AUSLANDER-REITEN THEORY FOR COMPLEX OF MODULES

by

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摘 要

对于一个给定的模范畴ModΛ, 可以考虑此范畴中的几乎可裂序列(又称Auslander-Reiten序列), 并建立相应的Auslander-Reiten公式, 此公式给出了几乎可裂序列中始端和末端的对应关系. 人们统称这些结论为经典Auslander-Reiten理论. 该理论已经拓展到三角范畴的研究中. 特别的, 通过Brown表示定理可以证明紧生成三角范畴中的Auslander-Reiten三角的存在性已经被人们所熟知, 但人们对其Auslander-Reiten三角的始端和末端的对应关系还知之甚少.

该论文研究了一类具体的紧生成三角范畴, 即由各分支均为内射Λ-模的复形所构成的同伦范畴$\mathbf{K}(\mathrm{Inj}\Lambda)$. 我们建立了该范畴中的Auslander-Reiten公式, 从而给出了该范畴中的Auslander-Reiten三角两端的复形之间准确的对应关系. 这一结论的意义不仅在于同伦范畴$\mathbf{K}(\mathrm{Inj}\Lambda)$本身, 还在于它包含了模范畴中Auslander-Reiten理论. 具体地说, 该公式包含了模范畴中经典的Auslander-Reiten公式作为其特例; 设M为一有限表示的不可分解非投射Λ-模, 考虑在$\mathbf{K}(\mathrm{Inj}\Lambda)$中以该模的内射分解为末端的Auslander-Reiten三角, 我们可以导出模范畴中以M为末端的几乎可裂序列. 实际上, 上述观点给出了一种新的计算模范畴中几乎可裂序列的方法.

我们还提出了另一种计算$\mathbf{K}(\mathrm{Inj}\Lambda)$中的Auslander-Reiten三角的方法: 这一方法主要是通过把Happel嵌入函子提升到同伦范畴$\mathbf{K}(\mathrm{Inj}\Lambda)$中, 然后构造该函子的右伴随函子. 另外, 我们研究了一类与$\mathbf{K}(\mathrm{Inj}\Lambda)$有密切联系的范畴, 复形范畴$\mathbf{C}(\mathrm{Inj}\Lambda)$. 这是一个正合范畴. 结合在$\mathbf{K}(\mathrm{Inj}\Lambda)$中的所得到地相关结论, 我们给出了该范畴中几乎可裂conflation存在性定理.

关键词: 三角范畴, 紧生成, Auslander-Reiten三角, Auslander-Reiten公式, 正合范畴, 几乎可裂conflation.

Abstract

Given a module category ModΛ, one considers its almost split sequences (or Auslander-Reiten sequences), and an Auslander-Reiten formula can be established, which describes the relationship for the two end terms of an almost split sequence. These results are called the classical Auslander-Reiten theory. The corresponding Auslander-Reiten theory for triangulated categories has been developed. A remarkable fact is that one can use the Brown representability theorem to show the existence of Auslander-Reiten triangles in a compactly generated triangulated category. However, in general, there is not much known about what the relationship between the two end terms of an Auslander-Reiten triangle should be.

In this thesis, a class of compactly generated triangulated categories $\mathbf{K}(\mathrm{Inj}\Lambda)$, the homotopy categories of complexes of injective Λ-modules, are investigated. The Auslander-Reiten formula for complexes is established, which tells precisely what the correspondence between the two end terms of its Auslander-Reiten triangles is. The significance of this result not only lies in the homotopy category $\mathbf{K}(\mathrm{Inj}\Lambda)$ itself, but also lies in the fact that it contains all information in module category in the following sense: it contains as a special case the classical Auslander-Reiten formula for modules; and the Auslander-Reiten triangle ending in the injective resolution of a finitely presented indecomposable non-projective module induces the classical almost split sequence in module category. In fact, a simple recipe for computing almost split sequences in the category ModΛ is provided which seems to be new.

Another method for computing Auslander-Reiten triangles in $\mathbf{K}(\mathrm{Inj}\Lambda)$ is provided, which is based on an extension of Happel's embedding functor to the homotopy category $\mathbf{K}(\mathrm{Inj}\Lambda)$, and a construction of its right adjoint. We also consider the category $\mathbf{C}(\mathrm{Inj}\Lambda)$ of complexes of injective Λ-modules. This is an exact category, which is in a close relation to the homotopy category $\mathbf{K}(\mathrm{Inj}\Lambda)$. By applying the results we have obtained in $\mathbf{K}(\mathrm{Inj}\Lambda)$, the existence theorem of almost split conflations in $\mathbf{C}(\mathrm{Inj}\Lambda)$ is deduced.

Key Words: triangulated category, compactly generated, Auslander-Reiten triangle, Auslander-Reiten formula, exact category, almost split conflation.

Contents

Chapter 1

Introduction

1.1 Background

Auslander-Reiten theory is one of the most important techniques for the investigation of categories which appear in representation theory. It was first established by Auslander and Reiten in the early seventies when they studied the finitely generated module category of an artin algebra.

Let Λ be an artin algebra over a commutative artinian ring k. An exact sequence of finitely generated Λ-modules $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called an almost split sequence if:

(i) it is not split;

(ii) every map $M \to C$ which is not a split epimorphism factors through g;

(iii) every map $A \to N$ which is not a split monomorphism factors through f.

Many people prefer to call such a sequence an Auslander-Reiten sequence.

The famous existence theorem [7] says that if C is an indecomposable module which is not projective, then there is an almost split sequence $0 \to A \to B \to C \to 0$ and any two such sequences are isomorphic; if A is an indecomposable Λ-module which is not injective, then there is an almost split sequence $0 \to A \to B \to C \to 0$ which is unique up to isomorphism. Thus the almost split sequence is an invariant of the indecomposable Λ-modules A and C.

Two fairly different existence proofs were given originally for artin algebras. One uses the hint about what the correspondence between the end terms should be. The classical Auslander-Reiten formula

$$D\mathrm{Ext}^1_\Lambda(M,N) \cong \overline{\mathrm{Hom}}_\Lambda(N, D\mathrm{Tr}M),$$

was given, and it has been shown that a nonzero element in the $\mathrm{End}(C)$-socle of $\mathrm{Ext}^1_\Lambda(C, D\mathrm{Tr}C)$ is an almost split sequence [7]. The other proof was given by showing that the simple functors from $\mathrm{mod}\Lambda$ to abelian groups are finitely presented, and $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an almost split sequence if and only if $0 \to (-,A) \to (-,B) \to (-,C) \to \mathrm{Coker}(-,g) \to 0$ is a minimal projective resolution of the simple functor $\mathrm{Coker}(-,g)$, see [6, 2] for details.

The existence theorem was extended to more general types of rings soon. Auslander himself first studied the existence theorem for a noetherian algebra over a complete local noetherian ring [1], and extended the finitely generated module category to the whole module category (note that an almost split sequence in the finitely generated module category of a ring is not necessary to be an almost split sequence in the whole module category anymore, see [60] for the counterexamples). The method he used is quite similar to the first one in the case of artin algebras. In 1978, he extended the existence theorem to an arbitrary ring finally through the functorial approach [2]. The notion of almost split sequences makes sense in any subcategories of an abelian category closed under extension, and the existence of almost split sequences in such subcategories of module categories was studied by Auslander and Smalø [11].

A series of important concepts were introduced following almost split sequences, which helped people to investigate the indecomposable modules in a global way. Irreducible morphisms were used to study the relations between two indecomposable modules, and to produce new classes of indecomposable modules from the old ones. Preprojective and preinjective modules were discussed to classify the indecomposable modules into some partitions. Ringel considered the Auslander-Reiten quiver in the mid seventies as a device to study all left and right almost split morphisms simultaneously, which confirms the fundamental role that Auslander-Reiten theory plays in the representation theory of artin algebras now. Much of the early work on hereditary algebras and self-injective algebras of finite representation type was concerned with describing their Auslander-Reiten quivers [56, 55]. The language of Auslander-Reiten theory leads to an easier criteria for algebras to be of finite representation type and a new description of Nakayama algebras, see [8, 10, 52] for more applications.

People tried to extend further the existence of almost split sequences, the central part of Auslander-Reiten theory, to cases other than the module categories. Similar results were found in modular group representations, the theory of orders and model theory of modules, see [4]. It was also related to algebraic geometry. For example, the existence of almost split sequences for vector bundles on Gorenstein projective curves and for coherent sheaves over nonsingular projective curves were proved in [3] and [5]. Recently, Jørgensen [30] proved that almost split sequences frequently exist in categories of quasi-coherent sheaves on schemes.

The Auslander-Reiten theory for complexes was initiated by Happel. In [23, 24], he introduced the notion of Auslander-Reiten triangles in triangulated categories, and characterized their existence in the bounded derived category $\mathbf{D}^b(\mathrm{mod}\Lambda)$ of an artin algebra. Since triangulated categories provide the natural setting for the investigation of several homological or representation theoretic problems not only in representation theory but also in algebraic geometry and algebraic topology, this developments provide strong motivation to consider Auslander-Reiten theory in more general triangulated categories. Krause proved in [36] the existence of Auslander-Reiten triangles in compactly generated triangulated categories, by using the Brown representability theorem. This is a big progress because lots of important triangulated categories are compactly generated, such as the homotopy category of spectra, the derived category of quasi-coherent sheaves over a quasi-compact separated scheme, and the unbounded derived category over a ring. More applications in algebraic topology was given by Jørgensen [29]. For instance, the existence of Auslander-Reiten triangles characterizes Poincaré duality spaces, and the Auslander-Reiten quiver is a sufficient invariant to distinguish spheres of different dimension apart.

Serre duality is an important property studied in algebraic geometry. It first appeared on cohomology groups. Bondal and Kapranov reformulated this duality in an elegant way in [16] and related the existence of Serre duality to the representability of cohomology functors. Let $\mathcal{T}$ be a k-linear triangulated category with finite-dimentional Hom's. Then $\mathcal{T}$ satisfies Serre duality if it has a so-called Serre functor. The latter is by definition an additive functor $F\colon \mathcal{T} \to \mathcal{T}$ such that there are isomorphisms

$$\mathrm{Hom}_{\mathcal{T}}(X, Y) \cong D\mathrm{Hom}_{\mathcal{T}}(Y, FX)$$

which are natural in X, Y. Here $D = \mathrm{Hom}_k(-, k)$. They proved that a triangulated category of finite type over k which is right and left-saturated has a Serre functor. More authors noticed the analogy between the Auslander-Reiten formula and Serre duality. It was pointed out by Reiten and van den Bergh that a triangulated category (finite over k, Krull-Schmidt) has Aulsander-Reiten triangles if and only if it has a Serre functor [53], and this result has been extended to a more general case by Beligiannis [15]. The Serre duality formula turned out to be equivalent to the Auslander-Reiten formula for Frobenius algebras [46] and further for self-injective algebras [47]. The proof of the existence of Auslander-Reiten triangles in compact generated triangulated categories also lies on such duality [36].

The notion of almost split conflations in exact categories are consider in [21] (almost split exact pairs in the terminology of [18]), which is a natural generalization of Auslander-Reiten sequences in abelian categories. Recently, a proof for the existence of almost split conflations in the exact category of complexes of fixed size has been provided in [13]. Beligiannis [14] presented a unified way to prove the existence of almost split morphisms, almost split sequences and almost split triangles in abstract homotopy categories.

1.2 Main results

This section is devoted to elaborating on the main results in the thesis. Most results appear in [43] and [44].

Let us consider the following embeddings:

$$\mathrm{Mod}\Lambda \hookrightarrow \mathbf{D}^b(\mathrm{Mod}\Lambda) \hookrightarrow \mathbf{K}(\mathrm{Inj}\Lambda).$$

For the first embedding, we identify a Λ-module with the complex concentrated in degree zero. While for the second embedding, we use the injective resolution functor $\mathbf{i}$, see Appendix B.

We would like to develop the Auslander-Reiten theory (i.e., the Auslander-Reiten formula and the existence of Auslander-Reiten triangles) in certen triangulated category consisting of complexes, which should be an analogy to the classical Auslander-Reiten theory in the module category.

The first step was partly done by Happel. He studied the case when Λ is a finite dimensional algebra over a field k, and proved that for each bounded complexes X, Y, with X the form $(P^i)_{i\in\mathbb{Z}}$, where all P^i are finitely generated projective Λ-modules, there is an isomorphism

$$D\mathrm{Hom}_{\mathbf{D}^b(\mathrm{Mod}\Lambda)}(X,Y) \cong \mathrm{Hom}_{\mathbf{D}^b(\mathrm{Mod}\Lambda)}(Y,\mathbf{t}X)$$

where $\mathbf{t}X \cong (P^i \otimes_\Lambda D\Lambda)_{i\in\mathsf{Z}}$. The existence theorem [23, 24] says that there exists an Auslander-Reiten triangle $X \to Y \to Z \to X[1]$ for each indecomposable $Z \in \mathbf{D}^b(\mathrm{mod}\Lambda)$ if Λ has finite global dimension. The inverse statement was pointed out to be also true by himself in [25] several yeas later. These results can be generalized to the case of artin algebras without difficulty.

Observe that if an artin algebra Λ is of infinite global dimension, then the bounded derived category $\mathbf{D}^b(\mathrm{mod}\Lambda)$ does not have enough Auslander-Reiten triangles. This make the "analogy" of Auslander-Reiten theory in $\mathbf{D}^b(\mathrm{mod}\Lambda)$ and Auslander-Reiten theory in the module category fail. For this reason, we choose to study certain "completion" of $\mathbf{D}^b(\mathrm{mod}\Lambda)$, the category $\mathbf{K}(\mathrm{Inj}\Lambda)$ of complexes of injective Λ-modules up to homotopy, which has much nicer properties, and on which we will somehow build up the the pursued "analogy".

Now let k be a commutative noetherian ring which is complete and local, and we fix a noetherian k-algebra Λ. Let D be the functor $\mathrm{Hom}_k(-,E)$, where $\mathfrak{m}$ denotes the unique maximal ideal of k and E is an injective envelope $E(k/\mathfrak{m})$. Note that the assumptions on Λ imply that $\mathbf{K}(\mathrm{Inj}\Lambda)$ is compactly generated as a triangulated category. The injective resolutions of all finitely generated modules generate the full subcategory of compact objects, which therefore is equivalent to the bounded derived category $\mathbf{D}^b(\mathrm{mod}\Lambda)$ of the category $\mathrm{mod}\Lambda$ of finitely generated Λ-modules. In this sense, we view $\mathbf{K}(\mathrm{Inj}\Lambda)$ as the "completion" of $\mathbf{D}^b(\mathrm{mod}\Lambda)$, as we promised above.

The analogy between the Auslander-Reiten formula and Serre duality lead us to combine this duality with Auslander-Reiten triangles. There is the following common setting for proving such duality formulas. Let $\mathcal{T}$ be a k-linear triangulated category which is compactly generated. Then one can apply Brown's representability theorem and has for any compact object X a representing object $\mathbf{t}X$ such that

$$D\mathrm{Hom}_{\mathcal{T}}(X,-) \cong \mathrm{Hom}_{\mathcal{T}}(-,\mathbf{t}X).$$

Here we take for $\mathcal{T}$ the category $\mathbf{K}(\mathrm{Inj}\Lambda)$. Then we can prove that $\mathbf{t}X = \mathbf{p}X \otimes_\Lambda D\Lambda$, where $\mathbf{p}X$ denotes the projective resolution of X. Our first main theorem states that

Theorem A *Let X and Y be complexes of injective Λ-modules. Suppose that $X^n = 0$ for $n \ll 0$, that $H^n X$ is finitely generated over Λ for all n, and that $H^n X = 0$ for $n \gg 0$. Then we have an isomorphism*

$$D\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(X, Y) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(Y, \mathbf{p}X \otimes_\Lambda D\Lambda)$$

which is natural in X and Y.

The significance of this formula not only lies in the homotopy category itself, but also lies in its containing all information in module category, that is, the classical Auslander-Reiten formula for modules can be deduced from this formula as a consequence.

Combining this formula with the existence theorem Krause obtained in [36] for compactly generated triangulated categories, we know that there is an Auslander-Reiten triangle

$$(\mathbf{p}Z \otimes_\Lambda D\Lambda)[-1] \to Y \to Z \to \mathbf{p}Z \otimes_\Lambda D\Lambda.$$

in $\mathbf{K}(\mathrm{Inj}\Lambda)$ for each indecomposable compact object Z.

In particular, an Auslander-Reiten triangle ending in the injective resolution of a finitely presented indecomposable non-projective module induces the classical almost split sequence in module category as the following result indicates. The theorem above provides a simple recipe for computing almost split sequences in the category $\mathrm{Mod}\Lambda$ of Λ-modules which seems to be new.

Theorem B *Let N be a finitely presented Λ-module which is indecomposable and non-projective. Then there exists an Auslander-Reiten triangle*

$$(\mathbf{p}N \otimes_\Lambda D\Lambda)[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} \mathbf{i}N \xrightarrow{\gamma} \mathbf{p}N \otimes_\Lambda D\Lambda$$

in $\mathbf{K}(\mathrm{Inj}\Lambda)$ which the 0-th cocycle functor Z^0 sends to an almost split sequence

$$0 \to D\mathrm{Tr}N \xrightarrow{Z^0\alpha} Z^0 Y \xrightarrow{Z^0\beta} N \to 0$$

in the category of Λ-modules.

There is another method for computing Auslander-Reiten triangles in $\mathbf{K}(\mathrm{Inj}\Lambda)$, which is based on an extension of Happel's embedding functor

$$\mathbf{D}^b(\mathrm{mod}\Lambda) \longrightarrow \underline{\mathrm{mod}}\hat{\Lambda}$$

and a construction of its right adjoint. For the definition of repetitive algebra see Appendix D. More precisely, we have our third main result.

Theorem C *The composite*

$$\underline{\mathrm{Mod}}\hat{\Lambda} \xrightarrow{\sim} \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\hat{\Lambda}) \xrightarrow{\mathrm{Hom}_{\hat{\Lambda}}(\Lambda,-)} \mathbf{K}(\mathrm{Inj}\Lambda)$$

has a fully faithful left adjoint

$$\mathbf{K}(\mathrm{Inj}\Lambda) \to \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\hat{\Lambda}) \xrightarrow{\sim} \underline{\mathrm{Mod}}\hat{\Lambda}$$

which extends Happel's functor

$$\mathbf{D}^b(\mathrm{mod}\Lambda) \xrightarrow{-\otimes_{\Lambda}\Lambda} \mathbf{D}^b(\mathrm{mod}\hat{\Lambda}) \to \underline{\mathrm{mod}}\hat{\Lambda}.$$

Here, $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\hat{\Lambda})$ denotes the full subcategory of $\mathbf{K}(\mathrm{Inj}\hat{\Lambda})$ formed by all acyclic complexes.

Let us explain how to use the adjoint above to reduce the computation of Auslander-Reiten triangles in $\mathbf{K}(\mathrm{Inj}\Lambda)$ to the problem of computing almost split sequences in $\mathrm{mod}\hat{\Lambda}$. Suppose that Λ is an artin algebra, and we fix an indecomposable compact object Z in $\mathbf{K}(\mathrm{Inj}\Lambda)$. Applying the embedding functor to this object we obtain an indecomposable non-projective $\hat{\Lambda}$-module Z'. We are familiar with almost split sequences in the module category $\mathrm{mod}\hat{\Lambda}$, and hence we know Auslander-Reiten triangles well in its stable category $\underline{\mathrm{mod}}\hat{\Lambda}$. Assume that $X' \to Y' \to Z' \to X'[1]$ is an Auslander-Reiten triangle in $\underline{\mathrm{mod}}\hat{\Lambda}$. Then applying the adjoint functor above to this triangle, we can prove that its image is the coproduct of an Auslander-Reiten triangle $X \to Y \to Z \to X[1]$ in $\mathbf{K}(\mathrm{Inj}\Lambda)$ and a trivial triangle $W \xrightarrow{\mathrm{id}} W \to 0 \to W[1]$.

Our last main theorem concerns on Auslander-Reiten conflations in the exact category $\mathbf{C}(\mathrm{Inj}\Lambda)$. Note that the homotopy category can be considered as the quotient of the category of complexes

$$\mathbf{C}(\mathrm{Inj}\Lambda) \longrightarrow \mathbf{K}(\mathrm{Inj}\Lambda).$$

We intend to lift the Aulsander-Reiten theory we have obtained in the homotopy category $\mathbf{K}(\mathrm{Inj}\Lambda)$ to the category $\mathbf{C}(\mathrm{Inj}\Lambda)$.

There are two ways one can provide such a lifting, as the following commutative diagram shows.

$$\begin{tikzcd}
\text{AR-formula in } \mathbf{K}(\mathrm{Inj}\Lambda) \arrow[r] \arrow[d] & \text{AR-triangles in } \mathbf{K}(\mathrm{Inj}\Lambda) \arrow[d] \\
\text{AR-formula in } \mathbf{C}(\mathrm{Inj}\Lambda) \arrow[r] & \text{almost split conflations in } \mathbf{C}(\mathrm{Inj}\Lambda)
\end{tikzcd}$$

The Auslander-Reiten formula in $\mathbf{K}(\mathrm{Inj}\Lambda)$ guarantees the existence of Auslander-Reiten triangles. In one way, we use the existence theorem in $\mathbf{K}(\mathrm{Inj}\Lambda)$ directly, and describe the case in $\mathbf{C}(\mathrm{Inj}\Lambda)$ by studying the relation between almost split conflations in a Frobenius category and Auslander-Reiten triangles in its stable category. We show that an almost split conflation corresponds to an Auslander-Reiten triangle under the canonical map, the the inverse is also true if we restrict the end terms to a proper subcategory.

In another way, we define a map τ and deduce the Auslander-Reiten formula in $\mathbf{C}(\mathrm{Inj}\Lambda)$ from the Auslander-Reiten formula in $\mathbf{K}(\mathrm{Inj}\Lambda)$. Using this formula, we prove the existence of almost split conflations directly, with τ the Auslander-Reiten translation.

The existence theorem in $\mathbf{C}(\mathrm{Inj}\Lambda)$ says that

Theorem D *Let Z be a non-projective indecomposable object in $\mathbf{C}^{+,b}(\mathrm{inj}\Lambda)$. Then there exists an almost split conflation in $\mathbf{C}(\mathrm{Inj}\Lambda)$ ending in Z.*

In particular, we note that in this case, the right term of an almost split conflation is not necessary to be a compact object. The compact objects in $\mathbf{C}(\mathrm{Inj}\Lambda)$ equals the full subcategory $\mathbf{C}^b(\mathrm{inj}\Lambda)$.

1.3 Organization

We begin in section 2 by recalling the definitions of triangulated category and exact category, and give some elementary properties for each. We still review some basic results about almost split sequences in a module category.

In section 3 we devote ourselves to constructing an Auslander-Reiten formula for a class of triangulated categories $\mathbf{K}(\mathrm{Inj}\Lambda)$, the homotopy category of complexes of injective Λ-modules. The key that the formula holds lies in the fact that $\mathbf{K}(\mathrm{Inj}\Lambda)$ is compactly generated, so the Brown representability theorem can be applied here. In subsection 3 we study some properties of Auslander-Reiten translation in this category, and point out that the formula we get contains as a special case the classical Auslander Reiten formula for modules.

In section 4, we provide two different methods to compute Auslander-Reiten triangles in $\mathbf{K}(\mathrm{Inj}\Lambda)$. In one way, the construction of Auslander-Reiten formula uniquely determines the Auslander-Reiten triangles, and moreover the almost split sequences in a module category can even be induced from them. In another way, the computation of Auslander-Reiten triangles can be deduced to computing almost split sequences in the module category of a repetitive algebra, by considering the right adjoint of Happel's functor.

We end the thesis in section 5 by lifting the Auslander-Reiten theory we have got in $\mathbf{K}(\mathrm{Inj}\Lambda)$ to the category $\mathbf{C}(\mathrm{Inj}\Lambda)$. Some properties of $\mathbf{C}(\mathrm{Inj}\Lambda)$ are investigated, including its indecomposable objects and compact objects, and the existence theorem of almost split conflations in this category is deduced.

Chapter 2

Preliminaries

The purpose of this chapter is to make preparations for the following chapters. In the first section, we recall the concept of a triangulated category. We pay more attention to a class of triangulated categories $\mathbf{K}(\mathcal{A})$. They can be also viewed as the stable category of Frobenius categories $\mathbf{C}(\mathcal{A})$, which we discuss in the next section. In the last section, we collect some results about almost split sequences in a module category $\mathrm{Mod}\Lambda$. These results inspire us to find the corresponding results in $\mathbf{K}(\mathrm{Inj}\Lambda)$ and lift them to $\mathbf{C}(\mathrm{Inj}\Lambda)$ further. Most proofs for the properties of triangulated categories we state here can be found in [26, 57, 58, 28, 31, 59, 50]. For more material about exact categories, we refer to [51, 21, 18] and [32, Appendix A.]. While the book [9] is very helpful to learn almost split sequences for artin algebras comprehensively.

2.1 Triangulated categories

Let $\mathcal{T}$ be an additive category with an autoequivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$. A *triangle* in $\mathcal{T}$ is a sequence (α, β, γ) of maps

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

and a *morphism* between two triangles (α, β, γ) and $(\alpha', \beta', \gamma')$ is a triple (ϕ_1, ϕ_2, ϕ_3) of maps in $\mathcal{T}$ making the following diagram commutative.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\phi_1"] & Y \arrow[r, "\beta"] \arrow[d, "\phi_2"] & Z \arrow[r, "\gamma"] \arrow[d, "\phi_3"] & \Sigma X \arrow[d, "\Sigma\phi_1"] \\
X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r, "\gamma'"] & \Sigma X'
\end{tikzcd}$$

The category $\mathcal{T}$ is called *triangulated* if it is equipped with a class of distinguished triangles (called *exact triangles*) satisfying the following conditions.

TR1 A triangle isomorphic to an exact triangle is exact. For each object X, the triangle $0 \to X \xrightarrow{\mathrm{id}} X \to 0$ is exact. Each map α fits into an exact triangle (α, β, γ).

TR2 A triangle (α, β, γ) is exact if and only if $(\beta, \gamma, -\Sigma\alpha)$ is exact.

TR3 Given two exact triangles (α, β, γ) and $(\alpha', \beta', \gamma')$, each pair of maps ϕ_1 and ϕ_2 satisfying $\phi_2 \circ \alpha = \alpha' \circ \phi_1$ can be completed to a morphism

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\phi_1"] & Y \arrow[r, "\beta"] \arrow[d, "\phi_2"] & Z \arrow[r, "\gamma"] \arrow[d, "\phi_3"] & \Sigma X \arrow[d, "\Sigma\phi_1"] \\ X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r, "\gamma'"] & \Sigma X' \end{tikzcd}$$

of triangles.

TR4 Given exact triangles $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1 \circ \alpha_1$, there exists an exact triangle $(\delta_1, \delta_2, \delta_3)$ making the following diagram commutative.

$$\begin{tikzcd} X \arrow[r, "\alpha_1"] \arrow[d, equal] & Y \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & U \arrow[r, "\alpha_3"] \arrow[d, "\delta_1"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "\gamma_1"] & Z \arrow[r, "\gamma_2"] \arrow[d, "\beta_2"] & V \arrow[r, "\gamma_3"] \arrow[d, "\delta_2"] & \Sigma X \arrow[d, "\Sigma\alpha_1"] \\ & W \arrow[r, equal] \arrow[d, "\beta_3"] & W \arrow[r, "\beta_3"] \arrow[d, "\delta_3"] & \Sigma Y \\ & \Sigma Y \arrow[r, "\Sigma\alpha_2"] & \Sigma U & \end{tikzcd}$$

The functor Σ in $\mathcal{T}$ is called the *suspension functor*.

Remark. *Let (ϕ_1, ϕ_2, ϕ_3) be a morphism between exact triangles. If two maps from $\{\phi_1, \phi_2, \phi_3\}$ are isomorphisms, then so is the third. Thus each map $X \xrightarrow{\alpha} Y$ can determine uniquely an exact triangle $X \xrightarrow{\alpha} Y \to Z \to \Sigma X$.*

Let $\mathcal{T}$ be a triangulated category. A non-empty full subcategory $\mathcal{U}$ is a *triangulated subcategory* if the following conditions hold.

TS1 $\Sigma^n X \in \mathcal{U}$ for all $X \in \mathcal{U}$ and $n \in \mathbb{Z}$.

TS2 Let $X \to Y \to Z \to \Sigma X$ be an exact triangle in $\mathcal{T}$. If two objects from $\{X, Y, Z\}$ belong to $\mathcal{U}$, then so does the third.

An *exact functor* $\mathcal{T} \to \mathcal{S}$ between triangulated categories is a pair (F, η) consisting of a (covariant) additive functor $F\colon \mathcal{T} \to \mathcal{S}$ and a natural isomorphism $\eta\colon F \circ \Sigma_{\mathcal{T}} \to \Sigma_{\mathcal{S}} \circ F$ such that for every exact triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{T}$ the triangle

$$FX \xrightarrow{F\alpha} FY \xrightarrow{F\beta} FZ \xrightarrow{\eta_X \circ F\gamma} \Sigma(FX)$$

is exact in $\mathcal{S}$.

Example 2.1.1. *Let $\mathcal{A}$ be an additive category. Denote by $\mathbf{C}(\mathcal{A})$ the category of cochain complexes*

$$\cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to \cdots$$

over $\mathcal{A}$. A cochain complex X is called left bounded *if $X^n = 0$ for $n \ll 0$. Dually, X is called* right bounded *if $X^n = 0$ for $n \gg 0$. If X is both left bounded and right bounded, then we call it a* bounded *complex. The bounded, left bounded, and right bounded complexes form full subcategories of $\mathbf{C}(\mathcal{A})$ that are denoted by $\mathbf{C}^b(\mathcal{A})$, $\mathbf{C}^+(\mathcal{A})$ and $\mathbf{C}^-(\mathcal{A})$, respectively. The are all additive categories.*

The homotopy category $\mathbf{K}(\mathcal{A})$ is defined as follows: its objects are all cochain complexes over $\mathcal{A}$, and the morphisms between two objects are classes of morphisms of complexes modulo null-homotopic morphisms, i.e., it is the quotient of $\mathbf{C}(\mathcal{A})$. This category has been proved to be a triangulated category, and the suspension functor Σ is just the shift functor, i.e., $\Sigma X = X[1]$ where $X[1]^n = X^{n+1}$ and differential $d^n_{X[1]} = -d^{n+1}_X$. The exact triangles in $\mathbf{K}(\mathcal{A})$ are those triangles with the form

$$X \xrightarrow{\alpha} Y \to \mathrm{cone}(\alpha) \to X[1]$$

where $\mathrm{cone}(\alpha)$ *is the complex with* $\mathrm{cone}(\alpha)^n = X^{n+1} \coprod Y^n$ *and differential* $\begin{bmatrix} -d^{n+1}_X & 0 \\ -\alpha^{n+1} & d^n_Y \end{bmatrix}$. *We call such a complex the* mapping cone *of α. Denote by $\mathbf{K}^b(\mathcal{A})$, $\mathbf{K}^+(\mathcal{A})$ and $\mathbf{K}^-(\mathcal{A})$ the image of this quotient restricted to $\mathbf{C}^b(\mathcal{A})$, $\mathbf{C}^+(\mathcal{A})$ and $\mathbf{C}^-(\mathcal{A})$, respectively. They are all triangulated subcategories of $\mathbf{K}(\mathcal{A})$.*

Derived categories are a special class of triangulated categories that people investigate. Let us recall some relevant notions first. Let S be a collection of morphisms in a category $\mathcal{C}$. A *localization of $\mathcal{C}$ with respect to S* is a category $S^{-1}\mathcal{C}$, together with a functor $Q\colon \mathcal{C} \to S^{-1}\mathcal{C}$ such that

L1 Qs is an isomorphism in $S^{-1}\mathcal{C}$ for every $s \in S$.

L2 Any functor $F\colon \mathcal{C} \to \mathcal{D}$ such that Fs is an isomorphism for all $s \in S$ factors uniquely through Q.

Ignoring set-theoretic problems, one can show that such a localization always exists, see [59, 10.3.3].

Now suppose that $\mathcal{A}$ is an abelian category. Then one defines, for each cochain complex X over $\mathcal{A}$ and each $n \in \mathbb{Z}$, the n-th *cohomology group* of X to be

$$H^n X = \mathrm{Ker} d^n / \mathrm{Im} d^{n-1}.$$

A map $\phi\colon X \to Y$ between complexes induces a map $H^n\colon H^n X \to H^n Y$ in each degree, and ϕ is a *quasi-isomorphism* if $H^n\phi$ is an isomorphism for all $n \in \mathbb{Z}$. The *derived category* $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ is defined to be the localization $S^{-1}\mathbf{K}(\mathcal{A})$ of the homotopy category $\mathbf{K}(\mathcal{A})$ at the collection S of quasi-isomorphisms. This means that the objects of $\mathbf{D}(\mathcal{A})$ are all complexes over $\mathcal{A}$, and morphisms in $\mathbf{D}(\mathcal{A})$ between two complexes are given by paths composed of morphisms of complexes and formal inverses of quasi-isomorphisms, modulo a suitable equivalence relation. Sometimes the morphism in $\mathbf{D}(\mathcal{A})$ can be realized as the morphism in $\mathbf{K}(\mathcal{A})$. For example, we have

Proposition 2.1.2. *If I is a left bounded cochain complex with injective components, then*

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X, I) \cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, I)$$

for every complex X. Dually, if P is a right bounded cochain complex with projectives components, then

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(P, X) \cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(P, X).$$

Note that $\mathbf{D}(\mathcal{A})$ is still a triangulated category, and the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is an exact functor. We write $\mathbf{D}^b(\mathcal{A})$, $\mathbf{D}^+(\mathcal{A})$ and $\mathbf{D}^-(\mathcal{A})$ for the full subcategories of $\mathbf{D}(\mathcal{A})$ corresponding to $\mathbf{K}^b(\mathcal{A})$, $\mathbf{K}^+(\mathcal{A})$ and $\mathbf{K}^-(\mathcal{A})$. It can be proved that they are all triangulated subcategories of $\mathbf{D}(\mathcal{A})$. Moreover, they are equivalent to certain subcategories of the homotopy category. Denote by $\mathbf{K}^{+,b}(\mathcal{A})$ the full subcategory of $\mathbf{K}(\mathcal{A})$ formed by the complex X which is left bounded and $H^n X = 0$ for $n \gg 0$. The subcategory $\mathbf{K}^{-,b}(\mathcal{A})$ is defined dually.

Theorem 2.1.3. *Let $\mathcal{A}$ be an abelian category with enough injectives, and denote by $\mathcal{I}$ the full subcategory formed by all injectives. Then the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ induces equivalences*

$$\mathbf{D}^+(\mathcal{A}) \cong \mathbf{K}^+(\mathcal{I}), \qquad \mathbf{D}^b(\mathcal{A}) \cong \mathbf{K}^{+,b}(\mathcal{I}).$$

Dually, let $\mathcal{A}$ be an abelian category with enough projectives, and denote by $\mathcal{P}$ the full subcategory formed by all projectives. Then

$$\mathbf{D}^-(\mathcal{A}) \cong \mathbf{K}^-(\mathcal{P}), \qquad \mathbf{D}^b(\mathcal{A}) \cong \mathbf{K}^{-,b}(\mathcal{P}).$$

One of the main motivations for introducing the derived category is the fact that the maps in $\mathbf{D}(\mathcal{A})$ provide all homological information on the object in $\mathcal{A}$. We mention a fundamental formula, which says that

$$\mathrm{Ext}^n_{\mathcal{A}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A, \Sigma^n B),$$

where A, B in $\mathcal{A}$ and $n \in \mathbb{Z}$. Here, we use the convention that $\mathrm{Ext}^n_{\mathcal{A}}(-,-)$ vanishes for $n < 0$.

2.2 Exact categories

Let $\mathcal{C}$ be an additive category. A pair (α, β) of composable morphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in $\mathcal{C}$ is called *exact* if α is the kernel of β and β is the cokernel of α.

Let $\mathcal{E}$ be a class of exact pairs $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ which is closed under isomorphisms. The morphisms α and β appearing in a pair (α, β) in $\mathcal{E}$ are called an *inflation* and a *deflation* of $\mathcal{E}$, respectively, and the pair (α, β) is called a *conflation*. The class $\mathcal{E}$ is said to be an *exact structure* on $\mathcal{C}$ and $(\mathcal{C}, \mathcal{E})$ an *exact category* if the following axioms are satisfied:

E1 The composition of two deflations is a deflation.

E2 For each f in $\mathcal{C}(Z', Z)$ and each deflation β in $\mathcal{C}(Y, Z)$, there is some Y' in $\mathcal{C}$, and f' in $\mathcal{C}(Y', Y)$ and a deflation β' in $\mathcal{C}(Y', Z')$ such that $\beta \circ f' = f \circ \beta'$.

E3 Identities are deflations. If $\beta \circ \gamma$ is a deflation, then so is β.

E3$^{\mathrm{op}}$ Identities are inflations. If $\gamma \circ \alpha$ is a deflation, then so is α.

This set of axioms is proved to be equivalent to the following one (see [18, Appendix]):

Ex0 The identity morphism of the zero object, id_0, is a deflation.

Ex1 The composition of two deflations is a deflation.

Ex1$^{\mathrm{op}}$ The composition of two inflations is a inflation.

Ex2 For each f in $\mathcal{C}(Z', Z)$ and each deflation β in $\mathcal{C}(Y, Z)$, there is a pull-back diagram

$$\begin{tikzcd} Y' \arrow[r, "\beta'"] \arrow[d, "f'"'] & Z' \arrow[d, "f"] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd}$$

where β' is a deflation.

Ex2$^{\mathrm{op}}$ For each f in $\mathcal{C}(X, X')$ and each inflation α in $\mathcal{C}(X, Y)$, there is a pushout diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \\ X' \arrow[r, "\alpha'"] & Y' \end{tikzcd}$$

where α' is an inflation.

Given an exact category $(\mathcal{C}, \mathcal{E})$, we may define two quotient categories. An object I of $\mathcal{C}$ is called *$\mathcal{E}$-injective* if each conflation $I \to Y \to Z$ in $\mathcal{E}$ splits. The *$\mathcal{E}$-projective* objects are defined dually. We denote by $\overline{\mathcal{C}}$ the *stable module category* of $\mathcal{C}$ modulo injectives. The objects of this category are the same as those of $\mathcal{C}$, and for any two objects X, Y, the morphism set is the the quotient

$$\mathrm{Hom}_{\overline{\mathcal{C}}}(X, Y) = \overline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\{X \to I \to Y \mid I \text{ is } \mathcal{E}\text{-injective}\}.$$

The element in this set is written as $\bar{f}$, where $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Analogously, the stable module category $\underline{\mathcal{C}}$ modulo projectives is defined.

An exact category $(\mathcal{C}, \mathcal{E})$ is said to *have enough injectives*, if for each $X \in \mathcal{C}$ there is an inflation $X \to IX$ with injective IX. If $\mathcal{C}$ also has enough projectives (i.e. for each $X \in \mathcal{C}$ there is a deflation $PX \to X$ with projective PX), and the classes of projectives and injectives coincide, we call $\mathcal{C}$ a *Frobenius category.* In this case, the quotient $\overline{\mathcal{C}} = \underline{\mathcal{C}}$ is called the stable category of $\mathcal{C}$.

Happel proved in [23] that the stable category $\overline{\mathcal{C}}$ of a Frobenius category $\mathcal{C}$ is a triangulated category, and the exact triangle $X \xrightarrow{\bar{\alpha}} Y \xrightarrow{\bar{\beta}} Z \xrightarrow{\bar{\gamma}} \Sigma X$ in $\overline{\mathcal{C}}$ can be obtained by the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, equal] & Y \arrow[r, "\beta"] \arrow[d] & Z \arrow[d, "\gamma"] \\ X \arrow[r] & IX \arrow[r] & \Sigma X \end{tikzcd}$$

where the two rows are conflations of $\mathcal{C}$.

Example 2.2.1. *Let $\mathcal{A}$ be an abelian category. Then it is an exact category, with all exact sequences the exact structure. In particular, a module category is an exact category. Moreover, a full additive subcategory closed under extension is an exact category.*

Example 2.2.2. *([33, Example 4.3, Example 5.3]) Let $\mathcal{A}$ be an additive category. Consider the category of cochain complex $\mathbf{C}(\mathcal{A})$. Endow $\mathbf{C}(\mathcal{A})$ with the class of all pairs $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ such that $X^n \xrightarrow{\alpha^n} Y^n \xrightarrow{\beta^n} Z^n$ is split for each $n \in \mathbb{Z}$, i.e., there is a commutative diagram*

$$\begin{tikzcd} X^n \arrow[r, "\alpha^n"] \arrow[d, equal] & Y^n \arrow[r, "\beta^n"] \arrow[d] & Z^n \arrow[d, equal] \\ X^n \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] & X^n \coprod Z^n \arrow[r, "{[0\ 1]}"] & Z^n. \end{tikzcd}$$

Then $\mathbf{C}(\mathcal{A})$ is an exact category.

Furthermore, we define

$$(IX)^n = X^n \amalg X^{n+1},\ d_{IX}^n = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \alpha^n = \begin{bmatrix} 1 \\ d_X^n \end{bmatrix},$$

$$(\Sigma X)^n = X^{n+1},\ d_{\Sigma X}^n = -d_X^{n+1},\ \beta^n = \begin{bmatrix} -d_X^n & 1 \end{bmatrix}.$$

It is easy to see that

$$X \xrightarrow{\alpha} IX \xrightarrow{\beta} \Sigma X$$

is a conflation with IX an $\mathcal{E}$-injective. Note that the inflation α splits if and only if X is homotopic to zero. Thus, a complex is $\mathcal{E}$-injective in $\mathbf{C}(\mathcal{A})$ if and only if it is homotopic to zero, i.e., the identity morphism of the complex is null homotopic. Since the complexes IX, with $X \in \mathbf{C}(\mathcal{A})$, are also projective, $\mathbf{C}(\mathcal{A})$ is a Frobenius category. Moreover, its stable category coincides with its homotopy category $\mathbf{K}(\mathcal{A})$, which hence is a triangulated category, and the triangulated structure coincides with the one we discussed in Example 2.1.1.

2.3 Almost split sequences in module categories

In this subsection, we collect some notions and elementary results about Auslander-Reiten sequences in module categories. These serve as the background and the main motivation for the work in this thesis.

Let $\mathcal{A}$ be an additive category. A morphism $f\colon A \to B$ in $\mathcal{A}$ is called a *section* if there exists a morphism $f'\colon B \to A$ such that $f' \circ f = \mathrm{id}_A$. Dually, $g\colon B \to C$ is called a *retraction* if there exists a morphism $g'\colon C \to B$ such that $g \circ g' = \mathrm{id}_C$. A morphism $f\colon A \to B$ in $\mathcal{A}$ is *left almost split*, if f is not a section and if every map $A \to N$ which is not a section factors through f. Dually, $g\colon B \to C$ is *right almost split*, if g is not a retraction and if every map $M \to C$ which is not a retraction factors through g. We say that $\mathcal{A}$ *has left almost split morphisms* if for all indecomposable A there exist left almost split morphisms staring from A, and say that $\mathcal{A}$ *has right almost split morphisms* if for all indecomposable C there exist right almost split morphisms ending in C. If $\mathcal{A}$ has both left and right almost split morphisms, we say that $\mathcal{A}$ *has almost split morphisms.*

A morphism $f\colon A \to B$ in $\mathcal{A}$ is *left minimal* if every endomorphism $\phi\colon B \to B$ satisfying $\phi \circ f = f$ is an isomorphism. Dually, $g\colon B \to C$ is *right minimal* if every endomorphism $\phi\colon B \to B$ satisfying $g \circ \phi = g$ is an isomorphism. We say that f is *minimal left almost split*, if it is both left minimal and left almost split; and g is *minimal right almost split*, if it is both right minimal and right almost split.

If $\mathcal{A}$ is moreover an abelian category, then a section is also called a *split monomorphism*, and a retraction is also called a *split epimorphism.* A short exact sequence $\varepsilon\colon 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ is said to be *split* if f is a *split monomorphism*, or equivalently, if g is a *split epimorphism.*

Proposition 2.3.1. *The following statements are equivalent for an exact sequence* $\varepsilon\colon 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *in* $\mathcal{A}$.

1. *f is left almost split and β is right almost split.*
2. *The ring* $\mathrm{End}_{\mathcal{A}}(C)$ *is local and f is left almost split.*
3. *The ring* $\mathrm{End}_{\mathcal{A}}(A)$ *is local and g is right almost split.*
4. *f is minimal left almost split.*

5. *g is minimal right almost split.*

Definition 2.3.2. *A short exact sequence ε in $\mathcal{A}$ is called an* almost split sequence (or Auslander-Reiten sequence)*, if it satisfies one of the equivalent conditions above.*

An abelian category $\mathcal{A}$ is said to *has almost split sequences* if

(i) $\mathcal{A}$ has almost split morphisms;

(ii) for any indecomposable non-injective A, there is an almost split sequence starting from A;

(iii) for any indecomposable non-projective C, there is an almost split sequence ending in C.

Note that both of the end terms are indecomposable for an almost split sequence, and an almost split sequence is unique determined by its end terms in some sense.

Proposition 2.3.3. *The following are equivalent for two almost split sequences $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ in $\mathcal{A}$.*

1. *The sequences are isomorphic in the sense that there is a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\wr"] & B \arrow[r] \arrow[d, "\wr"] & C \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

   with the vertical morphisms isomorphisms.

2. *$A \cong A'$.*

3. *$C \cong C'$.*

Hence given an almost split sequence $0 \to A \to B \to C \to 0$, then A is determined by C up to isomorphism, and we write it as τC. Similarly, we write C as $\tau^{-}A$. The operators τ and τ^{-} are called the *Auslander-Reiten translation* and *cotranslation* of $\mathcal{A}$, respectively.

The most interesting case we deal with is when $\mathcal{A}$ is a module category. Let k be a commutative noetherian ring which is complete and local.

We fix a noetherian k-algebra Λ, i.e., a k-algebra which is finitely generated as a module over k. We denote by $\mathrm{Mod}\Lambda$ the category of (right) Λ-modules, $\mathrm{Noeth}\Lambda$ and $\mathrm{Art}\Lambda$ the full subcategories of $\mathrm{Mod}\Lambda$ consisting of the the noetherian and artin Λ-modules, respectively. Note that the assumptions on Λ imply that $\mathrm{Noeth}\Lambda$ is equal to $\mathrm{mod}\Lambda$, the category of finitely generated Λ-modules. In particularly, if Λ is artinian rather than noetherian, then we have $\mathrm{Noeth}\Lambda = \mathrm{Art}\Lambda = \mathrm{mod}\Lambda$. Also, from the assumptions we know that every finitely generated Λ-module decomposes essentially uniquely into a finite coproduct of indecomposable modules with local endomorphism rings.

In addition, we fix an injective envelope $E = I(k/\mathfrak{m})$, where $\mathfrak{m}$ denotes the unique maximal ideal of k. We obtain a functor

$$D = \mathrm{Hom}_k(-, E)\colon \mathrm{Mod}k \longrightarrow \mathrm{Mod}k$$

which induces a functor between $\mathrm{Mod}\Lambda$ and $\mathrm{Mod}\Lambda^{\mathrm{op}}$. Moreover, this functor restricts to a duality $D\colon \mathrm{Noeth}\Lambda \to \mathrm{Art}\Lambda^{\mathrm{op}}$, which induces further a duality $\underline{\mathrm{Noeth}}\Lambda^{\mathrm{op}} \to \overline{\mathrm{Art}}\Lambda$ (recall the stable categories we mentioned in the previous section).

Let us continue with some definitions before introducing the main results.

Recall that a projective presentation of a module M is an exact sequence $P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \to 0$ with P_0 and P_1 projective Λ-modules. Moreover, if P_0 is a projective cover of M and P_1 is a projective cover of $\mathrm{Ker}\delta_0$, then we call this presentation a minimal projective presentation of M (note that the assumptions on Λ guarantee that every finitely generated module has a projective cover). A Λ-module M is *finitely presented* if it admits a projective presentation

$$P_1 \to P_0 \to M \to 0$$

such that P_0 and P_1 are finitely generated. Since Λ is an a noetherian algebra, the finitely presented Λ-modules coincide with finitely generated Λ-modules. The *transpose* $\mathrm{Tr}M$ relative to this presentation is the Λ^{op}-module which is defined by the exactness of the induced sequence

$$\mathrm{Hom}_\Lambda(P_0, \Lambda) \to \mathrm{Hom}_\Lambda(P_1, \Lambda) \to \mathrm{Tr}M \to 0.$$

Note that the presentation of M is minimal if and only if the corresponding presentation of $\mathrm{Tr}M$ is minimal. The construction of the transpose is natural up to maps factoring through a projective and hence induces a duality $\underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda^{\mathrm{op}}$. Since $\mathrm{mod} = \mathrm{Noeth}$, we get a composite $D\mathrm{Tr}\colon \underline{\mathrm{mod}}\Lambda \to \overline{\mathrm{Art}}\Lambda$ which is an equivalence with inverse $\mathrm{Tr}D$.

Auslander and Reiten established the existence of almost split sequences in ModΛ with DTr the Auslander-Reiten translation.

Theorem 2.3.4 (Auslander/Reiten).

1. *Let C be an indecomposable non-projective module in* modΛ*. Then there is an almost split sequence* $0 \to D\mathrm{Tr}C \to B \to C \to 0$ *in the category of* Λ*-modules.*

2. *Let A be an indecomposable non-injective module in* ArtΛ*. Then there is an almost split sequence* $0 \to A \to B \to \mathrm{Tr}DA \to 0$ *in the category of* Λ*-modules.*

This conclusion was originally proved by using the following classical Auslander-Reiten formula.

Theorem 2.3.5 (Auslander/Reiten). *Let M and N be Λ-modules and suppose that M is finitely presented. Then we have an isomorphism*

$$D\mathrm{Ext}^1_\Lambda(M,N) \cong \overline{\mathrm{Hom}}_\Lambda(N, D\mathrm{Tr}M).$$

Remark. *The existence of almost split sequences in module category was first investigated by Auslander and Reiten in [7] when Λ is an artin algebra, and stronger results were obtained in that case. The classical Auslander-Reiten formula was constructed, and the category* modΛ *was proved to has almost split sequences. These results were further extended to the whole module category over a noetherian algebra over a complete local noetherian ring [1], and the proof is similar to the case of an artin algebra. In 1978, Auslander generalized his results in [2] to the case of an arbitrary ring with a functorial method. The Auslander-Reiten formula holds for arbitrary ring, see [39] for a short proof.*

Chapter 3

The Auslander-Reiten formula

We begin this chapter by reviewing some properties of the triangulated category $\mathbf{K}(\mathrm{Inj}\Lambda)$. In particular, we point out that $\mathbf{K}(\mathrm{Inj}\Lambda)$ is compact generated. The Auslander-Reiten formula for complexes of modules is presented, which contains as a special case the classical Auslander-Reiten formula for modules.

3.1 The homotopy category of injectives

Throughout this chapter, we fix a noetherian algebra Λ over a commutative noetherian ring k. We consider the category $\mathrm{Mod}\Lambda$ of (right) Λ-modules and the following full subcategories:

$\mathrm{mod}\Lambda =$ the finitely presented Λ-modules,

$\mathrm{Inj}\Lambda =$ the injective Λ-modules,

$\mathrm{Proj}\Lambda =$ the projective Λ-modules,

$\mathrm{proj}\Lambda =$ the finitely generated projective Λ-modules.

We recall some basic properties of the homotopy category $\mathbf{K}(\mathrm{Inj}\Lambda)$. Note that all these properties can be extended to a more general case, see [42, Section 2].

Recall that an object X in an additive category is *compact* if every map $X \to \coprod_{i\in I} Y_i$ factors through $\coprod_{i\in J} Y_i$ for some finite $J \subseteq I$. Let $\mathcal{A}$ be an additive category with arbitrary coproducts, then we write $\mathcal{A}^c$ for the subcategory of $\mathcal{A}$ which is formed by all compact objects. In particular, for a triangulated category $\mathcal{T}$, the subcategory $\mathcal{T}^c$ is still a triangulated category

[50, Lemma 4.1.4]. The following equivalent statements are well-known, see for example [49, Lemma 3.2].

Lemma 3.1.1. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts, and $\mathcal{T}_0$ be a set of compact objects in $\mathcal{T}$. Then the following are equivalent.*

1. *An object $Y \in \mathcal{T}$ is zero provided that $\mathrm{Hom}_{\mathcal{T}}(\Sigma^n X, Y) = 0$ for all $X \in \mathcal{T}_0$ and $n \in \mathbb{Z}$.*

2. *The category $\mathcal{T}$ coincides with its smallest triangulated full subcategory which contains $\mathcal{T}_0$ and is closed under taking coproducts.*

A triangulated category satisfies one of the equivalent conditions above is called *compactly generated*. One of the reasons that people pay more attention to the category $\mathbf{K}(\mathrm{Inj}\Lambda)$ is that it is compactly generated. Recall that the translation functor Σ is just the shift functor in this case.

Lemma 3.1.2. *Let M be a Λ-module, and denote by $\mathbf{i}M$ its injective resolution. Let X be a complex of Λ-modules with injective components. Then the map $M \to \mathbf{i}M$ induces a bijection*

$$\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(\mathbf{i}M, X) \longrightarrow \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(M, X) \tag{3.1.1}$$

Therefore, $\mathbf{i}M$ is a compact object in $\mathbf{K}(\mathrm{Inj}\Lambda)$ if M is finitely generated.

Proof. Complete the map $M \to \mathbf{i}M$ to an exact triangle

$$\mathrm{cone}(\mathbf{i})[-1] \to M \xrightarrow{\mathbf{i}} \mathbf{i}M \to \mathrm{cone}(\mathbf{i}).$$

Using the cohomological functor $\mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(-, X)$ to this triangle (recall that an additive functor from a triangulated category $\mathcal{T}$ to an abelian category $\mathcal{A}$ is called *cohomological* if it sends each exact triangle in $\mathcal{T}$ to an exact sequence in $\mathcal{A}$), we obtain an exact sequence

$$\begin{aligned}\mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(\mathrm{cone}(\mathbf{i}), X) \longrightarrow \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(\mathbf{i}M, X) &\xrightarrow{\mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(\mathbf{i}, X)} \\ \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(M, X) &\longrightarrow \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(\mathrm{cone}(\mathbf{i})[-1], X).\end{aligned}$$

Note that

$$\mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(\mathrm{cone}(\mathbf{i}), X) = 0$$

since $\mathrm{cone}(\mathbf{i})$ is acyclic and left bounded, and all components of X are injective. Thus $\mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(\mathbf{i}, X)$ is an isomorphism.

Now assume that M is finitely generated, then M is a compact object in $\mathrm{Mod}\Lambda$. Moreover, it is compact in $\mathbf{C}(\mathrm{Mod}\Lambda)$ and further in $\mathbf{K}(\mathrm{Mod}\Lambda)$. The isomorphism (3.1.1) shows that $\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(\mathbf{i}M, -)$ preserves coproducts, hence $\mathbf{i}M$ is compact in $\mathbf{K}(\mathrm{Inj}\Lambda)$. $\square$

Lemma 3.1.3. *Let X be a non-zero object in $\mathbf{K}(\mathrm{Inj}\Lambda)$. Then there exists a finitely generated Λ-module M such that $\mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(M[n], X) \neq 0$ for some $n \in \mathbb{Z}$.*

Proof. Suppose first that $H^n X \neq 0$ for some n. Then there exists a non-zero map $f \colon \Lambda \to H^n X$, since Λ is a generator in the module category. The map f can be lifted to g because Λ is projective.

$$\begin{tikzcd} & \Lambda \arrow[dl, dotted, "\exists g"'] \arrow[d, "f"] & \\ Z^n X \arrow[r] & H^n X \arrow[r] & 0 \end{tikzcd}$$

Thus we obtain a chain map $\phi \colon \Lambda[-n] \to X$, and f is non-zero guarantees that ϕ is non-zero in $\mathbf{K}(\mathrm{Inj}\Lambda)$.

Now suppose that $H^n X = 0$ for all n. We can choose n such that $Z^n X$ is non-injective. Otherwise, X is the sum of complexes with the form $\cdots \to 0 \to Z^n X \xrightarrow{\mathrm{id}} Z^n X \to 0 \to \cdots$, which are zero objects in $\mathbf{K}(\mathrm{Inj}\Lambda)$. Using Baer's Lemma, we know there exists a right ideal $\mathfrak{a}$ of Λ and a Λ-module homomorphism $h \colon \mathfrak{a} \to Z^n X$ such that h can not be lifted to $\Lambda \to Z^n X$. This homomorphism induces a chain map $\mathfrak{a}[-n] \to X$ which is non-zero in $\mathbf{K}(\mathrm{Inj}\Lambda)$, and $\mathfrak{a}$ is finitely generated since Λ is noetherian. $\square$

Proposition 3.1.4. *Denote by $\mathbf{K}^c(\mathrm{Inj}\Lambda)$ the full subcategory of $\mathbf{K}(\mathrm{Inj}\Lambda)$ formed by the compact objects.*

1. *The triangulated category $\mathbf{K}(\mathrm{Inj}\Lambda)$ is compactly generated.*

2. *The canonical functor $\mathbf{K}(\mathrm{Mod}\Lambda) \to \mathbf{D}(\mathrm{Mod}\Lambda)$ induces an equivalence*

$$\mathbf{K}^c(\mathrm{Inj}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\mathrm{mod}\Lambda)$$

Proof. It follows from Lemma 3.1.2 and 3.1.3 that $\mathbf{K}(\mathrm{Inj}\Lambda)$ is compactly generated. Here we can set $\mathcal{T}_0 = \{\mathbf{i}M \mid M \in \mathrm{mod}\Lambda\}$. A standard argument shows that $\mathbf{K}^c(\mathrm{Inj}\Lambda)$ equals the thick subcategory of $\mathbf{K}(\mathrm{Inj}\Lambda)$ which is generated by the injective resolutions of finite generated Λ-module; see [48, Lemma 2.2]. The equivalence $\mathbf{K}^+(\mathrm{Inj}\Lambda) \to \mathbf{D}^+(\mathrm{Mod}\Lambda)$ restricts to an equivalence $\mathbf{K}^{+,b}(\mathrm{Inj}\Lambda) \to \mathbf{D}^b(\mathrm{Mod}\Lambda)$ (see Theorem 2.1.3) and identifies $\mathbf{K}^c(\mathrm{Inj}\Lambda)$ with $\mathbf{D}^b(\mathrm{mod}\Lambda)$. $\square$

Remark. *From the above proposition we know that an object in* $\mathbf{K}(\mathrm{Inj}\Lambda)$ *is compact if and only if it is isomorphic to a complex X satisfying*

(i) $X^n = 0$ *for* $n \ll 0$*,*

(ii) $H^n X$ *is finitely generated over* Λ *for all* n*, and*

(iii) $H^n X = 0$ *for* $n \gg 0$*.*

3.2 The Auslander-Reiten formula for complexes

Given a pair of complexes X, Y of modules over Λ or Λ^{op}, we denote by $\mathrm{Hom}_\Lambda(X, Y)$ and $X \otimes_\Lambda Y$ the total Hom and the total tensor product respectively, which are complexes of k-modules.

Let us consider the following commutative diagram

$$\begin{tikzcd}
 & \mathbf{D}^-(\mathrm{Mod}\Lambda) & \mathbf{K}^-(\mathrm{Proj}\Lambda) \arrow[l, "\sim"'] \\
\mathbf{K}^c(\mathrm{Inj}\Lambda) \arrow[r, "\sim"] \arrow[d, tail] & \mathbf{D}^b(\mathrm{mod}\Lambda) \arrow[u, tail] \arrow[d, tail] & \mathbf{K}^{-,b}(\mathrm{proj}\Lambda) \arrow[l, "\sim"'] \arrow[u, tail] \\
\mathbf{K}^+(\mathrm{Inj}\Lambda) \arrow[r, "\sim"] & \mathbf{D}^+(\mathrm{Mod}\Lambda) &
\end{tikzcd}$$

in which all horizontal functors are obtained by restricting the localization functor $\mathbf{K}(\mathrm{Mod}\Lambda) \to \mathbf{D}(\mathrm{Mod}\Lambda)$ to appropriate subcategories. We denote by

$$\pi\colon \mathbf{K}^c(\mathrm{Inj}\Lambda) \longrightarrow \mathbf{K}^{-,b}(\mathrm{proj}\Lambda)$$

the composite of the equivalence $\mathbf{K}^c(\mathrm{Inj}\Lambda) \to \mathbf{D}^b(\mathrm{mod}\Lambda)$ with a quasi-inverse of the equivalence $\mathbf{K}^{-,b}(\mathrm{proj}\Lambda) \to \mathbf{D}^b(\mathrm{mod}\Lambda)$. Note that $\pi X \cong \mathbf{p}X$, where $\mathbf{p}X$ denotes the projective resolution of X (see Appendix B).

The following isomorphism is the basis for all results we obtain in this thesis. Recall that $D = \mathrm{Hom}_k(-, E)$ where E is the injective envelope $E = I(k/\mathfrak{m})$.

Theorem 3.2.1. *Let X and Y be complexes of injective Λ-modules, and suppose that X is compact in $\mathbf{K}(\mathrm{Inj}\Lambda)$. Then we have an isomorphism*

$$D\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(X,Y) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(Y,(\pi X)\otimes_\Lambda D\Lambda) \tag{3.2.1}$$

which is natural in X and Y.

To prove the theorem, we need several lemmas.

Lemma 3.2.2. *Let X, X' be objects in a k-linear compactly generated triangulated category $\mathcal{T}$, and suppose that X is compact. If there is a natural isomorphism*

$$D\mathrm{Hom}_{\mathcal{T}}(X,Y) \cong \mathrm{Hom}_{\mathcal{T}}(Y,X')$$

for all compact $Y \in \mathcal{T}$, then $D\mathrm{Hom}_{\mathcal{T}}(X,-) \cong \mathrm{Hom}_{\mathcal{T}}(-,X')$.

Proof. We shall use Theorem 1.8 in [35], which states the following equivalent conditions for an object W in $\mathcal{T}$.

(1) The object $H_W = \mathrm{Hom}_{\mathcal{T}}(-,W)|_{\mathcal{T}^c}$ is injective in the category $\mathrm{Mod}\mathcal{T}^c$ of contravariant additive functors $\mathcal{T}^c \to \mathrm{Ab}$.

(2) The map $\mathrm{Hom}_{\mathcal{T}}(V,W) \to \mathrm{Hom}_{\mathrm{Mod}\mathcal{T}^c}(H_V,H_W)$ sending ϕ to H_ϕ is bijective for all V in $\mathcal{T}$.

Here, $\mathcal{T}^c$ denotes the full subcategory of compact objects in $\mathcal{T}$.

We apply Brown's representability theorem (see Appendix A) and obtain an object X'' such that

$$D\mathrm{Hom}_{\mathcal{T}}(X,-) \cong \mathrm{Hom}_{\mathcal{T}}(-,X''),$$

since X is compact. Note that $\mathrm{Hom}_{\mathcal{T}^c}(X,-)$ is a projective object in the category of covariant additive functors $\mathcal{T}^c \to \mathrm{Ab}$, by Yoneda's lemma. Hence both X' and X'' satisfy condition (1). We have an isomorphism

$$H_{X'} = \mathrm{Hom}_{\mathcal{T}}(-,X')|_{\mathcal{T}^c} \cong D\mathrm{Hom}_{\mathcal{T}}(X,-)|_{\mathcal{T}^c} \cong \mathrm{Hom}_{\mathcal{T}}(-,X'')|_{\mathcal{T}^c} = H_{X''},$$

and (2) implies that this isomorphism is induced by an isomorphism $X' \to X''$ in $\mathcal{T}$. We conclude that

$$D\mathrm{Hom}_{\mathcal{T}}(X,-) \cong \mathrm{Hom}_{\mathcal{T}}(-,X').$$

$\square$

Lemma 3.2.3. *Let X, Y be complexes in $\mathbf{C}(\mathrm{Mod}\Lambda)$. Then we have in $\mathbf{C}(\mathrm{Mod}k)$ a natural map*

$$Y \otimes_\Lambda \mathrm{Hom}_\Lambda(X, \Lambda) \longrightarrow \mathrm{Hom}_\Lambda(X, Y), \tag{3.2.2}$$

which is an isomorphism if $X \in \mathbf{C}^-(\mathrm{proj}\Lambda)$ and $Y \in \mathbf{C}^+(\mathrm{Mod}\Lambda)$.

Proof. Given Λ-modules M and N, we have a map

$$\Phi\colon N \otimes_\Lambda \mathrm{Hom}_\Lambda(M, \Lambda) \longrightarrow \mathrm{Hom}_\Lambda(M, N)$$

which is defined by

$$\Phi(n \otimes \phi)(m) = n\phi(m).$$

This map is an isomorphism if M is finitely generated projective and extends to an isomorphism of complexes provided that X and Y are bounded in the appropriate direction. $\square$

Lemma 3.2.4. *Let M, N be Λ-modules and suppose that M is finitely presented. Then there is an isomorphism*

$$M \otimes_\Lambda DN \cong D\mathrm{Hom}_\Lambda(M, N). \tag{3.2.3}$$

Proof. Define a map

$$\Psi\colon M \otimes_\Lambda \mathrm{Hom}_k(N, E) \longrightarrow \mathrm{Hom}_k(\mathrm{Hom}_\Lambda(M, N), E)$$
$$m \otimes f \mapsto (g \mapsto (f \circ g)(m)).$$

It is easy to see that Ψ is an isomorphism when $M = \Lambda$ and further $M = \Lambda^n$. For an arbitrary M, consider its presentation $\Lambda^n \to \Lambda^m \to M \to 0$. Note that both $- \otimes_\Lambda \mathrm{Hom}_k(N, E)$ and $\mathrm{Hom}_k(\mathrm{Hom}_\Lambda(-, N), E)$ are right exact, we have the following commutative diagram

$$\begin{tikzcd}
\Lambda^n \otimes_\Lambda DN \arrow[r] \arrow[d] & \Lambda^m \otimes_\Lambda DN \arrow[r] \arrow[d] & M \otimes_\Lambda DN \arrow[r] \arrow[d, "\Psi"] & 0 \\
D\mathrm{Hom}_\Lambda(\Lambda^n, N) \arrow[r] & D\mathrm{Hom}_\Lambda(\Lambda^m, N) \arrow[r] & D\mathrm{Hom}_\Lambda(M, N) \arrow[r] & 0
\end{tikzcd}$$

with the exact rows. We finish the proof by using Five Lemma. $\square$

We also need some well-known results here, see [26] or [59] for the proofs.

Lemma 3.2.5. *Let X, Y be complexes of Λ-modules. Then we have an isomorphism*
$$H^n\mathrm{Hom}_\Lambda(X,Y) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(X, Y[n]).$$
In particular,
$$H^0\mathrm{Hom}_\Lambda(X,Y) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(X, Y). \tag{3.2.4}$$

Lemma 3.2.6. *Let X be a complex of Λ-module. Given an injective Λ-module I and a projective Λ-module P, we have isomorphisms*
$$\mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(X, I) \cong \mathrm{Hom}_\Lambda(H^0X, I), \tag{3.2.5}$$
$$\mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(P, X) \cong \mathrm{Hom}_\Lambda(P, H^0X).$$

Proof of Theorem 3.2.1. We use the fact that $\mathbf{K}(\mathrm{Inj}\Lambda)$ is compactly generated. Therefore by Lemma 3.2.2 it is sufficient to verify the isomorphism for every compact object Y. Thus we suppose that $Y^n = 0$ for $n \ll 0$, and in particular $Y \cong \mathbf{i}Y$ in $\mathbf{K}(\mathrm{Inj}\Lambda)$ (see Appendix B, and note that $Y \in \mathbf{K}^+(\mathrm{Inj}\Lambda) \subseteq \mathbf{K}_{\mathrm{inj}}(\mathrm{Mod}\Lambda)$). We obtain the following sequence of isomorphisms, where short arguments are added on the right hand side.

$$\begin{aligned}
D\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(X,Y) &\cong \mathrm{Hom}_k(\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(X,\mathbf{i}Y), E) && Y \text{ compact}\\
&\cong \mathrm{Hom}_k(\mathrm{Hom}_{\mathbf{D}(\mathrm{Mod}\Lambda)}(X,Y), E) && \text{adjunction}\\
&\cong \mathrm{Hom}_k(\mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(\pi X,Y), E) && \text{adjunction}\\
&\cong \mathrm{Hom}_k(H^0\mathrm{Hom}_\Lambda(\pi X,Y), E) && \text{from (3.2.4)}\\
&\cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}k)}(\mathrm{Hom}_\Lambda(\pi X,Y), E) && \text{from (3.2.5)}\\
&\cong H^0\mathrm{Hom}_k(\mathrm{Hom}_\Lambda(\pi X,Y), E) && \text{from (3.2.4)}\\
&\cong H^0\mathrm{Hom}_k(Y \otimes_\Lambda \mathrm{Hom}_\Lambda(\pi X,\Lambda), E) && \text{from (3.2.2)}\\
&\cong H^0\mathrm{Hom}_\Lambda(Y, \mathrm{Hom}_k(\mathrm{Hom}_\Lambda(\pi X,\Lambda), E)) && \text{adjunction}\\
&\cong H^0\mathrm{Hom}_\Lambda(Y, (\pi X) \otimes_\Lambda \mathrm{Hom}_k(\Lambda, E)) && \text{from (3.2.3)}\\
&\cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(Y, (\pi X) \otimes_\Lambda D\Lambda) && \text{from (3.2.4)}
\end{aligned}$$

This isomorphism completes the proof. □

3.3 The Auslander-Reiten translation

In this section, we investigate the properties of the Auslander-Reiten translation for complexes of Λ-modules. The Auslander-Reiten translation $D\mathrm{Tr}$

for modules is obtained from the translation for complexes. In particular, we deduce the classical Auslander-Reiten formula.

Denote by $\mathbf{t}$ the composite of functors

$$\mathbf{t} : \mathbf{K}(\mathrm{Inj}\Lambda) \xrightarrow{\mathrm{can}} \mathbf{D}(\mathrm{Mod}\Lambda) \xrightarrow{\mathbf{p}} \mathbf{K}(\mathrm{Proj}\Lambda) \xrightarrow{-\otimes_\Lambda D\Lambda} \mathbf{K}(\mathrm{Inj}\Lambda)$$

The the isomorphism (3.2.1) can be rewritten as

$$D\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(X, Y) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(Y, \mathbf{t}X) \tag{3.3.1}$$

We have more properties about $\mathbf{t}$.

Proposition 3.3.1. *The functor* $\mathbf{t}$ *has the following properties.*

1. $\mathbf{t}$ *is exact and preserves all coproducts.*
2. *For compact objects* X, Y *in* $\mathbf{K}(\mathrm{Inj}\Lambda)$*, the natural map*
$$\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(X, Y) \longrightarrow \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(\mathbf{t}X, \mathbf{t}Y)$$
is bijective.
3. $\mathbf{t}$ *admits a right adjoint which is* $\mathbf{i} \circ \mathrm{Hom}_\Lambda(D\Lambda, -)$.

Proof. The property 1 is clear. Now we observe that for each pair X, Y of compact objects, the k-module $\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(X, Y)$ is finitely generated. Using the isomorphism (3.3.1) twice, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(X, Y) &\cong D^2\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(X, Y) \\ &\cong D\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(Y, \mathbf{t}X) \\ &\cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(\mathbf{t}X, \mathbf{t}Y). \end{aligned}$$

To prove 3, let X, Y be objects in $\mathbf{K}(\mathrm{Inj}\Lambda)$. Then we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(\mathbf{p}X \otimes_\Lambda D\Lambda, Y) &\cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(\mathbf{p}X, \mathrm{Hom}_\Lambda(D\Lambda, Y)) \\ &\cong \mathrm{Hom}_{\mathbf{D}(\mathrm{Mod}\Lambda)}(X, \mathrm{Hom}_\Lambda(D\Lambda, Y)) \\ &\cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(X, \mathbf{i} \circ \mathrm{Hom}_\Lambda(D\Lambda, Y)). \end{aligned}$$

Thus $\mathbf{t}$ and $\mathbf{i} \circ \mathrm{Hom}_\Lambda(D\Lambda, -)$ form an adjoint pair. □

Denote by $\mathbf{a}$ the composite of functors

$$\mathbf{a} \colon \mathrm{Mod}\Lambda \xrightarrow{\mathrm{inc}} \mathbf{D}(\mathrm{Mod}\Lambda) \xrightarrow{\mathbf{i}} \mathbf{K}(\mathrm{Inj}\Lambda) \xrightarrow{\mathbf{t}} \mathbf{K}(\mathrm{Inj}\Lambda) \xrightarrow{Z^{-1}} \overline{\mathrm{Mod}}\Lambda$$

Recall that Z^{-1} is the -1-th cocycle functor. We can also get some properties about $\mathbf{a}$.

Proposition 3.3.2. *The functor* $\mathbf{a}$ *has the following properties.*

1. $\mathbf{a}M \cong D\mathrm{Tr}M$ *for every finitely presented* Λ*-module* M*.*

2. $\mathbf{a}$ *preserves all coproducts.*

3. $\mathbf{a}$ *annihilates all projective* Λ*-modules and induces a functor* $\underline{\mathrm{Mod}}\Lambda \to \overline{\mathrm{Mod}}\Lambda$*.*

4. *Each exact sequence* $0 \to L \to M \to N \to 0$ *of* Λ*-modules induces a sequence*

$$0 \to \mathbf{a}L \to \mathbf{a}M \to \mathbf{a}N \to L \otimes_\Lambda D\Lambda \to M \otimes_\Lambda D\Lambda \to N \otimes_\Lambda D\Lambda \to 0$$

of Λ*-modules which is exact.*

Proof. 1. The functor $\mathbf{t}$ sends an injective resolution $\mathbf{i}M$ of M to $\mathbf{p}M \otimes_\Lambda D\Lambda$. Using (3.2.3), we have

$$\mathbf{p}M \otimes_\Lambda D\Lambda \cong D\mathrm{Hom}_\Lambda(\mathbf{p}M, \Lambda).$$

This implies

$$Z^{-1}(\mathbf{p}M \otimes_\Lambda D\Lambda) \cong D\mathrm{Tr}M. \tag{3.3.2}$$

2. First observe that $\coprod_i(\mathbf{i}M_i) \cong \mathbf{i}(\coprod_i M_i)$ for every family of Λ-modules M_i, since Λ is noetherian. Clearly, $\mathbf{t}$ and Z^{-1} preserve coproducts. Thus $\mathbf{a}$ preserves coproducts.

3. We have $\mathbf{t}(\mathbf{i}\Lambda) = D\Lambda$ and therefore $\mathbf{a}\Lambda = 0$ in $\overline{\mathrm{Mod}}\Lambda$. Thus $\mathbf{a}$ annihilates all projectives since it preserves coproducts.

4. An exact sequence $0 \to L \to M \to N \to 0$ induces an exact triangle $\mathbf{p}L \to \mathbf{p}M \to \mathbf{p}N \to (\mathbf{p}L)[1]$. This triangle can be represented by a sequence $0 \to \mathbf{p}L \to \mathbf{p}M \to \mathbf{p}N \to 0$ of complexes which is split exact in each degree. Now apply $- \otimes_\Lambda D\Lambda$ and use the Snake Lemma. □

We are now in the position to deduce the classical Auslander-Reiten formula for modules [7] from the formula for complexes.

Corollary 3.3.3 (Auslander/Reiten). *Let* M *and* N *be* Λ*-modules and suppose that* M *is finitely presented. Then we have an isomorphism*

$$D\mathrm{Ext}^1_\Lambda(M, N) \cong \overline{\mathrm{Hom}}_\Lambda(N, D\mathrm{Tr}M). \tag{3.3.3}$$

Proof. Let $\mathbf{i}M$ and $\mathbf{i}N$ be injective resolutions of M and N, respectively. We apply the Auslander-Reiten formula (3.2.1) and the formula (3.3.2) for the Auslander-Reiten translate. Thus we have

$$\begin{aligned}
D\mathrm{Ext}^1_\Lambda(M,N) &\cong D\mathrm{Hom}_{\mathbf{D}(\mathrm{Mod}\Lambda)}(M, N[1]) \\
&\cong D\mathrm{Hom}_{\mathbf{D}(\mathrm{Mod}\Lambda)}(\mathbf{i}M, \mathbf{i}N[1]) \\
&\cong D\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(\mathbf{i}M, (\mathbf{i}N)[1]) \\
&\cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(\mathbf{i}N, (\mathbf{p}M \otimes_\Lambda D\Lambda)[-1]) \\
&\cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(N, (\mathbf{p}M \otimes_\Lambda D\Lambda)[-1]),
\end{aligned}$$

where the last isomorphism comes from Lemma 3.1.2. Now consider the following composite of maps

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathbf{C}(\mathrm{Mod}\Lambda)}(N, (\mathbf{p}M \otimes_\Lambda D\Lambda)[-1]) & \longrightarrow & \mathrm{Hom}_\Lambda(N, D\mathrm{Tr}M) & \longrightarrow & \overline{\mathrm{Hom}}_\Lambda(N, D\mathrm{Tr}M) \\
\phi & \mapsto & Z^0\phi & \mapsto & \overline{Z^0\phi}
\end{array}$$

Note that $D\mathrm{Tr}M = Z^{-1}(\mathbf{p}M \otimes_\Lambda D\Lambda) = Z^0(\mathbf{p}M \otimes_\Lambda D\Lambda)[-1])$, so the map is clearly surjective. Furthermore, since $\mathbf{p}M \otimes_\Lambda D\Lambda$ is the complex with injective component, this composite induces a map

$$\Psi\colon \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\Lambda)}(N, (\mathbf{p}M \otimes_\Lambda D\Lambda)[-1]) \longrightarrow \overline{\mathrm{Hom}}_\Lambda(N, D\mathrm{Tr}M)$$

which is still surjective. Next we show that Ψ is also an injective map. This is clear, since a map $N \to D\mathrm{Tr}M$ factoring through an injective module N' comes from an element in $D\mathrm{Ext}^1_\Lambda(M, N')$ which vanishes, by the preceding isomorphisms we obtained. □

Remark. *This corollary also implies that the classical Auslander-Reiten formula for modules could be extended to a formula for a more general class of abelian categories, see [43] for details.*

Chapter 4

Auslander-Reiten triangles

In this chapter, two different approaches to produce Auslander-Reiten triangles in the category $\mathbf{K}(\mathrm{Inj}\Lambda)$ are presented, both of which are close related to the computation of almost split sequences for modules. One is by using Auslander-Reiten formula for complexes we have got in the previous section, and we show in addition that almost split sequences for modules over Λ can be obtained from Auslander-Reiten triangles in $\mathbf{K}(\mathrm{Inj}\Lambda)$. The other is based on the construction of a right adjoint for the fully faithful functor

$$\mathbf{K}(\mathrm{Inj}\Lambda) \to \underline{\mathrm{Mod}}\hat{\Lambda}$$

which is proved to preserve Auslander-Reiten triangles. Hence the computation of Auslander-Reiten triangles in $\mathbf{K}(\mathrm{Inj}\Lambda)$ can be deduced to the problem of computing almost split sequences for modules over $\hat{\Lambda}$.

4.1 An application of the Auslander-Reiten formula

Happel first introduced Auslander-Reiten triangles in [23, 25] and studied their existence in the derived category $\mathbf{D}^b(\mathrm{mod}\Lambda)$. The definition of an Auslander-Reiten triangle is analogous to the definition of an almost split sequence.

Definition 4.1.1. *An exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$ in a triangulated category is called an* Auslander-Reiten triangle*, if α is left almost split and β is right almost split.*

There are some equivalent descriptions about this definition.

Lemma 4.1.2. *Let $\varepsilon\colon X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ be an exact triangle and suppose that β is right almost split. Then the following are equivalent:*

1. *The endomorphism ring* $\mathrm{End}(X)$ *is local.*

2. *β is right minimal.*

3. *ε is an Auslander-Reiten triangle.*

We say that a triangulated category $\mathcal{T}$ *has right Auslander-Reiten triangles* if for all indecomposable $Z \in \mathcal{T}$ there is an Auslander-Reiten triangle ending in Z. Existence of left Auslander-Reiten triangles is defined in a similar way, and we say that $\mathcal{T}$ *has Auslander-Reiten triangles* if it has both right and left Auslander-Reiten triangles. Some people also prefers to say that $\mathcal{T}$ has Auslander-Reiten triangles if it has only right Auslander-Reiten triangles.

Happel proved that the derived category $\mathbf{D}^b(\mathrm{mod}\Lambda)$ has right Auslander-Reiten triangles when Λ is an artin algebra with finite global dimension.

Later the existence result was generalized to compactly generated triangulated categories by Krause in [36]. For completeness we include a brief proof here. We introduce a lemma first, whose proof can be found in [24].

Lemma 4.1.3. *Let $\mathcal{T}$ be a triangulated category and (α, β, γ) be an exact triangle. Then the following are equivalent:*

1. *$\gamma = 0$.*

2. *α is a section.*

3. *β is a retraction.*

Theorem 4.1.4 (Theorem 2.2 in [36]). *Let $\mathcal{T}$ be a triangulated category which is compactly generated. Let Z be a compact object in $\mathcal{T}$ and suppose that $\Gamma = \mathrm{End}_{\mathcal{T}}(Z)$ is local. Then there exists an AR-triangle*

$$\mathbf{t}Z[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \mathbf{t}Z,$$

where $\mathbf{t}Z$ is the object in $\mathcal{T}$ such that

$$\mathrm{Hom}_{\Gamma}(\mathrm{Hom}_{\mathcal{T}}(Z, -), I) \cong \mathrm{Hom}_{\mathcal{T}}(-, \mathbf{t}Z). \tag{4.1.1}$$

In addition, we denote by $\mu\colon \Gamma/\mathrm{rad}\Gamma \to I$ *an injective envelope in the category of* Γ*-modules, and* γ *is the map which corresponds under (4.1.1) to the canonical map*

$$\mathrm{Hom}_{\mathcal{T}}(Z,Z) \xrightarrow{\pi} \Gamma/\mathrm{rad}\Gamma \xrightarrow{\mu} I.$$

Proof. The map γ corresponds, by definition, under (4.1.1) to a non-zero map. Hence $\gamma \neq 0$ and β is not a retraction, by Lemma 4.1.3. Let $\phi\colon Y' \to Z$ be a map in $\mathcal{T}$ which is not a retraction. It follows that the image of the induced map

$$\mathrm{Hom}_{\mathcal{T}}(Z,\phi)\colon \mathrm{Hom}_{\mathcal{T}}(Z,Y') \longrightarrow \mathrm{Hom}_{\mathcal{T}}(Z,Z)$$

is contained in the radical of $\mathrm{End}_{\mathcal{T}}(Z)$. Therefore the composition with $\mu\circ\pi$ is zero. However, $\mu\circ\pi\circ\mathrm{Hom}_{\mathcal{T}}(Z,\phi)$ corresponds under (4.1.1) to the map $\gamma\circ\phi$, and this implies $\gamma\circ\phi = 0$. Thus ϕ factors through β, and β is right almost split. Applying Lemma 4.1.2, it is sufficient to show that the endomorphism ring of $\mathbf{t}Z[-1]$ is local. This ring is isomorphic to $\mathrm{End}_{\mathcal{T}}(\mathbf{t}Z)$, and applying the isomorphism twice we obtain

$$\mathrm{End}_{\mathcal{T}}(\mathbf{t}Z) \cong \mathrm{Hom}_{\Gamma}(\mathrm{Hom}_{\mathcal{T}}(Z,\mathbf{t}Z),I) \cong \mathrm{Hom}_{\Gamma}(\mathrm{Hom}_{\Gamma}(\mathrm{Hom}_{\mathcal{T}}(Z,Z),I),I) \cong \mathrm{End}_{\Gamma}(I).$$

The injective Γ-module I is indecomposable since $\Gamma/\mathrm{rad}\Gamma$ is simple, and therefore $\mathrm{End}_{\Gamma}(I)$ is local. $\square$

In the case of $\mathbf{K}(\mathrm{Inj}\Lambda)$, given a compact object Z in $\mathbf{K}(\mathrm{Inj}\Lambda)$ which is indecomposable, then $\Gamma = \mathrm{End}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(Z)$ is local, by using the fact that Λ is a noetherian algebra. Let $I = E(\Gamma/\mathrm{rad}\Gamma)$ and observe that the functor $\mathrm{Hom}_{\Gamma}(-,I)$ is isomorphic to $D = \mathrm{Hom}_k(-,E)$. Applying formula (3.2.1) to the above theorem, we can easily get

Proposition 4.1.5. *Let* Z *be a compact object in* $\mathbf{K}(\mathrm{Inj}\Lambda)$ *which is indecomposable. Then there exists an Auslander-Reiten triangle*

$$(\mathbf{p}Z \otimes_{\Lambda} D\Lambda)[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \mathbf{p}Z \otimes_{\Lambda} D\Lambda. \tag{4.1.2}$$

A remarkable consequence of the above result is that it yields a simple recipe for the construction of almost split sequences in the module category. Precisely, an Auslander-Reiten triangle ending in the injective resolution of a finitely presented indecomposable non-projective module induces an almost split sequence as follows.

Theorem 4.1.6. *Let N be a finitely presented Λ-module which is indecomposable and non-projective. Then there exists an Auslander-Reiten triangle*

$$(\mathbf{p}N \otimes_\Lambda D\Lambda)[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} \mathbf{i}N \xrightarrow{\gamma} \mathbf{p}N \otimes_\Lambda D\Lambda$$

in $\mathbf{K}(\mathrm{Inj}\Lambda)$ which the functor Z^0 sends to an almost split sequence

$$0 \to D\mathrm{Tr}N \xrightarrow{Z^0\alpha} Z^0Y \xrightarrow{Z^0\beta} N \to 0$$

in the category of Λ-modules.

Proof. The Auslander-Reiten triangle for $\mathbf{i}N$ is obtained from the triangle (4.1.2) by taking $Z = \mathbf{i}N$. Assume that the projective resolution $\mathbf{p}N$ is minimal. Using the fact that $\mathbf{K}(\mathrm{Inj}\Lambda)$ is the stable category of the Frobenius category $\mathbf{C}(\mathrm{Inj}\Lambda)$, we may take a proper Y such that

$$0 \to (\mathbf{p}N \otimes_\Lambda D\Lambda)[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} \mathbf{i}N \to 0$$

is a sequence of chain maps which is split exact in each degree.

The functor Z^0 takes this sequence to an exact sequence

$$0 \to Z^0(\mathbf{p}N \otimes_\Lambda D\Lambda)[-1] \xrightarrow{Z^0\alpha} Z^0Y \xrightarrow{Z^0\beta} N. \qquad (4.1.3)$$

It is clear that $Z^0\beta$ is not a retraction, otherwise the right inverse of $Z^0\beta$ can be lifted to the right inverse of β, contrary to the fact that β is right almost split. Observe also that Z^0 induces a bijection $\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(\mathbf{i}M, \mathbf{i}N) \to \mathrm{Hom}_\Lambda(M, N)$ for all M. Using this bijection twice, and combining again the fact that β is right almost split, we know that every map $M \to N$ which is not a retraction factors through $Z^0\beta$. Thus $Z^0\beta$ is right almost split. In particular, $Z^0\beta$ is an epimorphism since N is non-projective.

We complete the proof by considering the left term of (4.1.3). Applying (3.3.2) we know that

$$Z^0(\mathbf{p}N \otimes_\Lambda D\Lambda)[-1] \cong D\mathrm{Tr}N.$$

This module is indecomposable and has a local endomorphism ring. Here we use the fact that N is indecomposable and that the resolution $\mathbf{p}N$ is minimal. $\square$

Remark. *There is an analogue of Theorem 4.1.6 for a projective module N. In this case, we have $D\mathrm{Tr}N = 0$ and $Z^0\beta$ is the right almost split map ending in N.*

Note that the computation of almost split sequences is a classical problem in representation theory; see for instance [20] or [9]. In particular, the middle term is considered to be mysterious. However, we get the following

Corollary 4.1.7. *Let N be a finitely presented Λ-module which is indecomposable and non-projective. Denote by*

$$P_1 \xrightarrow{\delta_1} P_0 \to N \to 0 \quad \text{and} \quad 0 \to N \to I^0 \xrightarrow{\delta^0} I^1$$

a minimal projective presentation and an injective presentation of N respectively. Choose a non-zero k-linear map $\operatorname{End}_\Lambda(N) \to E$ annihilating the radical of $\operatorname{End}_\Lambda(N)$, and extend it to a k-linear map $\phi\colon \operatorname{Hom}_\Lambda(P_0, I^0) \to E$. Let $\bar{\phi}$ denote the image of ϕ under the isomorphism

$$D\operatorname{Hom}_\Lambda(P_0, I^0) \cong \operatorname{Hom}_\Lambda(I^0, P_0 \otimes_\Lambda D\Lambda).$$

Then we have a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r, "\alpha"] \arrow[d] & M \arrow[r, "\beta"] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1 \otimes_\Lambda D\Lambda \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] \arrow[d, "\delta_1\otimes 1"] & (P_1 \otimes_\Lambda D\Lambda) \amalg I^0 \arrow[r, "{[0\ 1]}"] \arrow[d, "\left[\begin{smallmatrix}\delta_1\otimes 1 & \bar{\phi}\\ 0 & \delta^0\end{smallmatrix}\right]"] & I^0 \arrow[r] \arrow[d, "\delta^0"] & 0 \\
0 \arrow[r] & P_0 \otimes_\Lambda D\Lambda \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] & (P_0 \otimes_\Lambda D\Lambda) \amalg I^1 \arrow[r, "{[0\ 1]}"] & I^1 \arrow[r] & 0
\end{tikzcd}$$

such that the upper row is an almost split sequence in the category of Λ-modules.

Example 4.1.8. *Let k be a field and $\Lambda = k[x]/(x^2)$. Let $\mathbf{i}S$ denote the injective resolution of the unique simple Λ-module $S = k[x]/(x)$. The corresponding Auslander-Reiten triangle in $\mathbf{K}(\operatorname{Inj}\Lambda)$ has the form*

$$\mathbf{p}S[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} \mathbf{i}S \xrightarrow{\gamma} \mathbf{p}S,$$

where γ denotes an arbitrary non-zero map. Viewing Λ as a complex concentrated in degree zero, the corresponding Auslander-Reiten triangle has the form

$$\Lambda[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} \Lambda \xrightarrow{\gamma} \Lambda,$$

where γ denotes the map induced by multiplication with x.

4.2 An adjoint of Happel's functor

In this section we assume that Λ is an artin k-algebra, that is, Λ is finitely generated as a module over a commutative artinian ring k. We denote by $\hat{\Lambda}$ its repetitive algebra, see Appendix D. We first extend Happel's functor

$$\mathbf{D}^b(\mathrm{mod}\Lambda) \longrightarrow \underline{\mathrm{mod}}\hat{\Lambda}$$

to a functor which is defined on unbounded complexes, and we also give a right adjoint. In the rest part of this section, this adjoint is used to reduce the computation of Auslander-Reiten triangles in $\mathbf{K}(\mathrm{Inj}\Lambda)$ to the problem of computing almost split sequences in $\mathrm{Mod}\hat{\Lambda}$.

Note that projective and injective modules over $\hat{\Lambda}$ coincide. We denote by $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\hat{\Lambda})$ the full subcategory of $\mathbf{K}(\mathrm{Inj}\hat{\Lambda})$ which is formed by all acyclic complexes. The following description of the stable category $\underline{\mathrm{Mod}}\hat{\Lambda}$ is well-known; see for instance [42, Proposition 7.2].

Lemma 4.2.1. *The functor $Z^0\colon \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\hat{\Lambda}) \to \underline{\mathrm{Mod}}\hat{\Lambda}$ is an equivalence of triangulated categories.*

We consider the algebra homomorphism

$$\phi\colon \hat{\Lambda} \longrightarrow \Lambda, \quad (x_{ij}) \mapsto x_{00},$$

and we view Λ as a bimodule ${}_{\Lambda}\Lambda_{\hat{\Lambda}}$ via ϕ. Let us explain the following diagram.

$$\begin{tikzcd}
\mathrm{mod}\Lambda \arrow[r, "-\otimes_{\Lambda}\Lambda"] \arrow[d, "\mathrm{inc}"] & \mathrm{mod}\hat{\Lambda} \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{inc}"] & \underline{\mathrm{mod}}\hat{\Lambda} \arrow[d, equal] \\
\mathbf{D}^b(\mathrm{mod}\Lambda) \arrow[r, "-\otimes_{\Lambda}\Lambda"] & \mathbf{D}^b(\mathrm{mod}\hat{\Lambda}) \arrow[r] & \underline{\mathrm{mod}}\hat{\Lambda} \\
\mathbf{K}^c(\mathrm{Inj}\Lambda) \arrow[r, "F_1^c"] \arrow[u, "\wr"'] \arrow[d, "\mathrm{inc}"] & \mathbf{K}^c(\mathrm{Inj}\hat{\Lambda}) \arrow[r, "F_2^c"] \arrow[u, "\wr"'] \arrow[d, "\mathrm{inc}"] & \mathbf{K}^c_{\mathrm{ac}}(\mathrm{Inj}\hat{\Lambda}) \arrow[u, "Z^0", "\wr"'] \arrow[d, "\mathrm{inc}"] \\
\mathbf{K}(\mathrm{Inj}\Lambda) \arrow[r, shift left, "F_1"] \arrow[d, shift left, "\mathrm{inc}"] & \mathbf{K}(\mathrm{Inj}\hat{\Lambda}) \arrow[l, shift left, "\mathrm{Hom}_{\hat{\Lambda}}(\Lambda,-)"] \arrow[r, shift left, "F_2"] \arrow[d, shift left, "\mathrm{inc}"] & \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\hat{\Lambda}) \arrow[l, shift left, "\mathrm{inc}"] \\
\mathbf{K}(\mathrm{Mod}\Lambda) \arrow[r, shift left, "-\otimes_{\Lambda}\Lambda"] \arrow[u, shift left, "\mathbf{j}_{\Lambda}"] & \mathbf{K}(\mathrm{Mod}\hat{\Lambda}) \arrow[l, shift left, "\mathrm{Hom}_{\hat{\Lambda}}(\Lambda,-)"] \arrow[u, shift left, "\mathbf{j}_{\hat{\Lambda}}"] &
\end{tikzcd}$$

The top squares show the construction of Happel's functor $\mathbf{D}^b(\mathrm{mod}\Lambda) \to \underline{\mathrm{mod}}\hat{\Lambda}$ for which we refer to [25, Subsection 2.5]. Note that $\hat{\Lambda}$ is a self-injective algebra, and $\mathbf{D}^b(\mathrm{mod}\hat{\Lambda}) \to \underline{\mathrm{mod}}\hat{\Lambda}$ is the loclalization sequence $\mathbf{K}^b(\mathrm{proj}\hat{\Lambda}) \to \mathbf{D}^b(\mathrm{mod}\hat{\Lambda}) \to \underline{\mathrm{mod}}\hat{\Lambda}$ as constructed in [54].

The bimodule ${}_{\Lambda}\Lambda_{\hat{\Lambda}}$ induces an adjoint pair of functors between $\mathbf{K}(\mathrm{Mod}\Lambda)$ and $\mathbf{K}(\mathrm{Mod}\hat{\Lambda})$. Moreover, the projectiveness of Λ as Λ-module implies that $\mathrm{Hom}_{\hat{\Lambda}}(\Lambda, -)$ takes injective $\hat{\Lambda}$-modules to injective Λ-modules. Thus we get an induced functor $\mathbf{K}(\mathrm{Inj}\hat{\Lambda}) \to \mathbf{K}(\mathrm{Inj}\Lambda)$. This functor preserves products and has therefore a left adjoint F_1, by Brown's representability theorem (see Appendix A). A left adjoint preserves compactness if the right adjoint preserves coproducts; see [49, Theorem 5.1]. Clearly, $\mathrm{Hom}_{\hat{\Lambda}}(\Lambda, -)$ preserves coproducts since Λ is finitely generated over $\hat{\Lambda}$. Thus F_1 induces a functor F_1^c, restricted to compact objects.

The inclusion $\mathbf{K}(\mathrm{Inj}\Lambda) \to \mathbf{K}(\mathrm{Mod}\Lambda)$ preserves products and has therefore a left adjoint $\mathbf{j}_{\Lambda}$, by Brown's representability theorem (see Appendix A). Note that $\mathbf{j}_{\Lambda}M = \mathbf{i}M$ is an injective resolution for every Λ-module M. We have the same for $\hat{\Lambda}$, of course. Thus we have

$$F_1 \circ \mathbf{j}_{\Lambda} = \mathbf{j}_{\hat{\Lambda}} \circ (- \otimes_{\Lambda} \Lambda).$$

It follows that F_1 takes the injective resolution of a Λ-module M to the injective resolution of the $\hat{\Lambda}$-module $M \otimes_{\Lambda} \Lambda$. This shows that F_1^c coincides with $- \otimes_{\Lambda} \Lambda$ when one passes to the derived category $\mathbf{D}^b(\mathrm{mod}\Lambda)$ via the canonical equivalence $\mathbf{K}^c(\mathrm{Inj}\Lambda) \to \mathbf{D}^b(\mathrm{mod}\Lambda)$.

The inclusion $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\hat{\Lambda}) \to \mathbf{K}(\mathrm{Inj}\hat{\Lambda})$ has a left adjoint F_2. This is because the sequence $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\hat{\Lambda}) \xrightarrow{\mathrm{inc}} \mathbf{K}(\mathrm{Inj}\hat{\Lambda}) \xrightarrow{Q} \mathbf{D}(\mathrm{Mod}\hat{\Lambda})$ is a colocalization sequence, where Q is the canonical functor $\mathbf{K}(\mathrm{Inj}\hat{\Lambda}) \xrightarrow{\mathrm{inc}} \mathbf{K}(\mathrm{Mod}\hat{\Lambda}) \xrightarrow{\mathrm{can}} \mathbf{D}(\mathrm{Mod}\hat{\Lambda})$ (see [42, Theorem 4.2]). This left adjoint admits an explicit description. For instance, it takes the injective resolution $\mathbf{i}M$ of a $\hat{\Lambda}$-module M to the mapping cone of the canonical map $\mathbf{p}M \to \mathbf{i}M$, which is a complete resolution of M. The functor F_2 preserves compactness and induces therefore a functor F_2^c, because its right adjoint preserves coproducts [49, Theorem 5.1].

The following lemma shows that the composite functor $F_2 \circ F_1$ is fully faithful.

Lemma 4.2.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be two triangulated categories with arbitrary coproducts, and suppose that $\mathcal{S}$ is compactly generated. Let $F\colon \mathcal{S} \to \mathcal{T}$ be an exact functor which admits a right adjoint $G\colon \mathcal{T} \to \mathcal{S}$. If F preserves compactness, and the restriction of F to $\mathcal{S}^c$ is fully faithful, then F is fully faithful.*

Proof. Fix an object $X \in \mathcal{S}^c$ and define a full subcategory of $\mathcal{S}$

$$\mathcal{S}_X := \{Y \in \mathcal{S} \mid \mathrm{Hom}_{\mathcal{S}}(X,Y) \cong \mathrm{Hom}_{\mathcal{T}}(FX,FY)\}.$$

By the Five Lemma, this subcategory is a triangulated subcategory which contains $\mathcal{S}^c$ and is closed under coproducts. Hence it coincides with $\mathcal{S}$, since $\mathcal{S}$ is compactly generated.

Now fix an object $Y \in \mathcal{S}$ and define a full subcategory of $\mathcal{S}$

$$\mathcal{S}_Y := \{X \in \mathcal{S} \mid \mathrm{Hom}_{\mathcal{S}}(X,Y) \cong \mathrm{Hom}_{\mathcal{T}}(FX,FY)\}.$$

The same argument shows that $\mathcal{S}_Y = \mathcal{S}$. Thus we complete the proof. $\square$

We summarizes our construction with the following theorem.

Theorem 4.2.3. *The composite*

$$\underline{\mathrm{Mod}}\hat{\Lambda} \xrightarrow{\sim} \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\hat{\Lambda}) \xrightarrow{\mathrm{Hom}_{\hat{\Lambda}}(\Lambda,-)} \mathbf{K}(\mathrm{Inj}\Lambda)$$

has a fully faithful left adjoint

$$\mathbf{K}(\mathrm{Inj}\Lambda) \xrightarrow{F_2 \circ F_1} \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\hat{\Lambda}) \xrightarrow{\sim} \underline{\mathrm{Mod}}\hat{\Lambda}$$

which extends Happel's functor

$$\mathbf{D}^b(\mathrm{mod}\Lambda) \xrightarrow{-\otimes_{\Lambda}\hat{\Lambda}} \mathbf{D}^b(\mathrm{mod}\hat{\Lambda}) \to \underline{\mathrm{mod}}\hat{\Lambda}.$$

The following result explains how to use the adjoint above to reduce the computation of Auslander-Reiten triangles in $\mathbf{K}(\mathrm{Inj}\Lambda)$ to the problem of computing almost split sequences in $\mathrm{mod}\hat{\Lambda}$.

Proposition 4.2.4. *Let $F\colon \mathcal{S} \to \mathcal{T}$ be a fully faithful exact functor between triangulated categories which admits a right adjoint $G\colon \mathcal{T} \to \mathcal{S}$. Suppose*

$$X_{\mathcal{S}} \xrightarrow{\alpha_{\mathcal{S}}} Y_{\mathcal{S}} \xrightarrow{\beta_{\mathcal{S}}} Z_{\mathcal{S}} \xrightarrow{\gamma_{\mathcal{S}}} X_{\mathcal{S}}[1] \quad \textit{and} \quad X_{\mathcal{T}} \xrightarrow{\alpha_{\mathcal{T}}} Y_{\mathcal{T}} \xrightarrow{\beta_{\mathcal{T}}} Z_{\mathcal{T}} \xrightarrow{\gamma_{\mathcal{T}}} X_{\mathcal{T}}[1]$$

are Auslander-Reiten triangles in $\mathcal{S}$ and $\mathcal{T}$ respectively, where $Z_{\mathcal{T}} = FZ_{\mathcal{S}}$. Then

$$GX_{\mathcal{T}} \xrightarrow{G\alpha_{\mathcal{T}}} GY_{\mathcal{T}} \xrightarrow{G\beta_{\mathcal{T}}} GZ_{\mathcal{T}} \xrightarrow{G\gamma_{\mathcal{T}}} GX_{\mathcal{T}}[1]$$

is the coproduct of $X_{\mathcal{S}} \xrightarrow{\alpha_{\mathcal{S}}} Y_{\mathcal{S}} \xrightarrow{\beta_{\mathcal{S}}} Z_{\mathcal{S}} \xrightarrow{\gamma_{\mathcal{S}}} X_{\mathcal{S}}[1]$ and a triangle $W \xrightarrow{\mathrm{id}} W \to 0 \to W[1]$.

Proof. We have a natural isomorphism $\mathrm{Id}_{\mathcal{S}} \cong G \circ F$ which we view as an identification. In particular, G induces a bijection

$$\mathrm{Hom}_{\mathcal{T}}(FX, Y) \to \mathrm{Hom}_{\mathcal{S}}((G \circ F)X, GY) \tag{4.2.1}$$

for all $X \in \mathcal{S}$ and $Y \in \mathcal{T}$. Next we observe that for any exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$, the map β is a retraction if and only if $\gamma = 0$, see Lemma 4.1.3.

The map $F\beta_{\mathcal{S}}$ is not a retraction since $F\gamma_{\mathcal{S}} \neq 0$. Thus $F\beta_{\mathcal{S}}$ factors through $\beta_{\mathcal{T}}$, and $G(F\beta_{\mathcal{S}}) = \beta_{\mathcal{S}}$ factors through $G\beta_{\mathcal{T}}$. We obtain the following commutative diagram.

$$\begin{tikzcd}
X_{\mathcal{S}} \arrow[r, "\alpha_{\mathcal{S}}"] \arrow[d, "\phi"] & Y_{\mathcal{S}} \arrow[r, "\beta_{\mathcal{S}}"] \arrow[d, "\psi"] & Z_{\mathcal{S}} \arrow[r, "\gamma_{\mathcal{S}}"] \arrow[d, equal] & X_{\mathcal{S}}[1] \arrow[d, "\phi[1]"] \\
GX_{\mathcal{T}} \arrow[r, "G\alpha_{\mathcal{T}}"] & GY_{\mathcal{T}} \arrow[r, "G\beta_{\mathcal{T}}"] & GZ_{\mathcal{T}} \arrow[r, "G\gamma_{\mathcal{T}}"] & GX_{\mathcal{S}}[1]
\end{tikzcd}$$

On the other hand, $G\beta_{\mathcal{T}}$ is not a retraction since the bijection (4.2.1) implies $G\gamma_{\mathcal{T}} \neq 0$. Thus $G\beta_{\mathcal{T}}$ factors through $\beta_{\mathcal{S}}$, and we obtain the following commutative diagram.

$$\begin{tikzcd}
GX_{\mathcal{T}} \arrow[r, "G\alpha_{\mathcal{T}}"] \arrow[d, "\phi'"] & GY_{\mathcal{T}} \arrow[r, "G\beta_{\mathcal{T}}"] \arrow[d, "\psi'"] & GZ_{\mathcal{T}} \arrow[r, "G\gamma_{\mathcal{T}}"] \arrow[d, equal] & GX_{\mathcal{T}}[1] \arrow[d, "\phi'[1]"] \\
X_{\mathcal{S}} \arrow[r, "\alpha_{\mathcal{S}}"] & Y_{\mathcal{S}} \arrow[r, "\beta_{\mathcal{S}}"] & Z_{\mathcal{S}} \arrow[r, "\gamma_{\mathcal{S}}"] & X_{\mathcal{S}}[1]
\end{tikzcd}$$

We have $\beta_{\mathcal{S}} \circ (\psi' \circ \psi) = \beta_{\mathcal{S}}$, and this implies that $\psi' \circ \psi$ is an isomorphism, since $\beta_{\mathcal{S}}$ is right minimal. In particular, $GY_{\mathcal{T}} = Y_{\mathcal{S}} \amalg W$ for some object W. It follows that

$$GX_{\mathcal{T}} \xrightarrow{G\alpha_{\mathcal{T}}} GY_{\mathcal{T}} \xrightarrow{G\beta_{\mathcal{T}}} GZ_{\mathcal{T}} \xrightarrow{\gamma_{\mathcal{T}}} GX_{\mathcal{T}}[1]$$

is the coproduct of $X_{\mathcal{S}} \xrightarrow{\alpha_{\mathcal{S}}} Y_{\mathcal{S}} \xrightarrow{\beta_{\mathcal{S}}} Z_{\mathcal{S}} \xrightarrow{\gamma_{\mathcal{S}}} X_{\mathcal{S}}[1]$ and the triangle $W \xrightarrow{\mathrm{id}} W \to 0 \to W[1]$. $\square$

Now suppose that Λ is an artin algebra. We fix an indecomposable compact object Z in $\mathbf{K}(\mathrm{Inj}\Lambda)$, and we want to compute the Auslander-Reiten triangle $X \to Y \to Z \to X[1]$. We apply Happel's functor

$$H \colon \mathbf{K}^c(\mathrm{Inj}\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\mathrm{mod}\Lambda) \to \underline{\mathrm{mod}}\hat{\Lambda}$$

and obtain an indecomposable non-projective $\hat{\Lambda}$-module $Z' = HZ$. For instance, if $Z = \mathbf{i}N$ is the injective resolution of an indecomposable Λ-module

N, then $H\mathbf{i}N = N$ where N is viewed as a $\hat{\Lambda}$-module via the canonical algebra homomorphism $\hat{\Lambda} \to \Lambda$. Now take the almost split sequence $0 \to D\mathrm{Tr}Z' \to Y' \to Z' \to 0$ in $\mathrm{Mod}\hat{\Lambda}$. This gives rise to an Auslander-Reiten triangle $D\mathrm{Tr}Z' \to Y' \to Z' \to D\mathrm{Tr}Z'[1]$ in $\underline{\mathrm{Mod}}\hat{\Lambda}$. We apply the composite

$$\underline{\mathrm{Mod}}\hat{\Lambda} \xrightarrow{\mathrm{sim}} \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\hat{\Lambda}) \xrightarrow{\mathrm{Hom}_{\hat{\Lambda}}(\Lambda,-)} \mathbf{K}(\mathrm{Inj}\Lambda).$$

It follows from Proposition 4.2.4 that the result is a coproduct of the Auslander-Reiten triangle $X \to Y \to Z \to X[1]$ and a split exact triangle.

Chapter 5

Almost split conflations

Throughout this chapter we fix an artin k-algebra Λ. In the previous chapter, Auslander-Reiten triangles in $\mathbf{K}(\mathrm{Inj}\Lambda)$ are investigated. In this chapter, we continue this work and lift the existence theorem to $\mathbf{C}(\mathrm{Inj}\Lambda)$, in which the corresponding notions are almost split conflations. There are two ways to provide such a lifting, as the following commutative diagram shows.

$$\begin{tikzcd} \text{AR-formula in } \mathbf{K}(\mathrm{Inj}\Lambda) \arrow[r] \arrow[d] & \text{AR-triangles in } \mathbf{K}(\mathrm{Inj}\Lambda) \arrow[d] \\ \text{AR-formula in } \mathbf{C}(\mathrm{Inj}\Lambda) \arrow[r] & \text{almost split conflations in } \mathbf{C}(\mathrm{Inj}\Lambda) \end{tikzcd}$$

In proposition 4.1.5, we use the Auslander-Reiten formula in $\mathbf{K}(\mathrm{Inj}\Lambda)$ to discuss the existence of Auslander-Reiten triangles. In section 2, we use the existence theorem in $\mathbf{K}(\mathrm{Inj}\Lambda)$ directly, and describe the case of $\mathbf{C}(\mathrm{Inj}\Lambda)$ by studying the relation between almost split conflations in a Frobenius category and Auslander-Reiten triangles in its stable category. While in section 3, we propose another method. A map τ is defined and an Auslander-Reiten formula in $\mathbf{C}(\mathrm{Inj}\Lambda)$ is deduced from the Auslander-Reiten formula in $\mathbf{K}(\mathrm{Inj}\Lambda)$. Using this formula, the existence of almost split conflations can be proved directly, with τ the Auslander-Reiten translation.

5.1 The category of complexes for injectives

This section is devoted to emphasizing some properties of the category $\mathbf{C}(\mathrm{Inj}\Lambda)$ for the later use.

Denote by $\mathcal{A}$ the additive category $\mathrm{Inj}\Lambda$ or $\mathrm{Proj}\Lambda$. We have discussed in Example 2.2.2 that $(\mathbf{C}(\mathcal{A}), \mathcal{E})$, the category of cochain complexes in $\mathcal{A}$, is an exact category, where $\mathcal{E}$ be the class of composable morphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ such that for each $n \in \mathbf{Z}$, the sequence $0 \to X^n \xrightarrow{\alpha^n} Y^n \xrightarrow{\beta^n} Z^n \to 0$ is split exact. Moreover, it is a Frobenius category, and the stable category coincides with its homotopy category $\mathbf{K}(\mathcal{A})$.

For $A \in \mathcal{A}$, consider the complex $J_i(A) = (J^s, d^s)$ with $J^s = 0$ if $s \neq i$, $s \neq i+1$, $J^i = J^{i+1} = A$, $d^i = \mathrm{id}_A$. It is not difficult to prove that all the indecomposable $\mathcal{E}$-projectives in $\mathbf{C}(\mathcal{A})$ are the complexes $J_i(A)$ with A indecomposable.

A complex X in some additive category is called *homotopically minimal*, if every map $\phi\colon X \to X$ of complexes is an isomorphism provided that ϕ is an isomorphism up to homotopy. From Appendix C we know that every complex in $\mathbf{C}(\mathcal{A})$ has a decomposition $X = X' \coprod X''$, such that X' is homotopically minimal, and X'' is null homotopic. Moreover, X' is unique up to isomorphism. Denote by $\mathbf{C}_{\mathcal{P}}(\mathcal{A})$ the full subcategory of $\mathbf{C}(\mathcal{A})$ whose objects are the X in $\mathbf{C}(\mathcal{A})$ with $X = X'$. Then we have the following useful lemma.

Lemma 5.1.1. *Let X be an object in $\mathbf{C}_{\mathcal{P}}(\mathcal{A})$. Then $\mathrm{End}_{\mathbf{C}(\mathcal{A})}(X)$ is a local ring if and only if $\mathrm{End}_{\mathbf{K}(\mathcal{A})}(X)$ is a local ring.*

Proof. Use that X is homotopically minimal. □

5.1.1 Indecomposable objects in $\mathbf{C}^{+,b}(\mathrm{inj}\Lambda)$

It is well known that the bounded derived category $\mathbf{D}^b(\mathrm{mod}\Lambda)$ is a Krull-Schmidt category (see [12, Corollary 2.10], and note that if Λ is an artin algebra, then $\mathbf{D}^b(\mathrm{mod}\Lambda)$ has split idempotents implies that $\mathbf{D}^b(\mathrm{mod}\Lambda)$ is Krull-Schmidt). Using the equivalence $\mathbf{K}^{+,b}(\mathrm{inj}\Lambda) \cong \mathbf{D}^b(\mathrm{mod}\Lambda)$, an object Z in $\mathbf{K}^{+,b}(\mathrm{inj}\Lambda)$ is indecomposable if and only if $\mathrm{End}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(Z)$ is a local ring. Furthermore, we have

Proposition 5.1.2. *Let Z be an object in $\mathbf{C}^{+,b}(\mathrm{inj}\Lambda)$. Then Z is indecomposable if and only if $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(Z)$ is a local ring.*

Proof. The sufficiency is obvious. For the necessity, if Z is $\mathcal{E}$-projective, then $Z = J_i(I)$ with I indecomposable, so $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(Z) \cong \mathrm{End}_\Lambda(I)$ is a local ring. If Z is not an $\mathcal{E}$-projective, then Z is homotopically minimal. We claim that Z is indecomposable in $\mathbf{K}^{+,b}(\mathrm{inj}\Lambda)$. Otherwise, there are two non-zero objects Z_1, Z_2 in $\mathbf{K}^{+,b}(\mathrm{inj}\Lambda)$, and an isomorphism $Z \cong Z_1 \coprod Z_2$ in $\mathbf{K}^{+,b}(\mathrm{inj}\Lambda)$, where Z_1 can be written as $Z_1' \coprod Z_1''$ with Z_1' homotopically minimal and Z_1'' null homotopy. Similarly, Z_2 can be written as $Z_2' \coprod Z_2''$. Hence $Z \cong Z_1' \coprod Z_2'$ in $\mathbf{C}^{+,b}(\mathrm{inj}\Lambda)$, a contradiction. Therefore $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(Z)$ is a local ring since $\mathrm{End}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(Z)$ is a local ring. $\square$

5.1.2 Compact objects

Let $\mathcal{A}$ be an additive category with arbitrary coproducts. Denote by $\mathcal{A}^c$ and $\mathbf{C}^c(\mathcal{A})$ the full subcategory of $\mathcal{A}$ and $\mathbf{C}(\mathcal{A})$ formed by all compact objects, respectively. We study the category $\mathbf{C}^c(\mathcal{A})$, and the compact objects in $\mathbf{C}(\mathrm{Inj}\Lambda)$ can be described explicitly as an immediate consequence.

Proposition 5.1.3.

$$\mathbf{C}^c(\mathcal{A}) = \mathbf{C}^b(\mathcal{A}^c).$$

Proof. On one hand, let X be a bounded complex with compact components and $f\colon X \to \coprod_{i\in I} Y_i$ be a chain map, then $f^s\colon X^s \to \coprod_{i\in I} Y_i^s$ factors through a finite subsum $\coprod_{i\in J_s} Y_i^s$. Hence f factors through the subsum indexed by the union of the J_s with $X^s \neq 0$, which implies that X is compact in $\mathbf{C}(\mathcal{A})$.

On the other hand, If $X = (X^s, d^s)$ is a compact object in $\mathbf{C}(\mathcal{A})$, then the morphism $\iota\colon X \to \coprod_{i=-\infty}^{+\infty} J_i(X^{i+1})$ which is given by

$$\begin{tikzcd}
X: & \cdots \arrow[r] & X^n \arrow[r, "d^n"] \arrow[d, "\left[\begin{smallmatrix}1\\ d^n\end{smallmatrix}\right]"'] & X^{n+1} \arrow[r] \arrow[d, "\left[\begin{smallmatrix}1\\ d^{n+1}\end{smallmatrix}\right]"] & \cdots \\
\coprod_{i=-\infty}^{+\infty} J_i(X^{i+1}): & \cdots \arrow[r] & X^n \coprod X^{n+1} \arrow[r, "\left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]"] & X^{n+1} \coprod X^{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

factors through a finite subsum. Hence only finitely many X^i are non-zero. This complete the proof that X is bounded.

Next we show that $X^s \in \mathcal{A}^c$ for each s. In fact, for any morphism $f\colon X^s \to \coprod_{i\in I} A_i$, we may consider the following chain map

$$\begin{tikzcd}
X: & \cdots \arrow[r] & X^{s-1} \arrow[r, "d^{s-1}"] \arrow[d, "fd^{s-1}"] & X^s \arrow[r, "d^s"] \arrow[d, "f"] & X^{s+1} \arrow[r] \arrow[d] & \cdots \\
\coprod_i J_{s-1}(A_i): & 0 \arrow[r] & \coprod_{i\in I} A_i \arrow[r, "\mathrm{id}"] & \coprod_{i\in I} A_i \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Again note that X is compact in $\mathbf{C}(\mathcal{A})$, we know f factors through finite subsum. $\square$

Corollary 5.1.4. $\mathbf{C}^c(\mathrm{Inj}\Lambda) = \mathbf{C}^b(\mathrm{inj}\Lambda)$

Remark. *Observe that* $\mathbf{C}^c(\mathrm{Inj}\Lambda) \subsetneq \mathbf{C}^{+,b}(\mathrm{inj}\Lambda)$, *and comparing this with the corresponding case in homotopy category* $\mathbf{K}^c(\mathrm{Inj}\Lambda) \cong \mathbf{K}^{+,b}(\mathrm{inj}\Lambda)$.

5.2 Almost split conflations

In this section, we consider the existence of almost split conflations in $\mathbf{C}(\mathrm{Inj}\Lambda)$ by studying its relation with the existence of Auslander-Reiten triangles in $\mathbf{K}(\mathrm{Inj}\Lambda)$.

Definition 5.2.1. *A conflation $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in an exact category is called an* almost split conflation*, if α is left almost split and β is right almost split.*

Note that the end terms X and Z of an almost split conflation are indecomposable objects with local endomorphism rings. Moreover, each end term determines an almost split conflation uniquely up to isomorphism.

We begin with some lemmas before stating the main theorem. Observe that idempotents split in the category $\mathbf{C}(\mathrm{Inj}\Lambda)$.

Lemma 5.2.2.

1. *Let Z be a complex in $\mathbf{C}_{\mathcal{P}}(\mathrm{Inj}\Lambda)$ with a local endomorphism ring. Then a morphism $\beta\colon Y \to Z$ is a retraction in $\mathbf{C}(\mathrm{Inj}\Lambda)$ if and only if $\bar{\beta}$ is a retraction in $\mathbf{K}(\mathrm{Inj}\Lambda)$.*

2. *Let X be a complex in $\mathbf{C}_{\mathcal{P}}(\mathrm{Inj}\Lambda)$ with a local endomorphism ring. Then a morphism $\alpha\colon X \to Y$ is a section in $\mathbf{C}(\mathrm{Inj}\Lambda)$ if and only if $\bar{\alpha}$ is a section in $\mathbf{K}(\mathrm{Inj}\Lambda)$.*

Proof. We only consider (1), and (2) can be proved dually. The necessity is obvious. For the sufficiency, assume that $\bar{\beta}$ is a retraction in $\mathbf{K}(\mathrm{Inj}\Lambda)$. Then there is a morphism $\rho\colon Z \to Y$ such that $\beta \circ \rho = 1 + t \circ s$, where $s\colon Z \to P$, $t\colon P \to Z$ and P is null homotopic. We claim that $t \circ s$ is not an isomorphism. Otherwise, there is a morphism u satisfying $u \circ t \circ s = \mathrm{id}_Z$, hence $s \circ u \circ t$ is an idempotent. Since all idempotents in $\mathbf{C}(\mathrm{Inj}\Lambda)$ split, we get that Z is a summand of P, which contradicts our assumption. The endomorphism ring $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(Z)$ is local implies that $1 + t \circ s$ is an isomorphism. Hence ρ is the right inverse of β, and β is a retraction. □

Lemma 5.2.3. *Let $\varepsilon\colon X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be a non-split conflation in $\mathbf{C}(\mathrm{Inj}\Lambda)$.*

1. *If $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(Z)$ is a local ring, then β is right almost split in $\mathbf{C}(\mathrm{Inj}\Lambda)$ if and only if $\bar{\beta}$ is right almost split in $\mathbf{K}(\mathrm{Inj}\Lambda)$.*

2. *If $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(X)$ is a local ring, then α is left almost split in $\mathbf{C}(\mathrm{Inj}\Lambda)$ if and only if $\bar{\alpha}$ is left almost split in $\mathbf{K}(\mathrm{Inj}\Lambda)$.*

Proof. We only consider (1), and (2) can be proved analogously. First we know that Z is indecomposable, since its endomorphism ring is local. The conflation ε is non-split implies that Z is a complex in $\mathbf{C}_{\mathcal{P}}(\mathrm{Inj}\Lambda)$. The necessity is easy, by using Lemma 5.2.2. For the sufficiency, assume that $\bar{\beta}$ is right almost split in $\mathbf{K}(\mathrm{Inj}\Lambda)$. Then β is not a retraction in $\mathbf{C}(\mathrm{Inj}\Lambda)$ since $\bar{\beta}$ is not a retraction in $\mathbf{K}(\mathrm{Inj}\Lambda)$. For any morphism $v\colon W \to Z$ in $\mathbf{C}(\mathrm{Inj}\Lambda)$ which is not a retraction, then by Lemma 5.2.2 we know that $\bar{v}$ is not a retraction in $\mathbf{K}(\mathrm{Inj}\Lambda)$, either. Hence there is a morphism $u\colon W \to Y$ such that $v - \beta \circ u = g \circ f$, where $f\colon W \to Q$, $g\colon Q \to Z$ and Q is null homotopic. Note that Q is $\mathcal{E}$-projective and β is a deflation, so g can be lifted to w.

$$\begin{tikzcd} & Q \arrow[dl, dotted, "\exists w"'] \arrow[d, "g"] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd}$$

Hence $v = \beta \circ u + g \circ f = \beta(u + w \circ f)$ and β is left almost split. □

From Lemma 5.1.1 and 5.2.3, we get immediately that

Proposition 5.2.4. *Let $\varepsilon\colon X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be an almost split conflation in $\mathbf{C}(\mathrm{Inj}\Lambda)$. Then $\bar{\varepsilon}\colon X \xrightarrow{\bar{\alpha}} Y \xrightarrow{\bar{\beta}} Z \xrightarrow{\bar{\gamma}} X[1]$ is an AR-triangle in $\mathbf{K}(\mathrm{Inj}\Lambda)$. Conversely, let $\varepsilon\colon X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be a conflation in $\mathbf{C}(\mathrm{Inj}\Lambda)$ with $X, Z \in \mathbf{C}_{\mathcal{P}}(\mathrm{Inj}\Lambda)$. If $\bar{\varepsilon}$ is an AR-triangle in $\mathbf{K}(\mathrm{Inj}\Lambda)$, then ε is an almost split conflation in $\mathbf{C}(\mathrm{Inj}\Lambda)$.*

Lemma 5.2.5. *Suppose $P \coprod X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \coprod Q$ is a conflation in $\mathbf{C}(\mathrm{Inj}\Lambda)$ with P and Q null homotopic. Then it has the form:*

$$\eta\colon \qquad P \coprod X \xrightarrow{\left[\begin{smallmatrix} \phi & 0 \\ 0 & \alpha_1 \\ 0 & 0 \end{smallmatrix}\right]} Y_1 \coprod Y_2 \coprod Y_3 \xrightarrow{\left[\begin{smallmatrix} 0 & \beta_1 & 0 \\ 0 & 0 & \psi \end{smallmatrix}\right]} Z \coprod Q$$

where ϕ and ψ are isomorphic and $\varepsilon\colon X \xrightarrow{\alpha_1} Y_2 \xrightarrow{\beta_1} Z$ is a conflation.

Proof. The proof is similar to [13, Lemma 9.2], □

The following main theorem asserts the existence of almost split conflations in $\mathbf{C}(\mathrm{Inj}\Lambda)$.

Theorem 5.2.6. *Let Z be a non-projective indecomposable object in $\mathbf{C}^{+,b}(\mathrm{inj}\Lambda)$. Then there exists an almost split conflation in $\mathbf{C}(\mathrm{Inj}\Lambda)$ ending in Z.*

Proof. From Proposition 5.1.2 we know that Z is indecomposable and compact in $\mathbf{K}(\mathrm{Inj}\Lambda)$. Hence by Proposition 4.1.5 there exists an Auslander-Reiten triangle

$$\bar{\theta}\colon X \xrightarrow{\bar{\alpha}} Y \xrightarrow{\bar{\beta}} Z \xrightarrow{\bar{\gamma}} X[1].$$

By the definition of exact triangles in $\mathbf{K}(\mathrm{Inj}\Lambda)$, we can find a conflation $\eta\colon M \xrightarrow{i} N \xrightarrow{p} L$ in $\mathbf{C}(\mathrm{Inj}\Lambda)$ such that $\bar{\theta}$ is isomorphic to $\bar{\eta}$ in $\mathbf{K}(\mathrm{Inj}\Lambda)$. From Lemma 5.2.5, we can choose a proper η such that $M, L \in \mathbf{C}_{\mathcal{P}}(\mathrm{Inj}\Lambda)$. Hence η is an almost split conflation in $\mathbf{C}(\mathrm{Inj}\Lambda)$, by proposition 5.2.4. Since Z is isomorphic to L in $\mathbf{K}(\mathrm{Inj}\Lambda)$, and both Z and L are objects in $\mathbf{C}_{\mathcal{P}}(\mathrm{Inj}\Lambda)$, we know that Z is isomorphic to L in $\mathbf{C}(\mathrm{Inj}\Lambda)$. □

5.3 The Auslander-Reiten translation

In section 4.1, the classical Auslander-Reiten formula for modules was extended to the homotopy category $\mathbf{K}(\mathrm{Inj}\Lambda)$. In this section, we will define a map τ in $\mathbf{C}(\mathrm{Inj}\Lambda)$, and give an analogous formula. Using this formula, the existence of almost split conflations can be proved directly, with τ the Auslander-Reiten translation.

Let X be a complex in $\mathbf{C}(\mathrm{Inj}\Lambda)$. From Appendix B we know that its projective resolution $\mathbf{p}X$ can be decomposed as $(\mathbf{p}X)' \coprod (\mathbf{p}X)''$, where $(\mathbf{p}X)'$ is homotopically minimal, and $(\mathbf{p}X)''$ is null homotopic. Call $(\mathbf{p}X)'$ the *minimal projective resolution* of X. Applying the tensor functor $-\otimes_\Lambda D\Lambda$ to every component of $(\mathbf{p}X)'[-1]$, we obtain a new complex

$$\tau X = (\mathbf{p}X)' \otimes_\Lambda D\Lambda[-1]$$

in $\mathbf{C}(\mathrm{Inj}\Lambda)$. The following proposition implies that τX is homotopically minimal in $\mathbf{C}(\mathrm{Inj}\Lambda)$.

Lemma 5.3.1. *Let $X \in \mathbf{C}(\mathrm{Proj}\Lambda)$ be homotopically minimal. Then $X \otimes_\Lambda D\Lambda$ is a homotopically minimal object in $\mathbf{C}(\mathrm{Inj}\Lambda)$.*

Proof. Note that the tensor functor $-\otimes_\Lambda D\Lambda$ induces an equivalence from $\mathrm{Proj}\Lambda$ to $\mathrm{Inj}\Lambda$, which can be extended to the equivalence on the categories of complexes $\mathbf{C}(\mathrm{Proj}\Lambda) \cong \mathbf{C}(\mathrm{Inj}\Lambda)$, hence induces further an equivalence $\mathbf{K}(\mathrm{Proj}\Lambda) \cong \mathbf{K}(\mathrm{Inj}\Lambda)$. Then the result is easy to prove. □

Lemma 5.3.2. *The map τ induces an endofunctor in $\mathbf{K}(\mathrm{Inj}\Lambda)$. Moreover, the restriction of τ to $\mathbf{K}^c(\mathrm{Inj}\Lambda)$ is fully faithful.*

Proof. The functor induced by τ is the composite

$$\mathbf{K}(\mathrm{Inj}\Lambda) \xrightarrow{\mathrm{can}} \mathbf{D}(\mathrm{Mod}\Lambda) \xrightarrow{\mathbf{p}} \mathbf{K}(\mathrm{Proj}\Lambda) \xrightarrow{-\otimes_\Lambda D\Lambda} \mathbf{K}(\mathrm{Inj}\Lambda) \xrightarrow{[-1]} \mathbf{K}(\mathrm{Inj}\Lambda)$$

In section 3.3 it has been proved that the restriction of τ to $\mathbf{K}^c(\mathrm{Inj}\Lambda)$ is fully faithful. □

Corollary 5.3.3. *Let X be a non-projective indecomposable object in $\mathbf{C}^{+,b}(\mathrm{inj}\Lambda)$. Then τX is indecomposable.*

Proof. Proposition 5.1.2 tells that the endomorphism ring $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(X)$ is local, so is $\mathrm{End}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(X)$. Since X is a compact object in $\mathbf{K}(\mathrm{Inj}\Lambda)$, by Lemma 5.3.2 we know that $\mathrm{End}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(\tau X)$ is a local ring. The object τX is homotopically minimal implies that $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(\tau X)$ is a local ring, by Lemma 5.1.1. Hence τX is indecomposable. □

Given objects X and Z in $\mathbf{C}(\mathrm{Inj}\Lambda)$, we denote by $\mathrm{Ext}^1_{\mathcal{E}}(Z,X)$ the set of all exact pairs $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in $\mathcal{E}$ modulo the equivalence relation which is defined in the following way. Two such pairs (α,β) and (α',β') are equivalent if there exists a commutative diagram as below:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, equal] & Y \arrow[r, "\beta"] \arrow[d] & Z \arrow[d, equal] \\ X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z'. \end{tikzcd}$$

then $\mathrm{Ext}^1_{\mathcal{E}}(Z,X)$ becomes an abelian group under Baer sum. Recall again that $\mathbf{C}(\mathrm{Inj}\Lambda)$ is a Frobenius category, so $\overline{\mathrm{Hom}}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(Z,X) \cong \underline{\mathrm{Hom}}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(Z,X) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(Z,X)$.

Lemma 5.3.4. *For arbitrary X , $Z \in \mathbf{C}(\mathrm{Inj}\Lambda)$, there is an isomorphism*

$$\mathrm{Ext}^1_{\mathcal{E}}(Z,X) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(Z,X[1])$$

which is natural in X and Z.

In section 3.3, we know that for X, Z in $\mathbf{K}(\mathrm{Inj}\Lambda)$ with Z compact, there is a natural isomorphism

$$D\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(Z,X) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\Lambda)}(X,\tau Z[1]).$$

Combining this with the isomorphism in 5.3.4 we get immediately

Proposition 5.3.5. *Let X be an object in $\mathbf{C}(\mathrm{Inj}\Lambda)$ and Z be an object in $\mathbf{C}^{+,b}(\mathrm{inj}\Lambda)$. Then we have an isomorphism*

$$\Phi_X\colon \mathrm{Ext}^1_{\mathcal{E}}(X,\tau Z) \xrightarrow{\sim} D\underline{\mathrm{Hom}}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(Z,X)$$

which is natural in X and Z.

Theorem 5.3.6. *Let Z be a non-projective indecomposable object in $\mathbf{C}^{+,b}(\mathrm{inj}\Lambda)$. Then there exists an almost split conflation*

$$\tau Z \to Y \to Z$$

in $\mathbf{C}(\mathrm{Inj}\Lambda)$.

We introduce some lemmas before proving the theorem.

Lemma 5.3.7. *Suppose that in an exact category $(\mathcal{C},\mathcal{E})$ there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"] & Y \arrow[r, "\beta"] \arrow[d, "g"] & Z \arrow[d, "h"] \\ X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \end{tikzcd}$$

such that (α,β) and (α',β') are conflations in $\mathcal{E}$. Then there exists a morphism $u\colon Y \to X'$ such that $u\alpha = f$ if and only if there exists a morphism $v\colon Z \to Y'$ such that $\beta' v = h$.

Lemma 5.3.8. *Let $\varepsilon\colon X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be a conflation in $(\mathbf{C}(\mathrm{Inj}\Lambda),\mathcal{E})$, then the following are equivalent:*

1. *ε is an almost split conflation.*

2. *β is right almost split and $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(X)$ is local.*

Proof of Theorem 5.3.6. Take for X the object Z in Proposition 5.3.5, we get an isomorphism

$$\Phi_Z\colon \mathrm{Ext}^1_{\mathcal{E}}(Z,\tau Z) \xrightarrow{\sim} D\underline{\mathrm{Hom}}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(Z,Z)$$

Note that $\Gamma = \underline{\mathrm{End}}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(Z)$ is a local ring, and finitely generated as a k-module. Let f be a non-zero map in $D\underline{\mathrm{Hom}}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(Z,Z)$ such that f vanishes on $\mathrm{rad}\Gamma$. Denote by $\eta_Z = \Phi^{-1}(f)$. We claim that η is the almost split conflation.

First, η is not split since it is non-zero. Let $u\colon W \to Z$ be an arbitrary morphism which is not a retraction, then its composition with any morphism from Z to W is in the radical of Γ, so $D\underline{\mathrm{Hom}}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(Z,u)(f) = 0$. Hence $\Phi_W \mathrm{Ext}^1_{\mathcal{E}}(u, \tau Z)(\eta) = 0$, by the naturalness of Φ. Since Φ_W is an isomorphism, $\mathrm{Ext}^1_{\mathcal{E}}(u, \tau Z)(\eta) = 0$. Lemma 5.3.7 implies that there exists a morphism $v\colon W \to Y$ such that $u = \beta v$, hence β is right almost split. Furthermore, in the proof of Corollary 5.3.3, we have seen that $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\Lambda)}(\tau Z)$ is a local ring. Using Lemma 5.3.8 we finish the proof. $\square$

Appendix A. Brown representability

The Brown representability theorem was first established by Brown [17] in homotopy theory. It asserts that representation functors $\mathrm{Hom}_{\mathcal{T}}(-, X)$ can be characterized as the cohomological functors taking coproducts to products. People followed him and generalized this theorem to triangulated categories through various approaches. Neeman pointed out in his paper [49] that the theorem holds for compactly generated triangulated categories. In his book [50] he introduced the concept of a well generated triangulated category. These categories naturally generalize compactly generated ones and they still satisfy the Brown representability. Neeman's result was improved by Krause [38] to perfectly generated triangulated categories; in [37] he compared his perfect generation with the well generation of Neeman.

Definition Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. Then $\mathcal{T}$ is said to be *perfectly generated*, if there exists a set $\mathcal{T}_0$ of objects satisfying:

PG1 An object $X \in \mathcal{T}$ is zero provided that $\mathrm{Hom}_{\mathcal{T}}(\Sigma^n S, X) = 0$ for all $n \in \mathbb{Z}$ and $S \in \mathcal{T}_0$.

PG2 Given a countable set of maps $X_i \to Y_i$ in $\mathcal{T}$ such that the map $\mathrm{Hom}_{\mathcal{T}}(S, X_i) \to \mathrm{Hom}_{\mathcal{T}}(S, Y_i)$ is surjective for all i and $S \in \mathcal{T}_0$, the induced map

$$\mathrm{Hom}_{\mathcal{T}}(S, \coprod_i X_i) \to \mathrm{Hom}_{\mathcal{T}}(S, \coprod_i Y_i)$$

is surjective for all $S \in \mathcal{T}_0$.

The category $\mathcal{T}$ is said to be *well generated* if the morphism set $X_i \to Y_i$ in condition [**PG2**] is an arbitrary set rather than a countable set, and in addition, the following condition holds.

WG The objects in $\mathcal{T}_0$ are α-small for some cardinal α.

Recall that an object S is *α-small* if every map $S \to \coprod_{i \in I} X_i$ in $\mathcal{T}$ factors through $\coprod_{i \in J} X_i$ for some $J \subseteq I$ with $\mathrm{card} J < \alpha$. When $\alpha = \aleph$, we say $\mathcal{T}$ is *compactly generated*.

The Brown representability theorem in Krause's paper [38] says that

Theorem A.1 *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts, and suppose that $\mathcal{T}$ is perfectly generated by a set of objects. Then a functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ is cohomological and sends all coproducts in $\mathcal{T}$ to products if and only if $F \cong \mathrm{Hom}_{\mathcal{T}}(-, X)$ for some object X in $\mathcal{T}$.*

An immediate consequence of this theorem is

Corollary A.2 *Let $\mathcal{T}$ be a perfectly generated triangulated category, and $\mathcal{S}$ be an arbitrary triangulated category. Then an exact functor $F\colon \mathcal{T} \to \mathcal{S}$ preserves all coproducts if and only if it has a right adjoint.*

Proof. Let s be an object in $\mathcal{S}$, and consider the functor $\mathrm{Hom}_{\mathcal{S}}(F(-), s)$. This functor is cohomological and sends all coproducts in $\mathcal{T}$ to products. Hence, by above theorem, this functor is representable; there is a $G(s) \in \mathcal{T}$ with

$$\mathrm{Hom}_{\mathcal{S}}(F(-), s) \cong \mathrm{Hom}_{\mathcal{T}}(-, G(s)).$$

From [45, IV Corollary 2] we know that G extends to a functor, right adjoint to F. □

There is the dual concept of a *perfectly cogenerated* triangulated category, and the dual Brown representability theorem. For convenience, we list them here.

Definition Let $\mathcal{T}$ be a triangulated category with arbitrary products. Then $\mathcal{T}$ is said to be *perfectly cogenerated*, if there exists a set $\mathcal{T}_0$ of objects satisfying:

PG1 An object $X \in \mathcal{T}$ is zero provided that $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^n S) = 0$ for all $n \in \mathbb{Z}$ and $S \in \mathcal{T}_0$.

PG2 Given a countable set of maps $X_i \to Y_i$ in $\mathcal{T}$ such that the map $\mathrm{Hom}_{\mathcal{T}}(Y_i, S) \to \mathrm{Hom}_{\mathcal{T}}(X_i, S)$ is surjective for all i and $S \in \mathcal{T}_0$, the induced map

$$\mathrm{Hom}_{\mathcal{T}}(\prod_i Y_i, S) \to \mathrm{Hom}_{\mathcal{T}}(\prod_i X_i, S)$$

is surjective for all $S \in \mathcal{T}_0$.

Theorem A.3 *Let $\mathcal{T}$ be a triangulated category with arbitrary products, and suppose that $\mathcal{T}$ is perfectly cogenerated by a set of objects. Then a functor*

$F\colon \mathcal{T} \to \mathrm{Ab}$ is cohomological and preserves all products if and only if $F \cong \mathrm{Hom}_{\mathcal{T}}(X, -)$ for some object X in $\mathcal{T}$.

Corollary A.4 *Let $\mathcal{T}$ be a perfectly cogenerated triangulated category, and $\mathcal{S}$ be an arbitrary triangulated category. Then an exact functor $F\colon \mathcal{T} \to \mathcal{S}$ preserves all products if and only if it has a left adjoint.*

Appendix B. Resolution of complexes

Resolutions are used to replace a complex in some abelian category $\mathcal{A}$ by another one which is quasi-isomorphic to the original one but easier to handle. Depending on properties of $\mathcal{A}$, injective and projective resolutions are constructed via Brown representability. We refer to [40] for the detailed proofs of all results in this appendix.

Injective resolutions. Let $\mathcal{A}$ be an abelian category. Suppose that $\mathcal{A}$ has arbitrary products which are exact, that is, for every family of exact sequences $X_i \to Y_i \to Z_i$ in $\mathcal{A}$, the sequence $\prod_i X_i \to \prod_i Y_i \to \prod_i Z_i$ is exact. Suppose in addition that $\mathcal{A}$ has an injective cogenerator which we denote by U.

Denote by $\mathbf{K}_{\mathrm{inj}}(\mathcal{A})$ the smallest full triangulated subcategory of $\mathbf{K}(\mathcal{A})$ which is closed under taking products and contains all injective objects of $\mathcal{A}$ (viewed as complexes concentrated in degree zero). Observe that $\mathbf{K}_{\mathrm{inj}}(\mathcal{A}) \subseteq \mathbf{K}(\mathrm{Inj}\mathcal{A})$

Lemma B.1 *The triangulated category $\mathbf{K}_{\mathrm{inj}}(\mathcal{A})$ is perfectly cogenerated by U. Therefore the inclusion $\mathbf{K}_{\mathrm{inj}}(\mathcal{A}) \hookrightarrow \mathbf{K}(\mathcal{A})$ has a left adjoint $\mathbf{i}\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}_{\mathrm{inj}}(\mathcal{A})$.*

Proposition B.2 *Let $\mathcal{A}$ be an abelian category. Suppose that $\mathcal{A}$ has an injective cogenerator and arbitrary products which are exact. Let X, Y be complexes in $\mathcal{A}$.*

1. *The natural map $X \to \mathbf{i}X$ is a quasi-isomorphism and we have natural isomorphisms*

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X, Y) \cong \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X, \mathbf{i}Y) \cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, \mathbf{i}Y)$$

2. *The composite*

$$\mathbf{K}_{\mathrm{inj}}(\mathcal{A}) \stackrel{\mathrm{inc}}{\to} \mathbf{K}(\mathcal{A}) \stackrel{\mathrm{can}}{\to} \mathbf{D}(\mathcal{A})$$

   is an equivalence of triangulated categories.

Remark. *The isomorphism in Proposition B.2 shows that the assignment $X \mapsto \mathbf{i}X$ induces a right adjoint for the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$.*

Example B.3 *Given an associative ring Λ, the category $\mathrm{Mod}\Lambda$ of Λ-modules has exact products and an injective cogenerator. so the above results apply.*

Projective resolutions. Let $\mathcal{A}$ be an abelian category. Suppose that $\mathcal{A}$ has arbitrary coproducts which are exact. Suppose in addition that $\mathcal{A}$ has a projective generator which we denote by S.

Denote by $\mathbf{K}_{\mathrm{proj}}(\mathcal{A})$ the smallest full triangulated subcategory of $\mathbf{K}(\mathcal{A})$ which is closed under taking coproducts and contains all projective objects of $\mathcal{A}$. We also have $\mathbf{K}_{\mathrm{proj}}(\mathcal{A}) \subseteq \mathbf{K}(\mathrm{Proj}\mathcal{A})$

Lemma B.4 *The triangulated category $\mathbf{K}_{\mathrm{proj}}(\mathcal{A})$ is perfectly generated by S. Therefore the inclusion $\mathbf{K}_{\mathrm{proj}}(\mathcal{A}) \hookrightarrow \mathbf{K}(\mathcal{A})$ has a right adjoint $\mathbf{p}\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}_{\mathrm{proj}}(\mathcal{A})$.*

Proposition B.5 *Let $\mathcal{A}$ be an abelian category. Suppose that $\mathcal{A}$ has an projective generator and arbitrary coproducts which are exact. Let X, Y be complexes in $\mathcal{A}$.*

1. *The natural map $\mathbf{p}X \to X$ is a quasi-isomorphism and we have natural isomorphisms*

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X,Y) \cong \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(\mathbf{p}X,Y) \cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(\mathbf{p}X,Y)$$

2. *The composite*

$$\mathbf{K}_{\mathrm{proj}}(\mathcal{A}) \stackrel{\mathrm{inc}}{\to} \mathbf{K}(\mathcal{A}) \stackrel{\mathrm{can}}{\to} \mathbf{D}(\mathcal{A})$$

is an equivalence of triangulated categories.

Remark. *The isomorphism in Proposition B.5 shows that the assignment $\mathbf{p}X \mapsto X$ induces a left adjoint for the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$.*

Example B.6 *Any module category $\mathrm{Mod}\Lambda$ has a projective generator and exact coproducts. so the above results apply.*

Appendix C. Homotopically minimal complexes

The results in this appendix is due to Krause [42, Appendix B].

A complex X in some additive category is called *homotopically minimal*, if every map $\phi\colon X \to X$ of complexes is an isomorphism provided that ϕ is an isomorphism up to homotopy. In this appendix, we show that each complex with injective components admits a decomposition $X = X' \coprod X''$ such that X' is homotopically minimal and X'' is null homotopic.

Let $\mathcal{A}$ be an abelian category, and suppose that $\mathcal{A}$ admits injective envelopes. Given a complex $X = (X^n, d^n)$ in $\mathcal{A}$ with injective components. For each $n \in \mathbb{Z}$, let $I(Z^nX)$ be the injective envelope of Z^nX, then we get a decomposition $X^n = V^n \coprod I(Z^nX)$. Note that the inclusion $d^n(V^n) \hookrightarrow B^{n+1}X \hookrightarrow Z^{n+1}X \hookrightarrow I(Z^{n+1}X)$ splits, we get another decomposition $I(Z^{n+1}X) = d^n(V^n) \coprod W^{n+1}$. Hence $X^n = V^n \coprod d^n(V^n) \coprod W^{n+1}$, and the complex X can be written as

$$\cdots \longrightarrow V^n \coprod d^{n-1}(V^{n-1}) \coprod W^n \xrightarrow{\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & \delta^n \\ 0 & 0 & d^n \end{smallmatrix}\right]} V^{n+1} \coprod d^n(V^n) \coprod W^{n+1} \longrightarrow \cdots$$

where $\delta^n\colon W^n \to d^n(V^n)$ satisfies $\delta^n d^{n-1} = 0$.

From the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & V^n \coprod d^{n-1}(V^{n-1}) \coprod W^n \arrow[r, "{\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & \delta^n \\ 0 & 0 & d^n \end{smallmatrix}\right]}"] \arrow[d, "{\left[\begin{smallmatrix} 1 & 0 & \delta^n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]}"'] & V^{n+1} \coprod d^n(V^n) \coprod W^{n+1} \arrow[r] \arrow[d, "{\left[\begin{smallmatrix} 1 & 0 & \delta^{n+1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]}"] & \cdots \\
\cdots \arrow[r] & V^n \coprod d^{n-1}(V^{n-1}) \coprod W^n \arrow[r, "{\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & d^n \end{smallmatrix}\right]}"] & V^{n+1} \coprod d^n(V^n) \coprod W^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

and note that the restriction of d^n on V^n is monomorphism, we know X is isomorphic to the complex $W \coprod (\coprod_{n\in\mathbb{Z}} X(n))$, where

$$W = \qquad \cdots \longrightarrow W^{n-1} \xrightarrow{d^{n-1}|_{W^{n-1}}} W^n \xrightarrow{d^n|_{W^n}} W^{n+1} \longrightarrow \cdots$$

$$X(n) = \qquad \cdots \longrightarrow 0 \longrightarrow V^n \xrightarrow{\simeq} d^n(V^n) \longrightarrow 0 \longrightarrow \cdots$$

It is easy to see that $W^n = I(Z^nW)$ for each $n \in \mathbb{Z}$.

Lemma C.1 *Let $\mathcal{A}$ be an abelian category with injective envelopes. Then the following are equivalent for a complex X in $\mathcal{A}$ with injective components.*

1. *The complex X is homotopically minimal.*

2. *The complex X has no non-zero direct factor which is null homotopic.*

3. *The canonical map $Z^nX \hookrightarrow X^n$ is an injective envelope.*

Proof. (1) $\Rightarrow$ (2) If $X = X' \coprod X''$ with X' null homotopic, then $X' \cong 0$ in the homotopy category. Denote by θ the composite of the projective map $X \twoheadrightarrow X''$ and the including may $X'' \hookrightarrow X$. Then θ is an isomorphism in the homotopy category. Since X is homotopically minimal, θ is also an isomorphism in the category of complexes, which implies that X' is zero.

(2) $\Rightarrow$ (3) Easy from the decomposition we get in the above discussion.

(3) $\Rightarrow$ (1) Let $\phi\colon X \to X$ be a chain map, and there is another chain map $\psi\colon X \to X$ such that $\psi \circ \phi$ and $\phi \circ \psi$ are chain homotopic to the identity id_X. Then, we have a family of maps $\rho^n\colon X^n \to X^{n-1}$ such that

$$\mathrm{id}_{X^n} = (\psi \circ \phi)^n + d^{n-1} \circ \rho^n + \rho^{n+1} \circ d^n.$$

We claim that ϕ is a monomorphism. Let $K = \mathrm{Ker}(\psi \circ \phi)$, and $L^n = K^n \cap Z^nX$. Then ρ^n identifies L^n with $\rho^n(L^n)$, and $\rho^n(L^n) \cap Z^{n-1}(X) = 0$, since $(d^{n-1} \circ \rho^n)L^n = L^n$. The assumption on $Z^{n-1}X$ implies $L^n = 0$. The same assumption on Z^nX implies $K^n = 0$. This completes the proof that ϕ is a monomorphism. Similarly we have that ψ is a monomorphism.

Next we show that ϕ is also an epimorphism. Consider the exact sequence $0 \to X \xrightarrow{\phi} X \to \mathrm{Coker}\phi \to 0$. The sequence is split exact in each degree because X has injective components. Hence it can be embedded into an exact triangle $X \xrightarrow{\phi} X \to \mathrm{Coker}\phi \to \Sigma X$ in the homotopy category. The map ϕ is isomorphic in $\mathbf{K}(\mathrm{Inj}\mathcal{A})$ implies that $\mathrm{Coker}\phi \cong 0$ in $\mathbf{K}(\mathrm{Inj}\mathcal{A})$, i.e. $\mathrm{Coker}\phi \cong 0$ is the sum of the complexes with the form $\cdots \longrightarrow 0 \longrightarrow I \xrightarrow{\cong} I \longrightarrow 0 \longrightarrow \cdots$. It follows that the sequence is also split exact in the category of complexes. Let $\phi'\colon X \to X$ be a left inverse of ϕ. Then $\mathrm{Ker}\phi' \cong \mathrm{Coker}\phi$. On the other hand, ϕ' is the inverse of ϕ in the homotopy category and therefore $\mathrm{Ker}\phi' = 0$ is a monomorphism by the first part of our proof. Thus ϕ is an epimorphism. $\square$

Proposition C.2 *Let $\mathcal{A}$ be an abelian category with injective envelopes. Then every complex X in $\mathcal{A}$ with injective components has a decomposition $X = X' \coprod X''$ such that X' is homotopically minimal and X'' is null homotopic. Given a second decomposition $X = Y' \coprod Y''$ such that Y' is homotopically minimal and Y'' is null homotopic, then X' is isomorphic to Y'.*

Dually, we have the following two results.

Lemma C.3 *Let $\mathcal{A}$ be an abelian category with projective covers. Then the following are equivalent for a complex X in $\mathcal{A}$ with projective components.*

1. *The complex X is homotopically minimal.*
2. *The complex X has no non-zero direct factor which is null homotopic.*
3. *The canonical map $X^n \twoheadrightarrow X^n/B^nX$ is a projective cover.*

Proposition C.4 *Let $\mathcal{A}$ be an abelian category with projective covers. Then every complex X in $\mathcal{A}$ with projective components has a decomposition $X = X' \coprod X''$ such that X' is homotopically minimal and X'' is null homotopic. Given a second decomposition $X = Y' \coprod Y''$ such that Y' is homotopically minimal and Y'' is null homotopic, then X' is isomorphic to Y'.*

Appendix D. Repetitive algebras

Let Λ be an artin k-algebra. The *repetitive algebra* associated with Λ is by definition the doubly infinite matrix algebra without identity,

$$\hat{\Lambda} = \begin{bmatrix} \ddots & & & & \\ \ddots & \Lambda & & & 0 \\ & D\Lambda & \Lambda & & \\ & & D\Lambda & \Lambda & \\ 0 & & & \ddots & \ddots \end{bmatrix}$$

in which matrices have only finitely many non-zero entries and the multiplication is induced from the canonical maps $\Lambda \otimes_\Lambda D\Lambda \to D\Lambda$, $D\Lambda \otimes_\Lambda \Lambda \to D\Lambda$, and the zero map $D\Lambda \otimes_\Lambda D\Lambda \to 0$. The notion of repetitive algebras was first introduced in [27] in connection with trivial extension algebras, for details, also see [24].

Let us recall the modules over a repetitive algebra. A $\hat{\Lambda}$-module X is a sequence $X = (X^n, f^n)$ of Λ-modules X^n and Λ-linear maps $f^n \colon X^n \to \mathrm{Hom}_\Lambda(D\Lambda, X^{n+1})$ satisfying $\mathrm{Hom}_\Lambda(D\Lambda, f^n) \circ f^{n-1} = 0$ for all $n \in \mathbb{Z}$. Sometimes we wrote (X^n, f^n) as

$$\cdots X^{n-1} \overset{f^{n-1}}{\sim} X^n \overset{f^n}{\sim} X^{n+1} \overset{f^{n+1}}{\sim} X^{n+2} \cdots .$$

A morphism $\phi \colon X = (X^n, f^n) \to Y = (Y^n, g^n)$ is a sequence $\phi = (\phi^n)$ of Λ-linear maps $\phi^n \colon X^n \to Y^n$ such that the following diagrams commute for all $n \in \mathbb{Z}$.

$$\begin{tikzcd} X^n \arrow[r, "f^n"] \arrow[d, "\phi^n"'] & \mathrm{Hom}_\Lambda(D\Lambda, X^{n+1}) \arrow[d, "\mathrm{Hom}_\Lambda(D\Lambda, \phi^{n+1})"] \\ Y^n \arrow[r, "g^n"] & \mathrm{Hom}_\Lambda(D\Lambda, Y^{n+1}) \end{tikzcd}$$

It is convenient to consider another equivalent description of $\hat{\Lambda}$-modules, by adjoint functors. Thus an $\hat{\Lambda}$-module X can also be written as $X = (X^n, \breve{f}^n)$, where X^n are Λ-modules and $\breve{f}^n$ are Λ-linear maps $\breve{f}^n \colon X^n \otimes_\Lambda D\Lambda \to X^{n+1}$ such that $\breve{f}^{n+1} \circ (\breve{f}^n \otimes 1) = 0$. We also write $(X^n, \breve{f}^n)$ as

$$\cdots X^{n-1} \overset{\breve{f}^{n-1}}{\sim} X^n \overset{\breve{f}^n}{\sim} X^{n+1} \overset{\breve{f}^{n+1}}{\sim} X^{n+2} \cdots$$

A morphism $\phi \colon X = (X^n, \breve{f}^n) \to Y = (Y^n, \breve{g}^n)$ is a sequence $\phi = (\phi^n)$ of Λ-linear maps $\phi^n \colon X^n \to Y^n$ such that the following diagrams commute for all $n \in \mathbb{Z}$.

$$\begin{tikzcd} X^n \otimes_\Lambda D\Lambda \arrow[r, "\breve{f}^n"] \arrow[d, "\phi^n \otimes 1"'] & X^{n+1} \arrow[d, "\phi^{n+1}"] \\ Y^n \otimes_\Lambda D\Lambda \arrow[r, "\breve{g}^n"] & Y^{n+1} \end{tikzcd}$$

We denote by Mod$\hat{\Lambda}$ the category of all $\hat{\Lambda}$-modules, and by mod$\hat{\Lambda}$ the category of all $\hat{\Lambda}$-modules $X = (X^n, f^n)$ such that $\dim_k(\oplus_n X^n) < \infty$.

Note that the indecomposable projective $\hat{\Lambda}$-modules are given by

$$X = \cdots 0 \sim X^n \overset{\mathrm{id}_{X^n}}{\sim} X^{n+1} \sim 0 \cdots$$

where X^{n+1} is an indecomposable injective Λ-module, and $X^n = \mathrm{Hom}_\Lambda(D\Lambda, X^{n+1})$ (hence X^n is an indecomposable projective Λ-module). Of course, X is also an indecomposable injective $\hat{\Lambda}$-module.

Let us end this appendix with a well-known result.

Theorem D.1 *Let Λ be an artin algebra with $\hat{\Lambda}$ its repetitive algebra. Then the categories* Mod$\hat{\Lambda}$ *and* mod$\hat{\Lambda}$ *are Frobenius categories. Moreover, the stable category* $\underline{\mathrm{Mod}}\hat{\Lambda}$ *is compactly generated and* $(\underline{\mathrm{Mod}}\hat{\Lambda})^c \simeq \underline{\mathrm{mod}}\hat{\Lambda}$.

Bibliography

[1] M. Auslander: Existence theorems for almost split sequences. Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975), pp. 1–44. Lecture Notes in Pure and Appl. Math., Vol. **26**, Dekker, New York, 1977.

[2] M. Auslander, Functors and morphisms determined by objects. Representation theory of artin algebras, Proceedings of the Philadelphia conference, Lecture Notes in Pure and Appl. Math., vol. 37, Dekker, New York, 1978, 1–244.

[3] M. Auslander, Rational singularities and almost split sequences. Trans. Amer. Math. Soc., vol. 293, **2**(1986), 511–531.

[4] M. Auslander, A survey of existence theorems for almost split sequences. Representations of algebras (Durham, 1985), 81–89, London Math. Soc. Lecture Note Ser., 116, Cambridge Univ. Press, Cambridge, 1986.

[5] M. Auslander, Almost split sequences and algebraic geometry. Representations of algebras (Durham, 1985), 165–179, London Math. Soc. Lecture Note Ser., 116, Cambridge Univ. Press, Cambridge, 1986.

[6] M. Auslander and I. Reiten, Stable equivalence of Dualizing R-varieties. Adv. Math. **12** (1974), 306–366.

[7] M. Auslander and I. Reiten, Representation theory of artin algebras III. Almost split sequences. Comm. Algebra **3** (1975), 239–294.

[8] M. Auslander and I. Reiten, Representation theory of artin algebras IV. Invariants given by almost split sequences. Comm. Algebra **5** (1977), 443–518.

[9] M. AUSLANDER, I. REITEN, AND S. O. SMALØ, Representation theory of Artin algebras. Cambridge University Press (1995).

[10] M. AUSLANDER AND S. O. SMALØ, Preprojective modules over artin algeras. J. Algebra **66** (1980), 61–122.

[11] M. AUSLANDER AND S. O. SMALØ, Almost split sequecnes in subcategories. J. Albegra **69** (1981), 426–454.

[12] P. BALMER AND M. SCHLICHTING, Idempotent completion of triangulated categories. J. Algebra **236**, no.2, (2001), 819–834.

[13] R. BAUTISTA, M. J. SOUTO SALORIO, AND R. ZUAZUA, Almost split conflations for complexes with fixed size. J. Algebra **287** (2005), 140–168.

[14] A. BELIGIANNIS, Purity and Almost Split Morphisms in Abstract Homotopy Categories: A Unified Approach via Brown Representability. Algebras and Representation Theory **5** (2002), 483–525.

[15] A. BELIGIANNIS, Auslander-Reiten Triangles, Ziegler Spectra and Gorenstein Rings. K-Theory **32** (2004), 1–82.

[16] A. I. BONDAL AND M. M. KAPRANOV, Representable Functors, Serre Functors, and Reconstructions. Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), 1183–1205.

[17] E. H. BROWN, Cohomology theories. Annals of Math. **75** (1962), 467–484.

[18] P. DRÄXLER, I. REITEN, S. O. SMALØ, Ø. SOLBERG AND WITH AN APPENDIX BY B. KELLER, Exact categories and vector space categories. Trans. of the A.M.S. **351**, no.2 (1999), 647–682.

[19] J. FRANKE, On the Brown representability theorem for triangulated categories. Topology **40** (2001), 667–680.

[20] P. GABRIEL, Auslander-Reiten sequences and representation-finite algebras. In: Representation Theory I. Proc. conf. Ottawa 1979, ed. V. Dlab and P. Gabriel, Springer Lecture Notes in Math. **831** (1980), 1–71.

[21] P. Gabriel, A. V .Roiter, Representations of finite dimensional algebras. Encyclopaedia of the Mathematical Sciences. **73**, A.I.Kostrikin and I.V.Shafareivch (Eds.), Algebra VIII, Springer,1992.

[22] P. Gabriel and A. V. Roiter, Representations of finite dimensional algebras, Springer-Verlag, Berlin, Heidelberg, 1997.

[23] D. Happel, On the derived category of a finite-dimensional algebras. Comment. Math. Helv. **62** (1987), 339–389.

[24] D. Happel, Triangulated categories in the representation theory of finite dimensional algebras. London Mathematical Society lecture note series, **vol. 119**, Cambridge University Press, 1988. MR **89e:**16035

[25] D. Happel, Auslander-Reiten triangles in derived categories of finite-dimensional algebras. Proc. Amer. Math. Soc. **112** (1991), 641–648.

[26] R. Hartshorne, Residue and duality, Lecture Notes in Math. 20, Springer-Verlag, 1966.

[27] D. Hughes and J. Waschbüsch, Trival extensions of tilted algebras. Proc. London Math. Soc. (3), **46**(1983), 347–364.

[28] B. Iversen, Cohomology of sheaves, Springer-Verlag, 1980.

[29] P. Jørgensen,Auslander-Reiten theory over topological spaces, Comment. Math. Helv. 79 (2004), 160–182.

[30] P. Jørgensen, Auslander-Reiten sequences on schemes, to appear in Ark. Mat.

[31] M. Kashiwara and P. SchapiraA, Sheaves on Manifold, Grundlehren der Mathematischen Wissenschaften, 292, Springer (1990).

[32] B. Keller, Chain complexes and stable categories. Manuscripta Math. **67**, no.4 (1990), 379–417.

[33] B. Keller, Derived categories and their uses. Handbook of algebra, **vol. 1**(1996), North-Holland, Amsterdam, 671–701.

[34] B. Keller, Deriving DG categories. Ann. Sci. École. Norm. Sup. **27** (1994), 63–102.

[35] H. Krause, Smashing subcategories and the telescope conjecture - an algebraic approach. Invent. Math. **139** (2000), 99–133.

[36] H. Krause, Auslander-Reiten theory via Brown representability. K-theory **20** (2000), 331–344.

[37] H. Krause, On Neeman's well generated triangulated categories. Doc. Math. **6** (2001), 121–126.

[38] H. Krause, A Brown representability theorem via coherent functors. topology **41** (2002), 853–861.

[39] H. Krause, A short proof for Auslander's defect formula. Linear Algebra Appl. **365** (2003), 267–270.

[40] H. Krause, Derived categories, resolutions, and Brown representability and Exercises (Summer School Chicago 2004).

[41] H. Krause, Auslander-Reiten triangles and a theorem of Zimmermann. Bull. London Math. Soc., **37** (2005), 361–372.

[42] H. Krause, The stable derived category of a noetherian scheme. Compos. Math. **141** (2005), 1128–1162.

[43] H. Krause and J. Le, The Auslander-Reiten formula for complexes of modules. Adv. Math., to appear.

[44] J. Le, Almost split conflations for complexes of modules. Comm. Algebra, to appear.

[45] S. MacLane, Categories for the working mathematician. Berlin, Heidelberg, New York: Springer-Verlag, 1971.

[46] A. Martsinkovsky and R. Martinez Villa, Cohomology of tails, Tate - Vogel cohomology, and noncommutative Serre duality over Koszul quiver algebras. Journal of Algebra, **280** (2005), 58–83.

[47] A. Martsinkovsky and R. Martinez Villa, Cohomology of tails and stable cohomology over Koszul quiver algebras. Fields Institute Communications, **45** (2005), 299–306.

[48] A. Neeman, The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel. Ann. Sci. École Norm. Sup., (4) **25** (1992), 547–566.

[49] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability. J. Amer. Math. Soc. **9** (1996), 205–236.

[50] A. Neeman, Triangulated Categories. Annals of Mathematics Studies, 148, Princeton University Press, Princeton.

[51] D. Quillen, Higher algebraic K-theory I, In: Lecture Notes in Math. 341, 85–147, Springer-Verlag, 1973.

[52] I. Reiten, The use of almost split sequences in the representation theory of Artin algebras. In: Representations of algebras (Puebla, 1980), pp. 29-104 Lecture Notes in Math., 944, Springer, Berlin-New York, 1982.

[53] I. Reiten and M. Van Den Bergh, Noetherian hereditary abelian categories satisfying Serre duality. J. Amer. Math. Soc. **15** (2002), 295–366.

[54] J. Rickard, Derived categories and stable equivalence. J. Pure Appl. Algebra **61** (1989), 303–317.

[55] C. Riedtmann, Algebren, Darstellungsköcher, Überlagerungen und zurück. Comm. Math. Helv., **5**(1980) 199–224.

[56] C. M. Ringel Finite dimensional hereditary algebras of wild representation type. Math. Z., **161** (1978) 235–255.

[57] J. L. Verdier, Catégories dérivées, etat 0, Springer Lecture Notes 569 (1977), 262-311.

[58] J. L. Verdier, Des catégories dérivées des catégories abéliennes. Astérisque **239** (1996).

[59] C. Weibel, An Introduction to Homological Algebra, Cambridge Studies in Adv. Math. 38, Cambridge Univ. Press, 19.

[60] W. Zimmermann Auslander-Reiten sequences over artin rings. J. Algebra 119 (1988), 366–392.