\text{st}}\colon \text{DS}_{\text{st}} \to \text{DS}_{\text{st}}$ by

$$F_{\text{st}}|_{D_{\text{st},s}(\widetilde{\mathcal{B}})} x := g^{-1}x$$

if $s = (\widetilde{\mathcal{B}}', g) \in \Sigma_{\text{st,red}}(\widetilde{\mathcal{B}})$ and $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. Further define $i\colon \Lambda_{\text{st,red}} \to \text{DS}_{\text{st}}$ by

$$i\big((a_n)_{n\in\mathbb{Z}}\big) := \gamma_v(\infty),$$

where $v \in \text{CS}'_{\text{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$ is the unit tangent vector with reduced coding sequence $(a_n)_{n\in\mathbb{N}}$. Prop. 6.8.11 shows that v is unique, and Prop. 6.8.22 shows that i only depends on $(a_n)_{n\in\mathbb{N}_0}$. Therefore

$$\Big(F_{\text{st}}, i, (D_s(\widetilde{\mathcal{B}}))_{\widetilde{\mathcal{B}}\in\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}, s\in\Sigma_{\text{st,red}}(\widetilde{\mathcal{B}})}\Big)$$

is a generating function for the future part of the symbolic dynamics $(\Lambda_{\text{st,red}}, \sigma)$.

Example 6.8.23. For the Hecke triangle group G_n and its family of shifted cells $\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} = \{\widetilde{\mathcal{B}}\}$ from Example 6.7.22 we have $I(\widetilde{\mathcal{B}}) = I_{\text{red}}(\widetilde{\mathcal{B}})$ and $\Sigma_{\text{red}} = \Sigma$. Obviously, the associated symbolic dynamics (Λ, σ) has a generating function for the future part. Recall the set bd from Sec. 6.5. Here we have $\text{bd} = G_n\infty = \mathbb{Q}$. Then

$$\text{DS} = \mathbb{R}^+ \quad \text{and} \quad \text{DS}_{\text{st}} = \mathbb{R}^+ \smallsetminus \mathbb{Q}.$$

Since there is only one (shifted) cell in SH, we omit $\widetilde{\mathcal{B}}$ from the notation in the following. We have

$$D_g = (g0, g\infty) \quad \text{and} \quad D_{\text{st},g} = (g0, g\infty) \smallsetminus \mathbb{Q} \quad \text{for } g \in \big\{U_n^k S \bigm| k = 1, \ldots, n-1\big\}.$$

The generating function for the future part of (Λ, σ) is $F\colon \text{DS} \to \text{DS}$,

$$F|_{D_g} x := g^{-1}x \quad \text{for } g \in \big\{U_n^k S \bigm| k = 1, \ldots, n-1\big\}..$$

For the symbolic dynamics $(\Lambda_{\text{st}}, \sigma)$ arising from the strong cross section $\widehat{\text{CS}}_{\text{st}}$ the generating function for the future part is $F_{\text{st}}\colon \text{DS}_{\text{st}} \to \text{DS}_{\text{st}}$,

$$F_{\text{st}}|_{D_{\text{st},g}} x := g^{-1}x \quad \text{for } g \in \big\{U_n^k S \bigm| k = 1, \ldots, n-1\big\}.$$

Example 6.8.24. Recall Example 6.8.3. If the shift map is $\mathbb{T}_2$, then the sets $I_{\text{red}}(\cdot)$ are pairwise disjoint and hence there is a generating function for the future part of the symbolic dynamics. In contrast, if the shift map is $\mathbb{T}_1$, the sets $I_{\text{red}}(\cdot)$ are not disjoint. Suppose that γ is a geodesic on H such that $\gamma(\infty) = \frac{1}{2}$. Then γ intersects $\text{CS}'_{\text{red}}\big(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}_1}\big)$ in, say, v. Example 6.8.14 shows that one cannot decide whether $v \in \text{CS}'\big(\widetilde{\mathcal{B}}_6\big)$ and hence is labeled with $\big(\widetilde{\mathcal{B}}_3, g_3\big)$, or whether $v \in \text{CS}'\big(\widetilde{\mathcal{B}}_1\big)$ and thus is labeled with $\big(\widetilde{\mathcal{B}}_4, g_4\big)$. This shows that the symbolic dynamics arising from the shift map $\mathbb{T}_1$ does not have a generating function for the future part.

7. Transfer operators

Suppose that (X, f) is a discrete dynamical system, where X is a set and f is a self-map of X. Further let $\psi\colon X \to \mathbb{C}$ be a function. The transfer operator $\mathcal{L}$ of (X, f) with potential ψ is defined by

$$\mathcal{L}\varphi(x) = \sum_{y \in f^{-1}(x)} e^{\psi(y)}\varphi(y)$$

with some space of complex-valued functions on X as domain of definition.

The main purposes of a transfer operator are to find invariant measures for the dynamical system and to provide, by means of Fredholm determinants, a relation to the dynamical zeta function of f (see, e.g., [CAM+08, Sec. 14], [Rue02], [May]). This involves a study of the spectral properties of the transfer operator $\mathcal{L}$, for which in turn one needs to investigate several properties of the dynamical system (X, f), the potential ψ and possible domains of definition for the transfer operator. All these questions we will leave for future work and we will define our transfer operators on a very general space, namely the set of all complex-valued functions on X. Further, we will only consider potentials of the type

$$\psi(y) = -\beta \log |f'(y)|,$$

where $\beta \in \mathbb{C}$.

Let Γ be a geometrically finite subgroup of $\mathrm{PSL}(2, \mathbb{R})$ of which ∞ is a cuspidal point and which satisfies (A2). Suppose that the set of relevant isometric spheres is non-empty. Fix a basal family $\mathbb{A}$ of precells in H and let $\mathbb{B}$ be the family of cells in H assigned to $\mathbb{A}$. Let $\mathbb{S}$ be a set of choices associated to $\mathbb{A}$ and suppose that $\widetilde{\mathbb{B}}_{\mathbb{S}}$ is the family of cells in SH associated to $\mathbb{A}$ and $\mathbb{S}$. Let $\mathbb{T}$ be a shift map for $\widetilde{\mathbb{B}}_{\mathbb{S}}$ and let $\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ denote the family of cells in SH associated to $\mathbb{A}$, $\mathbb{S}$ and $\mathbb{T}$.

We restrict ourselves to the strong reduced cross section $\widehat{\mathrm{CS}}_{\mathrm{st,red}}(\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}})$. Recall the discrete dynamical system $(\mathrm{DS}_{\mathrm{st}}, \widetilde{F}_{\mathrm{st}})$ from Sec. 6.8.2 as well as the set $\Sigma_{\mathrm{st,red}}$, its subsets $\Sigma_{\mathrm{st,red}}(\widetilde{\mathcal{B}})$ and the sets $\widetilde{D}_{\mathrm{st},s}(\widetilde{\mathcal{B}})$. The local inverses of $\widetilde{F}_{\mathrm{st}}$ are

$$\widetilde{F}_{\widetilde{\mathcal{B}},s} := \left(\widetilde{F}_{\mathrm{st}}|_{\widetilde{D}_{\mathrm{st},s}(\widetilde{\mathcal{B}})}\right)^{-1} : \begin{cases} \widetilde{F}_{\mathrm{st}}\big(\widetilde{D}_{\mathrm{st},s}(\widetilde{\mathcal{B}})\big) & \to & \widetilde{D}_{\mathrm{st},s}(\widetilde{\mathcal{B}}) \\ (x,y) & \mapsto & (gx, gy) \end{cases}$$

for $s \in \Sigma_{\mathrm{st,red}}(\widetilde{\mathcal{B}})$, $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. To abbreviate, we set

$$\widetilde{E}_{\widetilde{\mathcal{B}},s} := \widetilde{F}_{\mathrm{st}}\big(\widetilde{D}_{\mathrm{st},s}(\widetilde{\mathcal{B}})\big)$$

for $s \in \Sigma_{\mathrm{st,red}}(\widetilde{\mathcal{B}})$, $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$.

For two sets M, N let $\mathrm{Fct}(M,N)$ denote the set of functions from M to N. Then the transfer operator with parameter $\beta \in \mathbb{C}$

$$\mathcal{L}_\beta\colon \mathrm{Fct}(\widetilde{D}_{\mathrm{st}},\mathbb{C}) \to \mathrm{Fct}(\widetilde{D}_{\mathrm{st}},\mathbb{C})$$

associated to $\widetilde{F}_{\mathrm{st}}$ is given by (see [CAM$^+$08, Sec. 9.2])

$$(\mathcal{L}_\beta\varphi)(x,y) := \sum_{\widetilde{\mathcal{B}}\in\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} \sum_{s\in\Sigma_{\mathrm{st,red}}(\widetilde{\mathcal{B}})} \left|\det\left(\widetilde{F}'_{\widetilde{\mathcal{B}},s}(x,y)\right)\right|^\beta \varphi\left(\widetilde{F}_{\widetilde{\mathcal{B}},s}(x,y)\right)\chi_{\widetilde{E}_{\widetilde{\mathcal{B}},s}}(x,y),$$

where $\chi_{\widetilde{E}_{\widetilde{\mathcal{B}},s}}$ is the characteristic function of $\widetilde{E}_{\widetilde{\mathcal{B}},s}$. Note that the set $\widetilde{E}_{\widetilde{\mathcal{B}},s}$, the domain of definition of $\widetilde{F}_{\widetilde{\mathcal{B}},s}$, in general is not open. A priori, it is not even clear whether $E_{\widetilde{\mathcal{B}},s}$ is dense in itself. To avoid any problems with well-definedness, the derivative of $\widetilde{F}_{\widetilde{\mathcal{B}},s}$ shall be defined as the restriction to $\widetilde{E}_{\widetilde{\mathcal{B}},s}$ of the derivative of $F_{\widetilde{\mathcal{B}}}\colon \widetilde{F}(\widetilde{D}_s) \to \widetilde{D}_s$, $(x,y)\mapsto (gx,gy)$. Moreover, the maps $\widetilde{F}_{\widetilde{\mathcal{B}},s}$ and $\widetilde{F}'_{\widetilde{\mathcal{B}},s}$ are extended arbitrarily on $\widetilde{D}_{\mathrm{st}}\setminus\widetilde{E}_{\widetilde{\mathcal{B}},s}$.

Let $\beta\in\mathbb{C}$ and consider the map

$$\tau_{2,\beta}\colon \begin{cases} \Gamma\times\mathrm{Fct}(\widetilde{D}_{\mathrm{st}},\mathbb{C}) & \to \quad \mathrm{Fct}(\widetilde{D}_{\mathrm{st}},\mathbb{C}) \\ (g,\varphi) & \mapsto \quad \tau_{2,\beta}(g)\varphi \end{cases}$$

where

$$\tau_{2,\beta}(g^{-1})\varphi(x,y) := |g'(x)|^\beta |g'(y)|^\beta \varphi(gx,gy).$$

Since

$$\begin{aligned} \tau_{2,\beta}(h^{-1})\tau_{2,\beta}(g^{-1})\varphi(x,y) &= |h'(x)|^\beta|h'(y)|^\beta \tau_{2,\beta}(g^{-1})\varphi(hx,hy) \\ &= |h'(x)|^\beta|h'(y)|^\beta|g'(hx)|^\beta|g'(hy)|^\beta\varphi(ghx,ghy) \\ &= |(gh)'(x)|^\beta|(gh)'(y)|^\beta\varphi(ghx,ghy) \\ &= \tau_{2,\beta}\left((gh)^{-1}\right)\varphi(x,y) \end{aligned}$$

and

$$\tau_{2,\beta}(\mathrm{id})\varphi = \varphi,$$

the map $\tau_{2,\beta}$ is an action of Γ on $\mathrm{Fct}(\widetilde{D}_{\mathrm{st}},\mathbb{C})$. For $s = (\widetilde{\mathcal{B}}',g)\in\Sigma_{\mathrm{st,red}}(\widetilde{\mathcal{B}})$, $\widetilde{\mathcal{B}}\in\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$ and $(x,y)\in\widetilde{E}_{\widetilde{\mathcal{B}},s}$ we have

$$\left|\det\left(\widetilde{F}'_{\widetilde{\mathcal{B}},s}(x,y)\right)\right|^\beta = |g'(x)|^\beta|g'(y)|^\beta.$$

Therefore, the transfer operator becomes

$$\mathcal{L}_\beta\varphi = \sum_{\widetilde{\mathcal{B}}\in\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} \sum_{(\widetilde{\mathcal{B}}',g)\in\Sigma_{\mathrm{st,red}}(\widetilde{\mathcal{B}})} \chi_{\widetilde{E}_{\widetilde{\mathcal{B}},(\widetilde{\mathcal{B}}',g)}}\cdot\tau_{2,\beta}(g^{-1})\varphi.$$

Suppose now that the sets $I_{\mathrm{red}}(\widetilde{\mathcal{B}})$, $\widetilde{\mathcal{B}}\in\widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$, are pairwise disjoint so that the map F_{st} from Sec. 6.8.3 is a generating function for the future part of $(\Lambda_{\mathrm{st,red}},\sigma)$. Its local inverses are

$$F_{\widetilde{\mathcal{B}},s} := \left(F_{\mathrm{st}}|_{D_{\mathrm{st},s}(\widetilde{\mathcal{B}})}\right)^{-1} : \begin{cases} F_{\mathrm{st}}\left(D_{\mathrm{st},s}(\widetilde{\mathcal{B}})\right) & \to \quad D_{\mathrm{st},s}(\widetilde{\mathcal{B}}) \\ x & \mapsto \quad gx \end{cases}$$

for $s = (\widetilde{\mathcal{B}}', g) \in \Sigma_{\mathrm{st,red}}(\widetilde{\mathcal{B}})$, $\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}$. If we set

$$E_{\widetilde{\mathcal{B}},s} := F_{\mathrm{st}}\big(D_{\mathrm{st},s}(\widetilde{\mathcal{B}})\big),$$

then the transfer operator with parameter β associated to F_{st} is the map

$$\mathcal{L}_\beta \colon \operatorname{Fct}(\mathrm{DS}_{\mathrm{st}}, \mathbb{C}) \to \operatorname{Fct}(\mathrm{DS}_{\mathrm{st}}, \mathbb{C})$$

given by

$$(L_\beta \varphi)(x) := \sum_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} \sum_{s \in \Sigma_{\mathrm{st,red}}(\widetilde{\mathcal{B}})} \left|F'_{\widetilde{\mathcal{B}},s}\right|^\beta \varphi\big(F_{\widetilde{\mathcal{B}},s}(x)\big) \chi_{E_{\widetilde{\mathcal{B}},s}}(x).$$

As above we see that the map

$$\tau_\beta \colon \begin{cases} \Gamma \times \operatorname{Fct}(\mathrm{DS}_{\mathrm{st}}, \mathbb{C}) & \to \quad \operatorname{Fct}(\mathrm{DS}_{\mathrm{st}}, \mathbb{C}) \\ \qquad (g, \varphi) & \mapsto \quad \tau_\beta(g)\varphi \end{cases}$$

with

$$\tau_\beta(g^{-1})\varphi(x) = |g'(x)|^\beta \varphi(gx)$$

is a left action of Γ on $\operatorname{Fct}(\mathrm{DS}_{\mathrm{st}}, \mathbb{C})$. Then

$$\mathcal{L}_\beta \varphi = \sum_{\widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}}} \sum_{(\widetilde{\mathcal{B}}', g) \in \Sigma_{\mathrm{st,red}}(\widetilde{\mathcal{B}})} \chi_{E_{\widetilde{\mathcal{B}},(\widetilde{\mathcal{B}}',g)}} \tau_\beta(g^{-1})\varphi.$$

Note that for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we have $g'(x) = (cx+d)^{-2}$. Hence, the expression for $\mathcal{L}_\beta$ has a very simple structure. It seems reasonable to expect that this fact and the similarity of τ_β with principal series representation will allow and simplify a unified investigation of properties of $\mathcal{L}_\beta$.

Example 7.0.25. Recall from Example 6.8.23 the symbolic dynamics $(\Lambda_{\mathrm{st}}, \sigma)$ which we constructed for the Hecke triangle group G_n. The generating function for the future part of $(\Lambda_{\mathrm{st}}, \sigma)$ is given by $F_{\mathrm{st}} \colon \mathrm{DS}_{\mathrm{st}} \to \mathrm{DS}_{\mathrm{st}}$,

$$F_{\mathrm{st}}|_{D_{\mathrm{st},g}} x := g^{-1}x \quad \text{for } g \in \left\{U_n^k S \;\middle|\; k = 1, \ldots, n-1\right\}.,$$

where $\mathrm{DS}_{\mathrm{st}} = \mathbb{R}^+ \smallsetminus \mathbb{Q}$ and

$$D_{\mathrm{st},g} = (g0, g\infty) \smallsetminus \mathbb{Q}.$$

As in Example 6.8.23, we omit the (only) cell $\widetilde{\mathcal{B}}$ in SH from the notation. Then

$$F_{\mathrm{st}}\big(D_{\mathrm{st},g}\big) = \mathrm{DS}_{\mathrm{st}}$$

for each $g \in \left\{U_n^k S \;\middle|\; k = 1, \ldots, n-1\right\}$. Hence, the transfer operator with parameter β of F_{st} is $\mathcal{L}_\beta \colon \operatorname{Fct}(\mathrm{DS}_{\mathrm{st}}, \mathbb{C}) \to \operatorname{Fct}(\mathrm{DS}_{\mathrm{st}}, \mathbb{C})$,

$$\mathcal{L}_\beta = \sum_{k=1}^{n-1} \tau_\beta\big((U_n^k S)^{-1}\big).$$

8. The modular surface

The Hecke triangle group G_3 is the modular group $\mathrm{PSL}(2,\mathbb{Z})$. Set

$$g_1 := U_3^1 S = T_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g_2 := U_3^2 S = T_3 S T_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

With the basal family $\mathbb{A}$ of precells in H, the set $\mathbb{S}$ of choices associated to $\mathbb{A}$ and the shift map $\mathbb{T} \equiv \mathrm{id}$ as in Example 6.7.22, we get

$$\mathrm{DS}_{\mathrm{st}} = \mathbb{R}^+ \smallsetminus \mathbb{Q}, \quad D_{\mathrm{st},g_1} = (1,\infty) \smallsetminus \mathbb{Q} \quad \text{and} \quad D_{\mathrm{st},g_2} = (0,1) \smallsetminus \mathbb{Q}.$$

Example 6.8.23 shows that the generating function for the future part of the associated symbolic dynamics $(\Lambda_{\mathrm{st}}, \sigma)$ is given by $F\colon \mathrm{DS}_{\mathrm{st}} \to \mathrm{DS}_{\mathrm{st}}$,

$$\begin{aligned} F|_{D_{\mathrm{st},g_1}} &= g_1^{-1} \colon x \mapsto x - 1 \\ F|_{D_{\mathrm{st},g_2}} &= g_2^{-1} \colon x \mapsto \frac{x}{-x+1} = \frac{1}{\frac{1}{x} - 1}. \end{aligned}$$

This map and the symbolic dynamics are intimately related to the Farey map and the slow continued fraction algorithm (see [Ric81]). The transfer operator of F with parameter β is given by

$$\mathcal{L}_\beta = \tau_\beta\left(\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\right) + \tau_\beta\left(\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\right),$$

as shown in Example 7.0.25. The eigenfunctions of $\mathcal{L}_\beta$ for eigenvalue 1 are the solutions of the functional equation

$$f(x) = f(x+1) + (x+1)^{-2\beta} f\left(\tfrac{x}{1+x}\right). \tag{8.1}$$

Note that $x+1$ is positive for each $x \in \mathrm{DS}_{\mathrm{st}}$, hence $\left((x+1)^{-2}\right)^\beta = (x+1)^{-2\beta}$. Obviously, this functional equation can analytically be extended to $\mathbb{R}^+$ by the same formula. In [LZ01], Lewis and Zagier showed that the vector space of its real-analytic solutions of a certain decay is isomorphic to the vector space of Maass cusp forms for $\mathrm{PSL}(2,\mathbb{Z})$ with eigenvalue $\beta(1-\beta)$.

Originally, a connection between the geodesic flow on the modular surface $\mathrm{PSL}(2,\mathbb{Z})\backslash H$ and the functional equation (8.1) was established by the symbolic dynamics for this flow in [Ser85]. The generating function for the future part of this symbolic dynamics is the Gauß map. In [May91], Mayer investigated the transfer operator of the Gauß map. The space of its real-analytic eigenfunctions of certain decay with eigenvalue ± 1 is isomorphic to the subspace of solutions of (8.1) which are needed in the Lewis-Zagier correspondence.

8.1. The normalized symbolic dynamics

The coding sequences of the symbolic dynamics (Λ, σ) are bi-infinite sequences of g_1's and g_2's. We call a coding sequence $(a_n)_{n\in\mathbb{Z}} \in \Lambda$ *normalized* if $a_{-1} = g_1$ and $a_0 = g_2$, or vice versa. In other words, $(a_n)_{n\in\mathbb{Z}}$ is reduced if a_{-1} and a_0 are different labels. For the corresponding geodesic γ this is equivalent to $\gamma(\infty) > 1$ and $-1 < \gamma(-\infty) < 0$, or $0 < \gamma(\infty) < 1$ and $\gamma(-\infty) < -1$. We call geodesics with this property *normalized*. One easily proves the following lemma.

Lemma 8.1.1. *Each coding sequence in Λ is shift-equivalent to a normalized one. More precisely, for each $\lambda \in \Lambda$ there exists $n \in \mathbb{N}_0$ such that $\sigma^n(\lambda)$ is normalized.*

Let Λ_n denote the set of normalized coding sequences and $\widehat{\mathrm{CS}}_{\mathrm{st},n}$ the subset of $\widehat{\mathrm{CS}}_{\mathrm{st}}$ corresponding to Λ_n. Lemma 8.1.1 implies that $\widehat{\mathrm{CS}}_{\mathrm{st},n}$ is a strong cross section. Since there are only the two labels g_1, g_2, we can compactify (loss-free) the information contained in normalized coding sequences by counting the numbers of successive appearences of g_1's and g_2's and store only these numbers together with an element w in $\mathbb{Z}_2$ telling whether $a_0 = g_1$ (then $w = 0$) or $a_0 = g_2$ (then $w = 1$). Let $\widehat{\Lambda}_n \subseteq \mathbb{N}^{\mathbb{Z}} \times \mathbb{Z}_2$ denote these "condensed" coding sequence and let γ_λ be the geodesic corresponding to $\lambda \in \widehat{\Lambda}_n$.

For a sequence $(A_j)_{j\in\mathbb{N}_0}$ of matrices in $\mathrm{PSL}(2,\mathbb{R})$ and $z \in \overline{H}^g$ define

$$\prod_{j=0}^{\infty} A_j \cdot z := \lim_{n\to\infty} \left(\prod_{j=0}^{n} A_j \cdot z \right).$$

Further, let $[a_0, a_1, a_2, \ldots]$ denote the continued fraction

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

and set $g_n := g_{n \bmod 2}$.

Lemma 8.1.2. *Let $\lambda := ((n_j)_{j\in\mathbb{Z}}, w) \in \widehat{\Lambda}_n$. If $w = 0$, then*

$$\gamma_\lambda(\infty) = \prod_{j=0}^{\infty} g_{j+1}^{n_j} \cdot \infty = [n_0, n_1, n_2, \ldots],$$

$$\gamma_\lambda(-\infty) = \prod_{j=1}^{\infty} g_{j+1}^{-n_{-j}} \cdot \infty = -\frac{1}{[n_{-1}, n_{-2}, \ldots]}.$$

If $w = 1$, then

$$\gamma_\lambda(\infty) = \prod_{j=0}^{\infty} g_j^{n_j} \cdot \infty = \frac{1}{[n_0, n_1, n_2, \ldots]},$$

$$\gamma_\lambda(-\infty) = \prod_{j=1}^{\infty} g_j^{-n_{-j}} \cdot \infty = -[n_{-1}, n_{-2}, \ldots].$$

Proof. The statements are easily proved by induction. □

The theory of continued fractions shows that $\widehat{\Lambda}_n = \mathbb{N}^{\mathbb{Z}} \times \mathbb{Z}_2$. Define $\sigma_n\colon \widehat{\Lambda}_n \to \widehat{\Lambda}_n$ by

$$\sigma_n\big((a_n)_{n\in\mathbb{Z}}, w\big) := \big(\sigma((a_n)_{n\in\mathbb{Z}}), w+1\big).$$

Note that $\widetilde{\mathrm{DS}}_{\mathrm{st}} = \mathbb{R}^+ \smallsetminus \mathbb{Q} \times \mathbb{R}^- \smallsetminus \mathbb{Q}$. Let $i_n\colon \widehat{\Lambda}_n \to \widetilde{\mathrm{DS}}_{\mathrm{st}}$ be given by

$$i_n(\lambda) := \big(\gamma_\lambda(\infty), \gamma_\lambda(-\infty)\big).$$

Further set $\widetilde{F}_{\mathrm{st},n}\colon \widetilde{\mathrm{DS}}_{\mathrm{st}} \to \widetilde{\mathrm{DS}}_{\mathrm{st}}$

$$\widetilde{F}_{\mathrm{st},n}(x,y) := \begin{cases} \big(g_1^{-n}x, g_1^{-n}y\big) & \text{for } n < x < n+1, n \in \mathbb{N} \\ \big(g_2^{-n}x, g_2^{-n}y\big) & \text{for } \frac{1}{n+1} < x < \frac{1}{n}, n \in \mathbb{N}. \end{cases}$$

Then the diagram

$$\begin{tikzcd}
\widehat{\mathrm{CS}}_{\mathrm{st},n} \arrow[r, "R"] & \widehat{\mathrm{CS}}_{\mathrm{st},n} \\
\widehat{\Lambda}_n \arrow[u, "\mathrm{Cod}"] \arrow[r, "\sigma_n"] \arrow[d, "i_n"'] & \widehat{\Lambda}_n \arrow[u, "\mathrm{Cod}"'] \arrow[d, "i_n"] \\
\widetilde{\mathrm{DS}}_{\mathrm{st}} \arrow[r, "\widetilde{F}_{\mathrm{st},n}"] & \widetilde{\mathrm{DS}}_{\mathrm{st}}
\end{tikzcd}$$

commutes (and all horizontal maps are bijections).

8.2. The work of Series

We now show the relation of the symbolic dynamics and the cross section from Sec. 8.1 to those in [Ser85]. For simplicity, we restrict Series' work to geodesics that do not vanish into the cusp in either direction. This means that we restrict her symbolic dynamics to bi-infinite coding sequences and her cross section to the maximal strong cross section contained in it. Then her cross section is also $\widehat{\mathrm{CS}}_{\mathrm{st},n}$ and the symbolic dynamics is identical to $(\widehat{\Lambda}_n, \sigma_n)$, but she gives the "interpretation map" $i_S\colon \widehat{\Lambda}_n \to \mathbb{R}\times\mathbb{R}$,

$$i_S\big((n_j)_{j\in\mathbb{Z}}, w\big) := \begin{cases} \Big([n_0, n_1, \ldots], -\frac{1}{[n_{-1}, n_{-2}, \ldots]}\Big) & \text{if } w = 0 \\ \Big(-[n_0, n_1, \ldots], \frac{1}{[n_{-1}, n_{-2}, \ldots]}\Big) & \text{if } w = 1. \end{cases}$$

If $g\colon \widetilde{\mathrm{DS}}_{\mathrm{st}} \to \mathbb{R}\times\mathbb{R}$ is given by

$$g(x_1,x_2) := \begin{cases} (x_1, x_2) & \text{if } x_1 > 1 \\ S\cdot(x_1,x_2) = (-\frac{1}{x_1}, -\frac{1}{x_2}) & \text{if } 0 < x_1 < 1, \end{cases}$$

then

$$g \circ i_n = i_S.$$

The relation between i_n and i_S can also be seen on the level of the construction of the coding sequences. Let

$$\mathrm{CS}'_0 := \pi^{-1}\big(\operatorname{Cod}\big(\widehat{\Lambda}_n \cap (\mathbb{N}^{\mathbb{Z}} \times \{0\})\big)\big)$$

and

$$\mathrm{CS}'_1 := \pi^{-1}\big(\operatorname{Cod}\big(\widehat{\Lambda}_n \cap (\mathbb{N}^{\mathbb{Z}} \times \{1\})\big)\big).$$

Our coding sequences, the discrete dynamical system $(\widetilde{F}_{\mathrm{st},n}, \widetilde{\mathrm{DS}}_{\mathrm{st}})$ and the map i_n are constructed with respect to the set of representatives

$$\mathrm{CS}'_{\mathrm{st},n} := \pi^{-1}(\widehat{\mathrm{CS}}_{\mathrm{st},n}) \cap \mathrm{CS}' = \mathrm{CS}'_0 \cap \mathrm{CS}'_1$$

for $\widehat{\mathrm{CS}}_{\mathrm{st},n}$. Recall that CS' equals the set of unit tangent vectors based on the imaginary axis and pointing into $\{z \in H \mid \operatorname{Re} z > 0\}$. Series uses the same method but $\mathrm{CS}'_0 \cup S \cdot \mathrm{CS}'_1$ as set of representatives.

Bibliography

[Apa91] B. Apanasov, *Discrete groups in space and uniformization problems*, Mathematics and its Applications (Soviet Series), vol. 40, Kluwer Academic Publishers Group, Dordrecht, 1991, Translated and revised from the 1983 Russian original.

[Apa00] ———, *Conformal geometry of discrete groups and manifolds*, de Gruyter Expositions in Mathematics, vol. 32, Walter de Gruyter & Co., Berlin, 2000.

[Art24] E. Artin, *Ein mechanisches System mit quasiergodischen Bahnen*, Abh. Math. Sem. Univ. Hamburg **3** (1924), 170–175.

[BJ06] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 2006.

[Bor97] A. Borel, *Automorphic forms on* $\mathrm{SL}_2(\mathbf{R})$, Cambridge Tracts in Mathematics, vol. 130, Cambridge University Press, Cambridge, 1997.

[Bou98] N. Bourbaki, *Algebra I. Chapters 1–3*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation.

[CAM$^+$08] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay, *Chaos: Classical and quantum*, Niels Bohr Institute, Copenhagen, 2008, ChaosBook.org.

[CDKR91] M. Cowling, A. Dooley, A. Korányi, and F. Ricci, *H-type groups and Iwasawa decompositions*, Adv. Math. **87** (1991), no. 1, 1–41.

[CDKR98] ———, *An approach to symmetric spaces of rank one via groups of Heisenberg type*, J. Geom. Anal. **8** (1998), no. 2, 199–237.

[CG74] S. Chen and L. Greenberg, *Hyperbolic spaces*, Contributions to analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 1974, pp. 49–87.

[Cha04] C.-H. Chang, *Die Transferoperator-Methode für Quantenchaos auf den Modulflächen* $\Gamma\backslash\mathbb{H}$, Dissertation, TU Clausthal, 2004.

[CM01] C.-H. Chang and D. Mayer, *Eigenfunctions of the transfer operators and the period functions for modular groups*, Dynamical, spectral, and arithmetic zeta functions (San Antonio, TX, 1999), Contemp. Math., vol. 290, Amer. Math. Soc., Providence, RI, 2001, pp. 1–40.

[DH07] A. Deitmar and J. Hilgert, *A Lewis correspondence for submodular groups*, Forum Math. **19** (2007), no. 6, 1075–1099.

[Ebe96] P. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1996.

[EO73] P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973), 45–109.

[For72] L. Ford, *Automorphic functions*, Chelsea publishing company, New York, 1972.

[Gol99] W. Goldman, *Complex hyperbolic geometry*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1999, Oxford Science Publications.

[Had98] J. Hadamard, *Les surfaces à courbures opposées et leurs lignes géodésiques*, J. Math. Pures et Appl. **4** (1898), 27–73, see also: Oeuvres Complètes de Jacques Hadamard, vol. 2, Paris 1698, 729-775).

[Hel01] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001, Corrected reprint of the 1978 original.

[HP08] J. Hilgert and A. Pohl, *Symbolic dynamics for the geodesic flow on locally symmetric orbifolds of rank one*, Infinite Dimensional Harmonic Analysis IV, World Scientific, 2008.

[Kam03] S. Kamiya, *Generalized isometric spheres and fundamental domains for discrete subgroups of* $\mathrm{PU}(1, n; \mathbf{C})$, Proc. Japan Acad. Ser. A Math. Sci. **79** (2003), no. 5, 105–109.

[Kat92] S. Katok, *Fuchsian groups*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1992.

[KP03] I. Kim and J. Parker, *Geometry of quaternionic hyperbolic manifolds*, Math. Proc. Cambridge Philos. Soc. **135** (2003), no. 2, 291–320.

[KR] A. Korányi and F. Ricci, *A unified approach to compact symmetric spaces of rank one*, to appear in Colloquium Math.

[KR05] ———, *A classification-free construction of rank-one symmetric spaces*, Bull. Kerala Math. Assoc. (2005), no. Special Issue, 73–88 (2007).

[KU07] S. Katok and I. Ugarcovici, *Symbolic dynamics for the modular surface and beyond*, Bull. Amer. Math. Soc. (N.S.) **44** (2007), no. 1, 87–132 (electronic).

[LZ01] J. Lewis and D. Zagier, *Period functions for Maass wave forms. I*, Ann. of Math. (2) **153** (2001), no. 1, 191–258.

[Mar34] M. Martin, *A problem in arrangements*, Bull. Amer. Math. Soc. **40** (1934), no. 12, 859–864.

[Mar07] A. Marden, *Outer circles*, Cambridge University Press, Cambridge, 2007, An introduction to hyperbolic 3-manifolds.

[Mas71] B. Maskit, *On Poincaré's theorem for fundamental polygons*, Advances in Math. **7** (1971), 219–230.

[May] D. Mayer, *Transfer operators, the Selberg zeta function and the Lewis-Zagier theory of period functions*, unpublished lecture notes from a summer school in Günzburg.

[May91] ———, *The thermodynamic formalism approach to Selberg's zeta function for* $\mathrm{PSL}(2,\mathbf{Z})$, Bull. Amer. Math. Soc. (N.S.) **25** (1991), no. 1, 55–60.

[MH38] M. Morse and G. Hedlund, *Symbolic Dynamics*, Amer. J. Math. **60** (1938), no. 4, 815–866.

[Mos73] G. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton University Press, Princeton, N.J., 1973, Annals of Mathematics Studies, No. 78.

[Myr31] P. Myrberg, *Ein Approximationssatz für die Fuchsschen Gruppen*, Acta Math. **57** (1931), no. 1, 389–409.

[Nie27] J. Nielsen, *Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen*, Acta Math. **50** (1927), no. 1, 189–358.

[Par94] J. Parker, *On Ford isometric spheres in complex hyperbolic space*, Math. Proc. Cambridge Philos. Soc. **115** (1994), no. 3, 501–512.

[Rat06] J. Ratcliffe, *Foundations of hyperbolic manifolds*, second ed., Graduate Texts in Mathematics, vol. 149, Springer, New York, 2006.

[Ric81] I. Richards, *Continued fractions without tears*, Math. Mag. **54** (1981), no. 4, 163–171.

[Rob37] H. Robbins, *On a class of recurrent sequences*, Bull. Amer. Math. Soc. **43** (1937), no. 6, 413–417.

[Rue02] D. Ruelle, *Dynamical zeta functions and transfer operators*, Notices Amer. Math. Soc. **49** (2002), no. 8, 887–895.

[Ser85] C. Series, *The modular surface and continued fractions*, J. London Math. Soc. (2) **31** (1985), no. 1, 69–80.

[vQ79] B. von Querenburg, *Mengentheoretische Topologie*, second ed., Springer-Verlag, Berlin, 1979, Hochschultext.

[Vul99] L. Vulakh, *Farey polytopes and continued fractions associated with discrete hyperbolic groups*, Trans. Amer. Math. Soc. **351** (1999), no. 6, 2295–2323.

Index of Notations

Index of Definitions and Terminology