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Subcategories of Triangulated Categories and the Smashing Conjecture

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Abstract

In this thesis the global structure of three classes of algebraic triangulated categories is investigated by describing their thick, localizing and smashing subcategories and by analyzing the Smashing Conjecture. We show that the Smashing Conjecture for the stable module category of a self-injective artin algebra A is equivalent to the statement that a class of model categories associated with A is finitely generated. Smashing localizations of the derived category of a differential graded algebra are realized by morphisms of dg algebras. We use this theory to define a localization of a dg algebra with graded-commutative cohomology at a prime ideal of the cohomology ring. For a hereditary abelian category $\mathcal{A}$ we classify the thick subcategories and the localizing subcategories of the bounded and unbounded derived category of $\mathcal{A}$, respectively. As an application we prove that the Smashing Conjecture holds for the derived category of a hereditary artin algebra of finite representation type.

Zusammenfassung

In dieser Arbeit wird die globale Struktur von drei Klassen algebraischer triangulierter Kategorien untersucht. Dafür werden dicke, lokalisierende und *smashing* Unterkategorien beschrieben und die *Smashing Conjecture* wird analysiert. Wir zeigen, dass die *Smashing Conjecture* für die stabile Modulkategorie einer selbstinjektiven Artin Algebra A äquivalent dazu ist, dass eine Klasse von Modellkategorien, die zu A assoziiert ist, endlich erzeugt ist. *Smashing* Lokalisierungen der derivierten Kategorie einer differentiell graduierten Algebra werden als Morphismen von dg Algebren realisiert. Diese Theorie wird benutzt, um die Lokalisierung einer dg Algebra mit graduiert-kommutativer Kohomologie an einem Primideal des Kohomologierings zu definieren. Des Weiteren werden die dicken und lokalisierenden Unterkategorien der beschränkten beziehungsweise der unbeschränkten derivierten Kategorie einer erblichen abelschen Kategorie klassifiziert. Als eine Folgerung zeigen wir, dass die *Smashing Conjecture* für die derivierte Kategorie einer erblichen Artin Algebra vom endlichen Darstellungstyp wahr ist.

Contents

1 Introduction

In this thesis we study subcategories of triangulated categories and their finiteness properties that are related to the Smashing Conjecture. In particular we reformulate this conjecture for stable module categories, describe smashing subcategories of the derived category of a differential graded algebra, classify the thick and localizing subcategories of the derived category of a hereditary abelian category and prove the Smashing Conjecture for the derived category of a hereditary artin algebra of finite representation type.

A ring is a fundamental object in mathematics. It is a set together with a rule how to multiply and add elements. Interesting rings arise for instance from physics as endomorphism rings of Hilbert spaces, from number theory as rings of integers and from topology as the ring of stable homotopy groups of spheres. Many of these examples are very large and complicated. Since it is in general not possible to describe them easily as in terms of generators and relations, tools are needed to extract information from a ring. These tools are algebraic invariants as the center or more generally the Hochschild cohomology, or the Grothendieck group and its generalization, the algebraic K-theory. The center of the endomorphism ring of a configuration space contains the simultaneously measurable observables. The Grothendieck group controls parts of the representation theory of the ring and the higher algebraic K-groups contain deep number theoretic information.

Since we want to distinguish rings, the following question comes up: When do two rings R and S share the same properties, that is, agree in the algebraic invariants? To a given ring R it is possible to assign a *triangulated category* $\mathcal{D}(R)$, the derived category of R, which can be used to give an answer to the question. Two rings share the same properties if their derived categories are equivalent. So the derived category of a ring can be thought of as a "higher invariant" and motivates the study of triangulated categories.

The concept of a triangulated category is ubiquitous. In ring theory and homological algebra it arises as the derived category of a ring, in algebraic topology as the stable homotopy category, in representation theory of groups as the stable module category of a group algebra and in algebraic geometry as the derived category of sheaves on a scheme.

Classification is a main purpose of pure mathematics. For instance in representation theory we are interested in classifying all modules over a fixed ring R. It is known that the overwhelming part of rings is wild in the sense that it is not possible to determine all modules simultaneously. Triangulated categories provide a framework to classify objects in a weaker way. Two objects X and Y in a small triangulated category $\mathcal{T}$ are considered to be related if they generate the same thick subcategory, that is the collection of objects in $\mathcal{T}$ that can be constructed by the ambient structure of $\mathcal{T}$ starting with X. Since R-modules can be considered as objects in the derived category of R a classification of the thick subcategories leads to a classification of the R-modules. In a triangulated category with small coproducts the localizing subcategories are the analogs of the thick subcategories and are hence worth classifying.

If a localizing subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$ gives rise to a localization functor $L\colon \mathcal{T} \to \mathcal{T}$ such that $\mathcal{C}$ is the full subcategory of the objects that are annihilated by L and if L commutes with small coproducts, then the category $\mathcal{C}$ is called *smashing*. It is known that if $\mathcal{C}$ is of finite type, i.e., generated by compact objects in $\mathcal{T}$, then it is smashing. The Smashing Conjecture for a triangulated category with small coproducts $\mathcal{T}$ states the other direction: every smashing subcategory of $\mathcal{T}$ is of finite type. This conjecture originates from topology. The Smashing Conjecture for the stable homotopy category is a generalization of the Telescope Conjecture of Ravenel [Rav87b, 1.33] which has important consequences for the computation of the stable homotopy groups of spheres. Another reason for studying the Smashing Conjecture is its impact on non-commutative localization of rings and algebraic K-theory.

The stable module category $\underline{\mathrm{Mod}}(R)$ of a Frobenius algebra R is triangulated and possesses small coproducts. If R is artinian, then the Smashing Conjecture for $\underline{\mathrm{Mod}}(R)$ is equivalent to the statement that certain cotorsion pairs are of finite type [KS03]. Furthermore there is a connection between cotorsion pairs in $\mathrm{Mod}(R)$ and associated model structures on the module category [BR02, Hov02]. In fact every cotorsion pair $\mathfrak{X}$ gives rise to an associated model category $\mathrm{Mod}(R)_{\mathfrak{X}}$. In this thesis we extend this connection and find a reformulation of the Smashing Conjecture for the stable module category of a self-injective artin algebra in terms of model categories.

Theorem 1. *Let R be a self-injective artin algebra and $\mathrm{Mod}(R)$ be the category of arbitrary R-modules. The Smashing Conjecture for the stable module category of R is equivalent to the statement that for all cotorsion pairs $\mathfrak{X} = (\mathcal{C}, \mathcal{F})$ such that $\mathcal{C}$ and $\mathcal{F}$ are closed under filtered colimits the associated model category $\mathrm{Mod}(R)_{\mathfrak{X}}$ is finitely generated.*

The derived category of a differential graded algebra (dg algebra) plays a central role in the study of triangulated categories arising in algebra. Every algebraic triangulated category with small coproducts that is generated by a compact object is triangle equivalent to the derived category of a dg algebra [Kel94a].

Theorem 2. *Let A be a dg algebra and $\mathcal{C}$ be a smashing subcategory of $\mathcal{D}(A)$. If $L\colon \mathcal{D}(A) \to \mathcal{D}(A)$ is the localization functor associated with $\mathcal{C}$, then there are dg algebras A_L and A' and a diagram of morphisms of dg algebras $A \leftarrow A' \to A_L$ that induces up to isomorphism the canonical map $\mathcal{D}(A)(A, A) \to \mathcal{D}(A)(LA, LA)$ in cohomology. If A is cofibrant, then the canonical map is induced by a morphism $A \to A_L$ of dg algebras.*

Using Keller's result we deduce:

Corollary 1. *Let $\mathcal{T}$ be an algebraic triangulated category $\mathcal{T}$ with small coproducts that is generated by a compact object. If $\mathcal{C}$ is a smashing subcategory in $\mathcal{T}$ and $L\colon \mathcal{T} \to \mathcal{T}$ is the localization functor corresponding to $\mathcal{C}$, then there is a dg algebra A and a morphism of dg algebras $A \to A_L$ that induces L.*

Let A be a cofibrant dg algebra with graded-commutative cohomology ring H^*A and let $\mathfrak{p}$ be a prime ideal in H^*A that is generated by homogeneous elements. Since the full subcategory $\mathcal{C}_{\mathfrak{p}}$ of dg A-modules with the property that $H^*M_{\mathfrak{p}} = 0$ is smashing, there exists a localization functor $L_{\mathfrak{p}}\colon \mathcal{D}(A) \to \mathcal{D}(A)$ that annihilates $\mathcal{C}_{\mathfrak{p}}$. Define the localization of the dg algebra A at the prime by $A_{\mathfrak{p}} := A_{L_{\mathfrak{p}}}$.

Corollary 2. *The dg algebra morphism $A \to A_{\mathfrak{p}}$ of Theorem 2 induces the canonical map $H^*A \to (H^*A)_{\mathfrak{p}}$. Furthermore $\mathcal{D}(A_{\mathfrak{p}})$ can be characterized by a universal property.*

The derived category of a hereditary abelian category $\mathcal{A}$ is strongly related to $\mathcal{A}$ itself and can be described in a combinatorial way if $\mathcal{A}$ is the category of representations of a quiver.

We enhance the theory of classifications of thick subcategories [DHS88, Nee92, BCR97] to the field of representation theory of algebras:

Theorem 3. *For a hereditary abelian category $\mathcal{A}$ the zeroth homology group functor induces a one to one correspondence between the thick subcategories of the bounded derived category $\mathcal{D}^b(\mathcal{A})$ and the thick subcategories in $\mathcal{A}$.*

In particular Theorem 3 holds for the bounded derived category $\mathcal{D}^b(\mathrm{mod}(A))$ of the category of finitely presented modules over a hereditary algebra A. As an application we determine the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ in two examples combinatorially. Furthermore Theorem 3 implies that the thick subcategories of the category of representations of a Dynkin quiver are independent of the orientation of the quiver.

An analogous result holds in the full derived category.

Theorem 4. *Let $\mathcal{A}$ be a hereditary Grothendieck category. The localizing subcategories of $\mathcal{D}(\mathcal{A})$ correspond bijectively under the zeroth homology group functor to the thick subcategories that are closed under arbitrary direct sums in $\mathcal{A}$.*

As a consequence we are able to prove the Smashing Conjecture in the following case.

Theorem 5. *Let A be a hereditary artin algebra of finite representation type and let $\mathcal{D}(\mathrm{Mod}(A))$ be the derived category of the category of all A-modules. The Smashing Conjecture holds for $\mathcal{D}(\mathrm{Mod}(A))$.*

Outline In Section 2 the background that is necessary to formulate the Smashing Conjecture is presented. Furthermore we point out the two faces of localization: endofunctors and adjoint pairs of functors. Section 3 contains a historical overview on the Smashing Conjecture and the Telescope Conjecture. We explain in what sense the Smashing Conjecture is a generalization of the Telescope Conjecture. Furthermore results and applications of the Smashing Conjecture are described. We recall the language of model categories and the relation of abelian model categories with cotorsion pairs in Section 4. On the one hand we enhance this relation by proving that cofibrantly generated model categories determine cotorsion pairs that are cogenerated by a set and on the other hand we specialize this relation and use our result to obtain Theorem 1. In Section 5 we recall differential graded algebras and their derived category. We show Theorem 2 and draw the consequences Corollary 1 and Corollary 2. Finally Section 6 contains the classifications stated in Theorem 3 and Theorem 4, the illustration by two combinatorial examples and the proof of Theorem 5.

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2 Triangulated categories and their localization

In this section we study triangulated categories and various localizations of them. We pursuit two purposes besides introducing notation. The first is to systematically point out the parallels between subcategories and localizations of triangulated categories following [Kra06]. At the end the framework to formulate the smashing conjecture is provided.

Verdier's thesis [Ver96], the book of Weibel [Wei94] and the appendix in Margolis' book [Mar83] serve as references for triangulated categories as described in Paragraph 2.1. For the theory of localizations of triangulated categories developed in Paragraph 2.2, 2.3 and 2.4 the reader is referred to [Ver96, HPS97, Kra06].

2.1 Definitions and examples

Starting with some basics we define the notion of a triangulated category, state some properties and illustrate the concept with three examples that are important later on.

Let $\mathcal{T}$ be an additive category and $\Sigma\colon \mathcal{T} \to \mathcal{T}$ an additive endofunctor. A diagram $(a,b,c)\colon X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} \Sigma X$ in $\mathcal{T}$ is called *triangle*. A *morphism of triangles* $(a,b,c) \to (a',b',c')$ is given by the following commutative diagram in $\mathcal{T}$:

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f_1"] & Y \arrow[r, "b"] \arrow[d, "f_2"] & Z \arrow[r, "c"] \arrow[d, "f_3"] & \Sigma X \arrow[d, "\Sigma f_1"] \\
X' \arrow[r, "a'"] & Y' \arrow[r, "b'"] & Z' \arrow[r, "c'"] & \Sigma X'
\end{tikzcd}$$

Definition 2.1.1. A triple $(\mathcal{T}, \Sigma, \Delta)$ consisting of an additive category $\mathcal{T}$, an additive endo-equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$ called the *suspension functor* and a class of triangles Δ called the *exact triangles* is a *triangulated category*, if it satisfies the following axioms:

(TR1) A triangle isomorphic to an exact triangle is exact. The triangle $0 \to X \xrightarrow{=} X \to 0$ is exact, and every map $a\colon X \to Y$ can be completed to an exact triangle $X \xrightarrow{a} Y \to Z \to \Sigma X$.

(TR2) A triangle (a,b,c) is exact, if and only if $(b,c,-\Sigma a)$ is exact.

(TR3) If (a,b,c) and (a',b',c') are exact triangles, then morphisms f_1 and f_2 in $\mathcal{T}$ such that $f_2 \circ a = a' \circ f_1$ can be completed to a morphism of triangles

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f_1"] & Y \arrow[r, "b"] \arrow[d, "f_2"] & Z \arrow[r, "c"] \arrow[d, dotted, "f_3"] & \Sigma X \arrow[d, "\Sigma f_1"] \\
X' \arrow[r, "a'"] & Y' \arrow[r, "b'"] & Z' \arrow[r, "c'"] & \Sigma X'.
\end{tikzcd}$$

(TR4) If (a_1,a_2,a_3), (b_1,b_2,b_3) and (c_1,c_2,c_3) are exact triangles such that $c_1 = b_1 \circ a_1$, then there is an exact triangle (d_1, d_2, d_3) making the following diagram commutative

$$\begin{tikzcd}
X \arrow[r, "a_1"] \arrow[d, equal] & Y \arrow[r, "a_2"] \arrow[d, "b_1"] & U \arrow[r, "a_3"] \arrow[d, "d_1"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "c_1"] & Z \arrow[r, "c_2"] \arrow[d, "b_2"] & V \arrow[r, "b_3"] \arrow[d, "d_2"] & \Sigma X \arrow[d, "\Sigma a_1"] \\
& W \arrow[r, equal] \arrow[d, "b_3"] & W \arrow[r, "\Sigma b_3"] \arrow[d, "d_3"] & \Sigma Y \\
& \Sigma Y \arrow[r, "\Sigma a_2"] & \Sigma U. &
\end{tikzcd}$$

If $X \xrightarrow{a} B \to C \to \Sigma X$ is an exact triangle, then C is called the *cone of* a, and we sometimes write $\mathrm{cone}(a)$ for C.

Remark 2.1.2. The concept of a triangulated category was discovered independently by Verdier [Ver96] and Puppe [Pup62] who studied derived categories and stable homotopy theory, respectively.

The homomorphisms in a triangulated category $\mathcal{T}$ form a $\mathbb{Z}$-graded abelian group by setting

$$\mathcal{T}^n(X, Y) = \mathcal{T}(X, \Sigma^n Y).$$

The graded abelian group $\mathcal{T}^*(X, Y)$ is a graded right module over the graded ring $\mathrm{End}^*_{\mathcal{T}}(X) := \mathcal{T}^*(X, X)$.

There are plenty of examples of triangulated categories. We will concentrate on three types of triangulated categories which play a role in the Sections 6, 3 and 4: the derived category, the stable homotopy category and the stable module category.

Example 2.1.3. [Ver96, II Theorem 2.2.6, III Theorem 1.2.2] Let $\mathcal{A}$ be an abelian category. Then $\mathrm{Ch}(\mathcal{A})$ denotes the category of $\mathbb{Z}$-graded complexes in $\mathcal{A}$. Formal inversion of the quasi isomorphisms yields the *derived category* $\mathcal{D}(\mathcal{A})$. It can be constructed using calculus of fractions [GZ67, Ver96]. In general the homomorphisms between two objects in $\mathcal{D}(\mathcal{A})$ do not form a set but by adding the condition that $\mathcal{A}$ is a Grothendieck category [1] this set-theoretic problem can be overcome [Bek00]. We will write $\mathcal{D}(R)$, if $\mathcal{A}$ is the module category of a ring R. Starting with the category of bounded complexes $\mathrm{Ch}^b(\mathcal{A})$ in $\mathcal{A}$ we obtain the *bounded derived category* $\mathcal{D}^b(\mathcal{A})$. If $\mathcal{A}$ is the category of finitely presented modules $\mathrm{mod}(R)$ over a ring R, then the bounded derived category is denoted by $\mathcal{D}^b(R)$ [2].

The suspension functor is given by the *shift functor* [1]. It maps a complex (C, d_C) to the complex $(C[1], d_{C[1]})$ which is defined by $C[1]^n := C^{n+1}$ and $d^n_{C[1]} := -d^{n+1}_C$. Let $f\colon C \to D$ be a map of complexes. The *mapping cone* $\mathrm{cone}(f)$ of f is the complex with

[1]An abelian category $\mathcal{A}$ is called *Grothendieck category* if $\mathcal{A}$ is cocomplete, direct limits are exact and there is a generator in $\mathcal{A}$ [Ste75].

[2]If R is right coherent, then the module category is abelian. Otherwise $\mathcal{D}^b(R)$ can be defined by using that the module category is exact in the sense of Quillen.

$C^{n+1} \oplus D^n$ in degree n and the differential

$$d^n_{\mathrm{cone}(f)} = \begin{pmatrix} -d_C^{n+1} & 0 \\ f^{n+1} & d_D^n \end{pmatrix}.$$

Note that there is a canonical map $\mathrm{cone}(f) \to C[1]$. A triangle $X \to Y \to Z \to X[1]$ in the derived category is said to be *exact*, if it is isomorphic to the diagram $C \xrightarrow{f} D \to \mathrm{cone}(f) \to C[1]$ for some map f.

Example 2.1.4. [Hap88, I Theorem 2.6] Let R be a Frobenius ring, i.e., a ring with enough projective and injective modules such that the projectives and injectives coincide. Two morphisms f and g between R-modules are called *stably equivalent*, if their difference $f-g$ factors through a projective. Define the stable module category $\underline{\mathrm{Mod}}(R)$ to have the same objects as $\mathrm{Mod}(R)$ and $\underline{\mathrm{Mod}}(R)(M,N)$ to be the set of stable equivalence classes of morphisms between the R-modules M and N.

Let $X \in \mathrm{Mod}(R)$ and $X \to E$ be a monomorphism such that E is injective. Choose a module ΣM such that

$$0 \to M \to E \to \Sigma M \to 0$$

is exact. Then $\Sigma\colon \underline{\mathrm{Mod}}(R) \to \underline{\mathrm{Mod}}(R)$ is a well-defined equivalence of categories and serves as the suspension functor for the stable module category. A triangle in $\underline{\mathrm{Mod}}(R)$ is exact, if it is isomorphic to (a,b,c) in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "a"] \arrow[d, equal] & Y \arrow[r, "b"] \arrow[d] & Z \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & X \arrow[r] & E \arrow[r] & \Sigma X \arrow[r] & 0.
\end{tikzcd}$$

In the same way it is possible to construct a stable module category $\underline{\mathrm{mod}}(R)$ starting with the category of finitely presented modules $\mathrm{mod}(R)$.

Example 2.1.5. [Vog70, BF78, Hov99] A *spectrum* is a sequence of pointed spaces (simplicial sets or topological spaces) $E = \{E_n\}_{n\geq 0}$ together with structure maps $\sigma_n\colon \Sigma E_n \to E_{n+1}$. Here, Σ is the (topological) suspension functor that is defined on a pointed space X by $\Sigma X = \frac{X\times[0,1]}{X\times\{0,1\}\cup\{*\}\times[0,1]}$.

Spectra form a category by means of sequences of maps of spaces that commute with the structure maps. An important spectrum is the *sphere spectrum* $\mathbb{S}$ which is the n-dimensional sphere in degree n. More generally for a given space X the *suspension spectrum* $\Sigma^\infty X$ is in degree n the n-fold suspension $\Sigma^n X$ of X.

Let n be an integer; the n-th homotopy group of a spectrum E is defined as $\pi_n(E) := \mathrm{colim}_k\, \pi_{k+n}(E_n)$. The n-th stable homotopy group $\pi_*^s(X)$ of a space X is defined to be the n-the homotopy group of $\Sigma^\infty X$. Call a map of spectra *stable equivalence*, if it induces an isomorphism in all homotopy groups. The *stable homotopy category* $\mathcal{SHC}$ is the localization of the category of spectra with respect to stable equivalences (as for the derived category, it is not *a priori* clear that the "Hom-sets" in $\mathcal{SHC}$ are sets, but with the use of model categories [BF78] this can be shown).

The shift functor $[1]\colon \mathcal{SHC} \to \mathcal{SHC}$ maps a spectrum (E_n, σ_n) to (E_{n+1}, σ_{n+1}) where $E[1]_0 = *$ is the one point space. It turns out that the shift functor is isomorphic to the functor that suspends a spectrum level-wise. The shift functor serves as the suspension functor on the triangulated structure on $\mathcal{SHC}$. The exact triangles are the triangles isomorphic in $\mathcal{SHC}$ to the homotopy cofiber sequences.

In the categories $\mathcal{SHC}$, the derived category $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ that admits small coproducts and the stable module category $\underline{\mathrm{Mod}}(R)$ small coproducts exist and the coproduct can be used to define a finiteness condition.

Definition 2.1.6. Let $\mathcal{T}$ be a triangulated category with small coproducts. An object X in $\mathcal{T}$ is *compact*, if for each family of objects $\{Y_i\}_{i\in I}$ the canonical map

$$\bigoplus_{i\in I}\mathcal{T}(X,Y_i)\to\mathcal{T}(X,\bigoplus_{i\in I}Y_i)$$

is an isomorphism. We write $\mathcal{T}^c$ for the full subcategory of compact objects in $\mathcal{T}$.

The compact objects in $\mathcal{SHC}$ are the finite spectra. A spectrum E is *finite*, if there is an integer k such that the k-fold suspension $\Sigma^k E$ is homotopy equivalent to a suspension spectrum of a finite CW-complex (or finite simplicial set). A complex in $\mathcal{D}(R)$ is compact, if it is isomorphic, in the derived category, to a bounded complex of finitely generated projective modules. Such complexes are called *perfect*, and the full subcategory of perfect complexes is abbreviated with $\mathcal{D}^{\mathrm{per}}(R)$. Furthermore $\underline{\mathrm{Mod}}(R)^c=\underline{\mathrm{mod}}(R)$.

Now we turn to important properties of triangulated categories. The Hom-functor relates exact triangles to long exact sequences in an abelian category.

Proposition 2.1.7. *[Mar83, Appendix 2 Proposition 5] Let $X\to Y\to Z\to\Sigma X$ be an exact triangle and let A be an object. Then there is a long exact sequence of abelian groups*

$$\cdots\to\mathcal{T}(A,\Sigma^n X)\to\mathcal{T}(A,\Sigma^n Y)\to\mathcal{T}(A,\Sigma^n Z)\to\mathcal{T}(A,\Sigma^{n+1}X)\to\dots\quad .$$

Analogous to the Five-Lemma in abelian categories we have:

Proposition 2.1.8 (Five-Lemma). *[Mar83, Appendix 2 Proposition 6]*
Let $X\to Y\to Z\to\Sigma X$ and $X'\to Y'\to Z'\to\Sigma X'$ be exact triangles and let

$$\begin{tikzcd}
X \arrow[r]\arrow[d] & Y \arrow[r]\arrow[d] & Z \arrow[r]\arrow[d] & \Sigma X \arrow[d]\\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

be commutative. If two of the vertical arrows are isomorphisms, then so is the third.

As an easy consequence, we have

Corollary 2.1.9. *A map $f\colon X\to Y$ in $\mathcal{T}$ is an isomorphism, if and only if* $\mathrm{cone}(f)\cong 0$.

Proof. Let $\mathrm{cone}(f)\cong 0$. The following diagram determines a map of triangles

$$\begin{tikzcd}
\Sigma^{-1}(\mathrm{cone}(f)) \arrow[r]\arrow[d, "\cong"] & X \arrow[r, "f"]\arrow[d, "f"] & Y \arrow[r]\arrow[d, equal] & \mathrm{cone}(f) \arrow[d, "\cong"]\\
0 \arrow[r, equal] & Y \arrow[r, equal] & Y \arrow[r] & 0,
\end{tikzcd}$$

and the Five-Lemma yields that f is an isomorphism.

If $f\colon A \to B$ is an isomorphism in $\mathcal{T}$, then there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[r] & 0 \arrow[r] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "f"] & B \arrow[u, "\cong"', "f^{-1}"] \arrow[r] & \mathrm{cone}(f) \arrow[u, dotted] \arrow[r] & \Sigma A \end{tikzcd}$$

and by axiom (TR3) there is a fill-in map $\mathrm{cone}(f) \to 0$ making the diagram commute. Hence we get a map of triangles and by the Five-Lemma the fill-in map is an isomorphism. Therefore $\mathrm{cone}(f) \cong 0$. □

Definition 2.1.10. A functor $\mathcal{T} \to \mathcal{S}$ between triangulated categories is called *exact* or *triangle functor*, if it commutes with the suspensions and maps exact triangles to exact triangles. A functor $\mathcal{T} \to \mathcal{A}$ from a triangulated category to an abelian category is called *cohomological*, if it maps exact triangles to long exact sequences of abelian groups.

Proposition 2.1.7 tells us that the Hom-functor

$$\mathcal{T}^*(A, -)\colon \mathcal{T} \to \mathrm{Mod}(\mathcal{T}^*(A, A))$$

is cohomological.

Definition 2.1.11. (i) A small triangulated category $\mathcal{T}$ is *generated* by a set of objects S, if $\mathcal{T}$ is the smallest subcategory containing S that is closed under triangles, suspensions and direct summands.

(ii) A triangulated category with small coproducts $\mathcal{S}$ is *generated* by a set S' of objects, if $\mathcal{S}$ is the smallest subcategory containing S' that is closed under triangles, shifts and direct sums.

(iii) A triangulated category is *compactly generated*, if it is generated by a set of compact objects.

The sphere spectrum $\mathbb{S}$ generates $\mathcal{SHC}^c$. For a Frobenius algebra A over a commutative ring k the category $\underline{\mathrm{mod}}(A)$ is generated by k. The triangulated categories with small coproducts $\mathcal{D}(R)$, $\mathcal{SHC}$ and $\underline{\mathrm{Mod}}(A)$ are generated by the ring R, the sphere spectrum $\mathbb{S}$ and the ring k, respectively.

The following theorem characterizes the Hom-functors among the cohomological functors.

Brown Representability Theorem 2.1.12. *[Nee96, Theorem 3.1] Let $\mathcal{T}$ be a triangulated category with small coproducts which is compactly generated and let $\mathcal{A}b$ be the category of abelian groups. A functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{A}b$ is cohomological and sends small coproducts to small products, if and only if there is an object X in $\mathcal{T}$ such that F is isomorphic to* $\mathrm{Hom}_{\mathcal{T}}(-, X)$.

Corollary 2.1.13. *Let $\mathcal{T}$ be a compactly generated triangulated category with small coproducts. An exact functor $F\colon \mathcal{T} \to \mathcal{S}$ has a right adjoint, if and only if it commutes with arbitrary direct sums.*

Proof. The right adjoint is defined to map an object Y in $\mathcal{S}$ to the representing object of the cohomological functor $\mathrm{Hom}_{\mathcal{S}}(F(-), Y)\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{A}b$. □

In the following three subsections, we return to our purpose and describe subcategories and their relation to localization.

2.2 Thick subcategories and Verdier-quotients

Thick subcategories and the related Verdier-localizations are introduced in this part. Throughout this subsection let $\mathcal{T}$ be a triangulated category.

Definition 2.2.1. A full subcategory $\mathcal{C} \subset \mathcal{T}$ is called *triangulated*, if it is closed under forming triangles and suspensions in $\mathcal{T}$. If furthermore $\mathcal{C}$ is closed under retracts, it is called *thick*.

Lemma 2.2.2. *The subcategory of compact objects in a triangulated category $\mathcal{T}$ with direct sums is thick.*

Proof. The category $\mathcal{T}^c$ is obviously closed under retracts and suspensions. It is closed under forming triangles in $\mathcal{T}$ because of Proposition 2.1.7 and the Five-Lemma for abelian categories. □

Example 2.2.3. Let $F\colon \mathcal{T} \to \mathcal{S}$ be exact and let $\ker(F)$ be the full subcategory $\{X \in \mathcal{T} \mid F(X) = 0\}$ in $\mathcal{T}$. Then $\ker(F)$ is thick. Also the kernel of a cohomological functor $H\colon \mathcal{T} \to \mathcal{A}$, which is defined as $\bigcap_{n\in\mathbb{Z}} \ker(H \circ \Sigma^n)$, is thick.

Theorem 2.2.4. *[Ver96, II.2.1.8] Let $\mathcal{C} \subset \mathcal{T}$ be a triangulated subcategory. Then there is a triangulated category $\mathcal{T}/\mathcal{C}$ and an exact functor $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{C}$ annihilating $\mathcal{C}$ that satisfy:*

(i) The kernel of Q is the smallest thick subcategory containing $\mathcal{C}$.

(ii) The functor Q is universal among the exact functors annihilating $\mathcal{C}$, i.e, if $F\colon \mathcal{T} \to \mathcal{S}$ is exact such that $F(C) = 0$ for all objects $C \in \mathcal{C}$, then there is a unique exact functor G such that

$$\begin{tikzcd} \mathcal{T} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{S} \\ \mathcal{T}/\mathcal{C} \arrow[ur, dotted, "G"'] & \end{tikzcd}$$

commutes.

(iii) For every cohomological functor $H\colon \mathcal{T} \to \mathcal{A}$ annihilating $\mathcal{C}$ there is a unique cohomological functor $H'\colon \mathcal{T}/\mathcal{C} \to \mathcal{A}$ such that $H = H' \circ Q$.

The category $\mathcal{T}/\mathcal{C}$ can be constructed as a *category of fractions* $\mathcal{T}[S^{-1}]$ such that $S = \{\sigma \mid \mathrm{cone}(\sigma) \in \mathcal{C}\}$ is the class of morphisms which are inverted and the quotient functor Q is the canonical functor $\mathcal{T} \to \mathcal{T}[S^{-1}]$. We call $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{C}$ *Verdier-localization* or *quotient functor.*

2.3 Localizations

In this part localizations of triangulated categories are defined, and we clarify the relations between localization functors and localizing subcategories. As an example cohomological localization is discussed.

Let $\mathcal{T}$ be a triangulated category with small coproducts.

Definition 2.3.1. A full triangulated subcategory $\mathcal{C}$ in $\mathcal{T}$ is called *localizing*, if it is closed under direct sums.

Proposition 2.3.2. *[Nee01, 1.6.8] Every localizing subcategory in a triangulated category $\mathcal{T}$ with all small coproducts is thick.*

Let $\mathcal{C} \subset \mathcal{T}$ be localizing. Since $\mathcal{C}$ is also thick we can construct the Verdier-quotient functor $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{C}$. *A priori* the category $\mathcal{T}/\mathcal{C}$ is a *large* category, that is the morphisms between two objects do not necessarily form a set. The following lemma gives a necessary and sufficient condition for $\mathcal{T}/\mathcal{C}$ to be a category. It can be proved by using the Brown Representability Theorem 2.1.12.

Lemma 2.3.3. *[Ric00][Theorem 5.1] Let $\mathcal{C}$ be a localizing subcategory in a compactly generated triangulated category with small coproducts $\mathcal{T}$. The following statements are equivalent:*

(i) The maps between two objects in $\mathcal{T}/\mathcal{C}$ form a set.

(ii) The quotient functor $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{C}$ has a right adjoint.

So it is natural to investigate the existence of a right adjoint of the quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{C}$. The following lemmas provide the necessary background on adjoint functors to address this question.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be a pair of adjoint functors such that F is the left adjoint. Let $\psi\colon \text{id} \to G\circ F$ be the unit and $\phi\colon F\circ G \to \text{id}$ the counit of the adjunction. Let $S' = \{\sigma \in \text{Mor}(\mathcal{C}) \mid F(\sigma) \text{ is invertible}\}$. Using calculus of fractions [GZ67] it is possible to construct a functor $Q_{S'}\colon \mathcal{C} \to \mathcal{C}[(S')^{-1}]$ that is universal among the functors that invert elements of S'.

Lemma 2.3.4. *[GZ67, I.1.3] The following assertions are equivalent:*

(i) The functor $\bar{F}\colon \mathcal{C}[(S')^{-1}] \to \mathcal{D}$ with $F = \bar{F} \circ Q_{S'}$ is an equivalence.

(ii) The functor G is fully faithful.

(iii) The counit $\phi\colon F \circ G \to \text{id}_{\mathcal{D}}$ is invertible.

Now let (F, G) be an adjoint pair satisfying the conditions of Lemma 2.3.4. Let $L = G \circ F\colon \mathcal{C} \to \mathcal{C}$ and $\psi\colon \text{id}_{\mathcal{C}} \to G \circ F$ be the counit of the adjunction. The following lemma clarifies when, starting with a pair (L, ψ), we can recover the pair of adjoint functors (F, G).

Lemma 2.3.5. *[Kra06, Lemma 2.2] Let $L\colon \mathcal{C} \to \mathcal{C}$ be a functor and $\psi\colon \text{id}_{\mathcal{C}} \to L$ be a natural transformation. Then the following statement are equivalent:*

(i) $L\psi\colon L \to L^2$ is invertible and $L\psi = \psi L$.

(ii) There is an adjoint pair of functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

such that F is the left adjoint, G is fully faithful, $L = G \circ F$ and $\psi\colon \text{id}_{\mathcal{C}} \to G \circ F$ is the unit of adjunction.

Proof. We indicate how to translate the data of (i) in (ii) and vice versa. Starting with an endofunctor $L\colon \mathcal{C} \to \mathcal{C}$ and a natural transformation ψ, define $\mathcal{D}$ to be the full subcategory with objects $\{X \in \mathcal{C} \mid \psi_X : X \xrightarrow{\cong} LX\}$. Let $F\colon \mathcal{C} \to \mathcal{D}$ be given by L and $G\colon \mathcal{D} \to \mathcal{C}$ be the inclusion. Conversely, if an adjoint pair (F, G) is given, let $L = G \circ F$ and $\psi\colon \mathrm{id}_C \to G \circ F$ be the unit of the adjunction. $\square$

Let $\mathcal{C}$ be a localizing subcategory in $\mathcal{T}$ and $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{C}$ the quotient functor. The question concerning the existence of a right adjoint functor for Q can now be answered by specializing Lemma 2.3.4 and Lemma 2.3.5 to $F = Q$.

Corollary 2.3.6. *The following statements are equivalent:*

(i) The quotient functor $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{C}$ has a right adjoint R.

(ii) The quotient functor $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{C}$ has a right adjoint R which is fully faithful.

(iii) The quotient functor $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{C}$ has a right adjoint R, and the unit $\phi\colon Q \circ R \to \mathrm{id}_{\mathcal{C}}$ is invertible.

If the right adjoint R of Q exists, we set $L := R \circ Q$ and let $\psi\colon \mathrm{id}_{\mathcal{C}} \to L$ be the unit of the adjunction. Then $L\psi = \psi L$, and $L\psi\colon L \to L^2$ is invertible.

Proof. We will show that the sets $S = \{\sigma \in \mathrm{Mor}(\mathcal{T}) \mid \mathrm{cone}(\sigma) \in \mathcal{C}\}$ and $S' = \{\tau \in \mathrm{Mor}(\mathcal{T}) \mid Q(\tau) \text{ is invertible}\}$ are the same. Then $Q = Q_{S'}$ and the equivalence of the three assertions follows from Lemma 2.3.4. The last assertion is a consequence of Lemma 2.3.5.

The inclusion $S \subset S'$ holds by definition. For the other inclusion, let $\tau\colon X \to Y$ such that $Q(\tau)$ is invertible. By Corollary 2.1.9 $\mathrm{cone}(Q\tau) \cong 0$. Since Q is an exact functor, it follows that $\mathrm{cone}(Q\tau) \cong Q(\mathrm{cone}(\tau))$. Therefore $\mathrm{cone}(\tau)$ is in the kernel of Q which is equal to $\mathcal{C}$. $\square$

This corollary motivates

Definition 2.3.7. Let $\mathcal{T}$ be a triangulated category with small coproducts. Let in addition (L, ψ) be a pair consisting of an exact endofunctor $L\colon \mathcal{T} \to \mathcal{T}$ and a natural transformation $\psi\colon \mathrm{id}_{\mathcal{T}} \to L$. The pair (L, ψ) is called *localization functor* or just *localization*, if $L\psi\colon L \to L^2$ is invertible, the natural transformation ψ commutes with the suspension functor and $L\psi = \psi L$.

Sometimes we suppress the natural transformation ψ of a localization functor in our notation. Two important classes of objects are associated with a localization.

Definition 2.3.8. Let $L\colon \mathcal{T} \to \mathcal{T}$ be a localization. An object $X \in \mathcal{T}$ is called *L-local*, if the localization morphism $\psi_X : X \to LX$ is an isomorphism. The object X is called *L-acyclic*, if $LX = 0$. The full subcategory of L-local objects is denoted by $\mathcal{T}_L$ and $\ker(L)$ is the full subcategory of L-acyclics.

Since the category $\ker(L)$ is localizing a localization functor determines a localizing subcategory.

The following four assertions deal with the properties of localizations of triangulated categories. They are well-known and were studied for instance in [HPS97]. We refer to [Kra06] since we need slightly more general statements.

Lemma 2.3.9. *[Kra06, Lemma 2.5] Let X be an object in $\mathcal{T}$ and L a localization. Then the following assertions are equivalent:*

(i) X is L-local.

(ii) There is an object X' in $\mathcal{T}$ such that $X \cong LX'$.

(iii) For all $f\colon Y \to Z$ with the property that $L(f)$ is an isomorphism, the map

$$f^*\colon \operatorname{Hom}_{\mathcal{T}}(Z, X) \to \operatorname{Hom}_{\mathcal{T}}(Y, X)$$

is an isomorphism of abelian groups.

Lemma 2.3.10. *[Kra06, Proposition 2.7] The inclusion functor $\mathcal{T}_L \to \mathcal{T}$ is right adjoint to the functor $\mathcal{T} \to \mathcal{T}_L$ sending an object X to LX.*

Therefore the localization L is determined by the category of L-local objects.

The L-acyclic and L-local objects are related by the following proposition.

Proposition 2.3.11. *[Kra06, Lemma 2.8]*

(i) An object $X \in \mathcal{T}$ is L-acyclic, if and only if $\operatorname{Hom}_{\mathcal{T}}(X, Y) = 0$ for all L-local objects Y.

(ii) The functor L induces an equivalence of triangulated categories

$$\mathcal{T}/\ker(L) \xrightarrow{\simeq} \mathcal{T}_L.$$

The L-acyclic objects determine the L-locals up to equivalence. Therefore the localization functor L is determined by $\ker(L)$.

The converse is not known in general. If $\mathcal{C} \subset \mathcal{T}$ is a localizing subcategory, then it is not clear whether a localization functor $L\colon \mathcal{T} \to \mathcal{T}$ with $\mathcal{C} = \ker(L)$ exists. Nevertheless it is possible to make some assertions. Casacuberta, Gutiérrez and Rosický [CGR06] have shown that such a functor exists by adding Vopěnka's Principle to the axioms of set theory. Miller gave a construction of a localization functor on the stable homotopy category assuming that the given localizing subcategory is generated by objects which are compact in $\mathcal{SHC}$ [Mil92]. Such localizations are called *finite* and will be studied in the next section.

In the remaining part of this section we introduce localization functors that are induced by localizations in the homology as an important example.

Let $\mathcal{T}$ be a triangulated category with small coproducts which is generated by a compact object A and let $\Gamma := \mathcal{T}^*(A, A)$ be the graded endomorphism ring. Denote by $\operatorname{Mod}_{\mathrm{gr}}(\Gamma)$ the category of graded Γ-modules and let $H^*\colon \mathcal{T} \to \operatorname{Mod}_{\mathrm{gr}}(\Lambda)$ be the cohomological functor $\mathcal{T}^*(C, -)$. The functor H^* relates the triangulated category $\mathcal{T}$ to the abelian category $\operatorname{Mod}_{\mathrm{gr}}(\Gamma)$.

Definition 2.3.12. Let $\mathcal{A}$ be an abelian category. A pair (L, ψ) consisting of an exact endofunctor $L\colon \mathcal{A} \to \mathcal{A}$ and a natural transformation $\psi\colon \mathrm{id}_{\mathcal{A}} \to L$ is called *localization functor*, if $L\psi = \psi L$, the natural transformation ψ commutes with the suspension functor and $L\psi\colon L \to L^2$ is an isomorphism.

Theorem 2.3.13. *[Kra06, Theorem 3.1] Let (L,ψ) be a localization functor on* $\mathrm{Mod}_{\mathrm{gr}}(\Gamma)$*. Then there is a localization $(\tilde{L},\tilde{\psi})$ on $\mathcal{T}$ such that*

$$\begin{tikzcd} \mathcal{T} \arrow[r, dotted, "\tilde{L}"] \arrow[d, "H^*"] & \mathcal{T} \arrow[d, "H^*"] \\ \mathrm{Mod}_{\mathrm{gr}}(\Gamma) \arrow[r, "L"] & \mathrm{Mod}_{\mathrm{gr}}(\Gamma) \end{tikzcd}$$

commutes up to natural isomorphism. More precisely, $LH^*\psi$, $\psi H^*\tilde{L}$ and

$$LH^* \xrightarrow{LH^*\tilde{\psi}} LH^*\tilde{L} \xrightarrow{(\psi H^*\tilde{L})^{-1}} H^*\tilde{L}$$

are invertible. Furthermore $\tilde{L}X = 0$, if and only if $LH^*X = 0$. If X is $\tilde{L}$-local, then H^*X is L-local, and if H^* reflects isomorphisms, then the converse also holds.

Remark 2.3.14. [Kra06, Rem. 2.4] Let $L\colon \mathrm{Mod}_{\mathrm{gr}}\Gamma \to \mathrm{Mod}_{\mathrm{gr}}\Gamma$ be an exact localization functor and denote by $\hat{L}\colon \mathcal{T} \to \mathcal{T}$ the exact localization functor which exists by Theorem 2.3.13. Write $\mathcal{C}$ for the $\hat{L}$-acyclic objects. By Lemma 2.3.5 $\hat{L}$ and L give rise to adjoint pairs of functors

$$\begin{tikzcd} \mathcal{T} \arrow[r, shift right, "Q"'] & \mathcal{T}/\mathcal{C} \arrow[l, shift right, "R"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathrm{Mod}_{\mathrm{gr}}\Gamma \arrow[r, shift right, "F"'] & (\mathrm{Mod}_{\mathrm{gr}}\Gamma)_L \arrow[l, shift right, "G"'] \end{tikzcd}$$

satisfying $\hat{L} = R \circ Q$ and $L = G \circ F$. The diagram below commutes up to natural isomorphism.

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\mathcal{T}(A,-)^*"] \arrow[d, "Q"'] & \mathrm{Mod}_{\mathrm{gr}}\Gamma \arrow[d, "F"] \\ \mathcal{T}/\mathcal{C} \arrow[r, "\mathcal{T}/\mathcal{C}(QA,-)^*"] \arrow[d, "R"'] & (\mathrm{Mod}_{\mathrm{gr}}\Gamma)_L \arrow[d, "G"] \\ \mathcal{T} \arrow[r, "\mathcal{T}(A,-)^*"] & \mathrm{Mod}_{\mathrm{gr}}\Gamma \end{tikzcd}$$

The following two results are joint work with Birgit Huber [BH07].

Proposition 2.3.15. *Suppose that the ring $\mathcal{T}(A,A)^*$ is graded-commutative and let $L\colon \mathrm{Mod}_{\mathrm{gr}}\mathcal{T}(A,A)^* \to \mathrm{Mod}_{\mathrm{gr}}\mathcal{T}(A,A)^*$ be a localization with respect to a multiplicatively closed subset of homogeneous elements $S \subseteq \mathcal{T}(A,A)^*$. If $\mathcal{C} = \ker \hat{L}$, then the diagram*

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\mathcal{T}(A,-)^*"] \arrow[d, "Q"'] & \mathrm{Mod}_{\mathrm{gr}}\mathcal{T}(A,A)^* \arrow[d, "\mathrm{can}"] \\ \mathcal{T}/\mathcal{C} \arrow[r, "\mathcal{T}/\mathcal{C}(QA,-)^*"] & \mathrm{Mod}_{\mathrm{gr}}S^{-1}\mathcal{T}(A,A)^* \end{tikzcd}$$

commutes up to natural isomorphism. Furthermore $\mathcal{T}/\mathcal{C}(QA,QA)^*$ and $S^{-1}\mathcal{T}(A,A)^*$ are isomorphic not only as graded $\mathcal{T}(A,A)^*$-modules, but also as graded rings.

Proof. The diagram commutes by Remark 2.3.14. Writing again H^* for $\mathcal{T}(A,-)^*$ the naturality of $H^*\Psi$ yields a commutative square

$$\begin{tikzcd} H^*A \arrow[r, "\Psi H^*A"] \arrow[d, "H^*\hat{\Psi}A"'] & LH^*A \arrow[d, "\cong"', "LH^*\hat{\Psi}A"] \\ H^*\hat{L}A \arrow[r, "\Psi H^*\hat{L}A", "\cong"'] & LH^*\hat{L}A \end{tikzcd}$$

in which the lower and the right hand side morphism are bijective by Theorem 2.3.13. Now note that $H^*\hat{\Psi}A$ is up to isomorphism given by the canonical map

$$Q\colon \mathcal{T}(A,A)^* \to \mathcal{T}/\mathcal{C}(QA,QA)^*, \quad f \mapsto Qf,$$

and that ΨH^*A equals up to isomorphism the canonical ring homomorphism

$$\mathrm{can}\colon \mathcal{T}(A,A)^* \to S^{-1}\mathcal{T}(A,A)^*.$$

Since $Q\colon \mathcal{T}/(A,A)^* \to \mathcal{T}/\mathcal{C}(QA,QA)^*$ is a multiplicative map inverting all elements in S, we obtain a ring homomorphism $r\colon S^{-1}\mathcal{T}(A,A)^* \to \mathcal{T}/\mathcal{C}(QA,QA)^*$ which makes the upper triangle in the modified diagram

$$\begin{tikzcd} \mathcal{T}(A,A)^* \arrow[r, "\mathrm{can}"] \arrow[d, "Q"'] & S^{-1}\mathcal{T}(A,A)^* \arrow[d, "\cong"', "S^{-1}Q"] \arrow[dl, dotted, "r"'] \\ \mathcal{T}/\mathcal{C}(QA,QA)^* \arrow[r, "\nu", "\cong"'] & S^{-1}\mathcal{T}/\mathcal{C}(QA,QA)^* \end{tikzcd}$$

commute.

The lower triangle commutes by the following argument. Since both the maps $\nu \circ r$ and $S^{-1}Q$ make the following diagram of $\mathcal{T}(A,A)^*$-modules

$$\begin{tikzcd} \mathcal{T}(A,A)^* \arrow[r, "\mathrm{can}"] \arrow[d, "\nu\circ Q"'] & S^{-1}\mathcal{T}(A,A)^* \arrow[dl, shift right, "\nu\circ r"'] \arrow[dl, shift left, "S^{-1}Q"] \\ S^{-1}\mathcal{T}/\mathcal{C}(QA,QA)^* & \end{tikzcd}$$

commute, the universal property of localization of modules implies that $\nu \circ r = S^{-1}Q$. Hence r is an isomorphism. ☐

Proposition 2.3.16. *Suppose that the ring $\mathcal{T}(A,A)^*$ is graded-commutative and let $L\colon \mathrm{Mod}_{\mathrm{gr}}\,\mathcal{T}(A,A)^* \to \mathrm{Mod}_{\mathrm{gr}}\,\mathcal{T}^*(A,A)^*$ be localization with respect to a multiplicatively closed subset of homogeneous elements $S \subseteq \mathcal{T}(A,A)^*$. If the compact object $A \in \mathcal{T}$ is a generator, then the category $\mathcal{C} = \ker \hat{L}$ is generated by compact objects of $\mathcal{T}$.*

Proof. We show that $\mathcal{C}$ is generated by $\{\mathrm{cone}(\sigma) \mid \sigma\colon A \to A[n] \in S,\, n \in \mathbb{Z}\}$. On this purpose consider an arbitrary object $M \in \mathcal{C}$ and prove that $\mathcal{T}(\mathrm{cone}(\sigma), M)^* = 0$ for all $\sigma \in S$ implies $M = 0$.

Every triangle

$$A \xrightarrow{\sigma} A[n] \to \mathrm{cone}(\sigma) \to A[1]$$

gives rise to an exact sequence

$$\mathcal{T}(\mathrm{cone}(\sigma), M)^* \to \mathcal{T}(A[n], M)^* \xrightarrow{\mathcal{T}(\sigma,M)^*} \mathcal{T}(A, M)^* \to \mathcal{T}(\mathrm{cone}(\sigma)[-1], M)^*.$$

By assumption we have $\mathcal{T}(\mathrm{cone}(\sigma)[-1], M)^* = 0 = \mathcal{T}(\mathrm{cone}(\sigma), M)^*$. Hence the map

$$\mathcal{T}(A[n], M)^* \xrightarrow{\mathcal{T}(\sigma,M)^*} \mathcal{T}(A, M)^*$$

is an isomorphism for all $\sigma \in S$ and thus, $\mathcal{T}(A, M)^*$ is L-local. On the other hand, $\mathcal{T}(A, M)^*$ is L-acyclic. It follows that $\mathcal{T}(A, M)^* = 0$ and hence $M = 0$. □

2.4 Smashing and finite localizations

In this part two classes of localizations are introduced: the smashing and the finite localizations. We investigate their relations which lead to the Smashing Conjecture.

Let $\mathcal{T}$ be a triangulated category with small coproducts.

Definition 2.4.1. A localization $L\colon \mathcal{T} \to \mathcal{T}$ is called *smashing*, if L commutes with small coproducts.

Let (Q, R) be the adjoint pair corresponding to L according to Lemma 2.3.5. Then L is smashing, if and only if R commutes with direct sums. Therefore the following definition is sensible:

Definition 2.4.2. A localizing subcategory $\mathcal{C} \subset \mathcal{T}$ is called *smashing*, if the quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{C}$ has a right adjoint that commutes with direct sums.

In the following we describe the origin of the name “smashing”: let $\mathcal{T}$ be equipped with a symmetric monoidal product compatible with the triangulation, i.e., a pair $(- \wedge -, S)$ such that $-\wedge-\colon \mathcal{T}\times\mathcal{T} \to \mathcal{T}$ is a functor that is exact in both variables and that commutes with direct sums. The unit S is asked to be a compact generator of $\mathcal{T}$. An example of such a category is the stable homotopy category together with the smash product $- \wedge -$ and the sphere spectrum $\mathbb{S}$ as the unit [Ada74, EKMM97, HSS00]. The derived category $\mathcal{D}(R)$ of a commutative ring R together with the derived tensor product over R and the ring R, considered as a complex, that is concentrated in degree 0 is an algebraic example.

Lemma 2.4.3. *Let $\mathcal{C}$ be localizing in $\mathcal{T}$. If C is an object in $\mathcal{C}$, then for all objects $X \in \mathcal{T}$ the product $C \wedge X$ is in $\mathcal{C}$.*

Proof. Consider the full subcategory $\mathcal{C}_C := \{X \in \mathcal{T} \mid C \wedge X \in \mathcal{C}\}$ in $\mathcal{T}$. The unit S is in $\mathcal{C}_C$. Since the smash product is exact and preserves direct sums $\mathcal{C}_C$ is also closed under exact triangles and arbitrary direct sums. Therefore, $\mathcal{C}_C \subset \mathcal{T}$ is localizing and contains S. Since S is a compact generator, $\mathcal{C}_C = \mathcal{T}$. □

If (L, ψ) is a localization on $\mathcal{T}$ and X is an object in $\mathcal{T}$, then there is a canonical map

$$\alpha_X\colon LS \wedge X \to LX.$$

To define it, first consider the exact triangle $CS \to S \xrightarrow{\psi_S} LS \to \Sigma CS$. Smashing with X yields the exact triangle

$$CS \wedge X \to S \wedge X \to LS \wedge X \to \Sigma CS \wedge X.$$

Now $CS \wedge X$ is L-acyclic by Lemma 2.4.3 and by Corollary 2.1.9 it follows that $L(\psi_S \wedge X)\colon LX \to L(LS \wedge X)$ is an isomorphism in $\mathcal{T}$. We define α_X to be the composition

$$LS \wedge X \xrightarrow{\psi_{LS\wedge X}} L(LS \wedge X) \xrightarrow{(L(\psi_S \wedge X))^{-1}} LX.$$

Now we are able to characterize smashing localizations:

Proposition 2.4.4. *[HPS97, 3.3.2] The following assertions are equivalent:*

(i) The functor $L\colon \mathcal{T} \to \mathcal{T}$ is a smashing localization.

(ii) The natural map $\alpha_X\colon LS \wedge X \to LX$ is an isomorphism.

(iii) The category $\mathcal{T}_L$ of L-local objects is localizing.

The name "smashing" originates from assertion (ii). For a smashing localization functor L *applying L is the same as smashing with LS.*

We give a class of examples of smashing localizations.

Proposition 2.4.5. *Let $L\colon \mathcal{T} \to \mathcal{T}$ be a localization functor. If the L-acyclics* $\ker(L)$ *are generated by a set of objects $\{C_i\}_{i\in I}$ which are compact in $\mathcal{T}$, then the localization L is smashing.*

Proof. Let $\mathcal{C} := \ker(L)$ be the category of L-acyclics, and let $\{X_\alpha \mid \alpha \in A\}$ be a family of objects in $\mathcal{T}$. We first show that $\bigoplus_{\alpha\in A} LX_\alpha$ is L-local. Since $\{C_i\}_{i\in I}$ is a set of compact generators of $\mathcal{C}$ by Proposition 2.3.11 it is enough to show that $\mathcal{T}(C_i, \bigoplus_{\alpha\in A} LX_\alpha) = 0$ for all $i \in I$. Since each C_i is compact there is an isomorphism

$$\mathcal{T}(C_i, \bigoplus_{\alpha\in A} LX_\alpha) \cong \bigoplus_{\alpha\in A} \mathcal{T}(C_i, LX_\alpha).$$

The abelian group $\mathcal{T}(C_i, LX_\alpha) = 0$ for all $i \in I$ because C_i is L-acyclic and LX_α is L-local. Hence $\bigoplus_\alpha LX_\alpha$ is L-local. Therefore the map

$$\psi_{\bigoplus_{i\in I} LX_i}\colon \bigoplus_{i\in I} LX_i \to L(\bigoplus_{i\in I} LX_i)$$

is an isomorphism.

To end the proof we show that the map

$$L(\bigoplus_{i\in I} \psi_{X_i})\colon L(\bigoplus_{i\in I} X_i) \to L(\bigoplus_{i\in I} LX_i)$$

is an isomorphism. For each $i \in I$ there is an exact triangle

$$CX_i \to X_i \to LX_i \to \Sigma CX_i$$

such that $CX_i \in \mathcal{C}$. Since coproducts are exact, the following triangle

$$\bigoplus_{i\in I} CX_i \to \bigoplus_{i\in I} X_i \to \bigoplus_{i\in I} LX_i \to \bigoplus_{i\in I} \Sigma CX_i$$

is exact. The object $\bigoplus_I CX_i$ is L-acyclic because $\mathcal{C}$ is localizing. As L annihilates L-acyclic objects it follows that $L(\bigoplus_{i\in I}\psi_{X_i})$ is an isomorphism. Hence the composition

$$(\psi_{\bigoplus_{i\in I} LX_i})^{-1}\circ L(\bigoplus_{i\in I}\psi_{X_i})\colon L(\bigoplus_{i\in I}X_i)\to\bigoplus_{i\in I}LX_i$$

is an isomorphism and L commutes with direct sums. Therefore L is smashing. $\square$

Definition 2.4.6. A localization $L\colon\mathcal{T}\to\mathcal{T}$ is called a *finite localization*, if the category of L-acyclic objects is generated as a triangulated category with small coproducts by a set of objects that are compact in $\mathcal{T}$. A localizing subcategory $\mathcal{C}\subset\mathcal{T}$ is said to be of *finite type*, if $\mathcal{C}=\ker(L)$ for a finite localization functor L.

As we have seen in Proposition 2.4.5 every finite localization is smashing. We state the converse in the following conjecture which was formulated for the first time by Neeman [Nee92].

Smashing Conjecture 2.4.7. In a triangulated category with small coproducts, every smashing localization is finite.

We consider the Smashing Conjecture as an assertion on a fixed triangulated category with small coproducts $\mathcal{T}$ rather than a statement about *all* triangulated categories with small coproducts. Therefore we use the terminology "the Smashing Conjecture for $\mathcal{T}$" in the sequel. The Smashing Conjecture is sometimes called Telescope Conjecture due to its origin. We choose this name to avoid confusion.

In the next chapter we discuss the origin, examples, results and applications of the Smashing Conjecture.

3 The Telescope Conjecture and the Smashing Conjecture

This section gives an overview on the history, results and applications of the Smashing Conjecture 2.4.7. The aim of Paragraph 3.1 is to show that the Smashing Conjecture is a generalization of Ravenel's Telescope Conjecture of stable homotopy theory which was the starting point of the investigation. In Paragraph 3.2 and 3.3 an example of a triangulated category for which the Smashing Conjecture is valid and an example for which it does not hold are introduced. In 3.4 Krause and Solberg's reformulation of the Smashing Conjecture for stable module categories and a result by Angeleri-Hügel, Šaroch and Trlifaj are stated. At the end we describe general results on the Smashing Conjecture in Paragraph 3.5 and point out applications to chain lifting problems, non-commutative localization of rings and algebraic K-theory. During this section we give no proofs and do not go into the details in order to present the topic streamlined and compact.

3.1 The Telescope Conjecture in stable homotopy theory

We recall basic notions of stable homotopy theory, in particular Bousfield localization, the p-local stable homotopy category and two examples of spectra. Having these it is possible to describe the Periodicity Theorem, define the mapping telescope and state the Telescope Conjecture in its very first version. In addition we indicate a motivation for the telescope conjecture.

Recall from the previous section that the stable homotopy category $\mathcal{SHC}$ is a triangulated category with small coproducts that is compactly generated by the sphere spectrum $\mathbb{S}$. Furthermore it is symmetric monoidal by means of the smash product $- \wedge -$ and the unit $\mathbb{S}$.

Spectra are strongly related to generalized cohomology theories. Given a generalized cohomology theory

$$E^*\colon \mathcal{SHC}^{\mathrm{op}} \to (\text{graded abelian groups})$$

by the Brown Representability Theorem 2.1.12 there is a spectrum E such that $E^n(-) \cong \mathrm{Hom}_{\mathcal{SHC}}(-, \Sigma^n E)$.

On the other hand a spectrum E gives rise to a generalized homology theory

$$E_*\colon \mathcal{SHC} \to (\text{graded abelian groups})$$

by setting $E_n(F) := \pi_n(E \wedge F)$. For example, the *Moore spectrum* $M\mathbb{Z}_{(p)}$, which is defined as the cone of $\mathbb{S} \xrightarrow{p\cdot \mathrm{id}} \mathbb{S}$, is the spectrum representing the generalized homology theory $\pi_*(-) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Bousfield [Bou79] showed that for a spectrum E there is a localization functor $L_E\colon \mathcal{SHC} \to \mathcal{SHC}$ such that the L_E-acyclic objects are the E_*-acyclics. That is, $L_E(F) = 0$, if and only if $E_*(F) = 0$ for any spectrum F. It is obtained as the fibrant replacement in a model structure on the category of spectra in which the weak equivalences are the isomorphisms in the represented homology theory E_*. We will call L_E sometimes *E_*-localization*.

For a prime p the *p-local stable homotopy category $\mathcal{SHC}_p$* is defined to be the category of $L_{M\mathbb{Z}_{(p)}}$-local objects.

A monoid R in $\mathcal{SHC}$ with respect to the smash product is called *ring spectrum*. The multiplication on R

$$R \wedge R \to R$$

induces a ring structure on $\pi_* R$. So a ring spectrum encodes a "blueprint" for a discrete ring. The ring of homotopy groups $\pi_*(R)$ is often called *coefficients* since $R^*(\mathbb{S}) \cong \pi_*(R)$. It plays an important role for the represented cohomology theory R^*.

One typical example for a ring spectrum arises from classical algebra. Given an associative ring with unit, there is a ring spectrum HR, the *Eilenberg-Mac Lane spectrum*, with the property that $\pi_0 HR \cong R$. It represents singular (co-)homology with coefficients in R. Another example of a ring spectrum is the sphere spectrum $\mathbb{S}$. The ring structure $\mathbb{S} \wedge \mathbb{S} \to \mathbb{S}$ is the canonical isomorphism which exists since $\mathbb{S}$ is the unit of the monoidal structure. This spectrum represents the homology theory given by the stable homotopy groups $\pi_*(-)$.

From now on we fix a prime $p \in \mathbb{Z}$ and work in the p-local stable homotopy category. We discuss two examples of spectra that play an important role in stable homotopy theory, the Johnson-Wilson spectra $E(n)$ and the Morava K-theories $K(n)$ for $n \geq 0$. Both are p-local spectra, and we follow the usual convention of suppressing the prime p in the notation. We describe important properties of these spectra and their ring of stable homotopy groups.

The Morava K-theory for $n = 0$ is $K(0) = H\mathbb{Q}$. For $n = \infty$ we define $K(\infty)$ to be the Eilenberg-Mac Lane spectrum $H\mathbb{Z}_{(p)}$. The spectrum $K(n)$ for $n \neq 0$ has the coefficients

$$\pi_*(K(n)) = \mathbb{F}_p[v_n, v_n^{-1}],$$

where v_n is in degree $2p^n - 2$. The Morava K-theories have the following remarkable properties:

Proposition 3.1.1. *[HS98, Propositions 1.4, 1.5, 1.9]*

(i) *$K(n)$ is a ring spectrum and a skew field object, i.e., all modules over a $K(n)$ are free.*

(ii) *There is a Küneth-isomorphism*

$$K(n)_*(X \wedge Y) \cong K(n)_*(X) \otimes_{K(n)_*(pt)} K(n)_*(Y)$$

for two p-local spectra X and Y.

(iii) *A skew field object in $\mathcal{SHC}_p$ is the direct sum of shifted copies of Morava K-theories.*

The Morava K-theories are constructed from the cobordism spectrum MU by localizing and taking quotients. Similarly the *Johnson-Wilson spectrum* $E(n)$ for $n > 0$ can be defined. Its homotopy is the following ring: $\pi_* E(n) = \mathbb{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$. See [HS98], [EKMM97, V.4] and [Wei05] for a construction.

The Johnson-Wilson spectra and the Morava K-theories are also related via their localization functors. In fact, $E(n)_*$-localization is isomorphic to $(\bigoplus_{i=1}^n K(i))_*$-localization. Let L_n denote the $E(n)_*$-localization. The functor L_n^f is the *finite localization functor associated with* $E(n)$. It can be constructed as finite localization in the sense of Miller [Mil92] with respect to the set $\{X \in \mathcal{SHC}_p \text{ finite} \,|\, E(n)_*(X) = 0\}$.

Periodicity and the mapping telescope

There are very readable papers [Rav92b, Rav93] and the book [Rav92a] by Ravenel which can serve as introduction to this subject. In order to formulate the Telescope Conjecture we need a result by Hopkins, Devinatz and Smith [DHS88] and some notation.

Again we fix a prime p.

Definition 3.1.2. A finite p-local spectrum E is called *of type* n, if n is the smallest integer such that $K(n)_*(E) \neq 0$. A map $\Sigma^d E \to E$ is called v_n*-map*, if it induces an isomorphism in the n-th Morava K-theory and the zero-map in all other Morava K-theories.

The mod-n Moore spectrum is of type n. Examples of v_n-maps arise as self maps of Moore spectra. Recall that $\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}]$. A v_n-map induces the multiplication by some power of v_n in the n-th Morava K-theory, whence its name.

Let $i \geq 0$. Then f^i denotes the iteration of i copies of a map $f\colon X \to X$.

Theorem 3.1.3 (Periodicity Theorem). *[HS98, Theorem 9] Let E be a spectrum of type n. Then there is a non null-homotopic v_n-map $\Sigma^d E \to E$.*

Furthermore given two spectra E_1 and E_2 with v_n-maps f_1 and f_2 then for every map $g\colon E_1 \to E_2$ there is a commutative square in $\mathcal{SHC}$:

$$\begin{tikzcd} \Sigma^{id_1} E_1 \arrow[r, "\Sigma^{id_1} g"] \arrow[d, "f_1^i"] & \Sigma^{jd_2} E_2 \arrow[d, "f_2^j"] \\ E_1 \arrow[r, "g"] & E_2. \end{tikzcd}$$

The second part of the Periodicity Theorem implies that a v_n-map is unique in the sense that some iterations of two v_n-maps are homotopic.

Definition 3.1.4. Let E be a finite p-local spectrum of type n. Then the *mapping telescope* $\hat{E}$ is defined to be the homotopy colimit of the iteration of the v_n-map $\Sigma^{-d} f\colon E \to \Sigma^{-d} E$

$$E \xrightarrow{\Sigma^{-d} f} \Sigma^{-d} E \xrightarrow{\Sigma^{-2d} f} \Sigma^{-2d} E \to \dots.$$

The mapping telescope is well-defined because two choices of a v_n-map are homotopic up to respective iterates.

The Telescope Conjecture and some applications

We state the telescope conjecture of Ravenel in four equivalent formulations and discuss its relevance.

Recall that two spectra E and F are *Bousfield equivalent*, if for all spectra X the equation $E_* X = 0$ holds, if and only if $F_* X = 0$, i.e., the E_*-localizations and F_*-localization coincide. The *Bousfield class* $\langle E \rangle$ of a spectrum E is the equivalence class of E with respect to this equivalence relation.

Telescope Conjecture 3.1.5. *[Rav87b, 1.33] Let E be a p-local spectrum of type n. Then $\langle \hat{E} \rangle = \langle K(n) \rangle$. That is, localization with respect to the mapping telescope of a p-local spectrum of type n is the same as $K(n)_*$-localization for the Morava K-theory $K(n)$.*

The Telescope Conjecture describes the behavior of localization with respect to the *mapping telescope* of a spectrum. This is the origin of its name.

Let $L_n \colon \mathcal{SHC}_p \to \mathcal{SHC}_p$ be the localization functor associated with the Johnson-Wilson spectrum $E(n)$. Let $X \in \mathcal{SHC}_p$ be of type n. The canonical map $X \to L_n X$ factors through the telescope $X \to \hat{X} \xrightarrow{\lambda} L_n X$.

Proposition 3.1.6. *[MRS01, 1.13] The Telescope Conjecture is equivalent to the statement that for all $X \in \mathcal{SHC}_p$ the map $\lambda \colon \hat{X} \to L_n X$ is an isomorphism in the stable homotopy category.*

A major problem in algebraic topology is the computation of stable homotopy groups of spaces and in particular the stable homotopy groups of spheres. For example there is not a single non-contractible finite CW-complex for which the stable homotopy groups are entirely known. A motivation for Ravenel to state the Telescope Conjecture was its consequences on computation: One device to organize a computation is the Adams-Novikov spectral sequence (ANSS). It exists for every space X and may or may not converge to the stable homotopy groups of this space. It is possible to compute the E_2-term of this spectral sequence for the mapping telescope of a type n spectrum [Dav95, Rav86]. The ANSS collapses, but it is not clear, if the spectral sequence converges. It does always converge for the localized spectra $L_n X$ [Rav87a]. So the Telescope Conjecture would imply that the ANSS of $\hat{X}$ converges to $\pi_*^s \hat{X}$ and would yield a possibility to compute some stable homotopy groups.

Proposition 3.1.7. *[MRS01, 1.13] Let X be a spectrum of type n. The Telescope Conjecture is equivalent to the assertion that the ANSS for $\hat{X}$ converges to $\pi_* \hat{X}$.*

The Telescope Conjecture was verified for $n = 1$ and $p = 2$ by Mahowald [Mah82] and for $n = 1$ at all odd primes by Miller [Mil81]. Ravenel announced a counterexample for $n = 2$ [Rav92b] but withdrew it later on [MRS01]. Despite the lack of counterexamples, the Telescope Conjecture is commonly not expected to hold for all n and p.

In order to relate the Telescope Conjecture 3.1.5 to the Smashing Conjecture 2.4.7 we discuss some properties of the Johnson-Wilson spectra. Let $E(n)$ be the Johnson-Wilson spectrum and L_n^f the finite localization with respect to $E(n)$. It is known by [Rav92a, Theorem 7.5.6] that the functor $L_n \colon \mathcal{SHC} \to \mathcal{SHC}$ is smashing in the sense of Definition 2.4.1. In [MRS01, 1.13] there is a reformulation of the Telescope Conjecture 3.1.5 in terms of these localizations:

Proposition 3.1.8. *[MRS01, 1.13] The Telescope Conjecture is equivalent to the statement that $L_n = L_n^f$, if $L_{n-1} = L_{n-1}^f$.*

In particular if $L_n = L_n^f$ for all n, then the Telescope Conjecture is true for all n. So the Smashing Conjecture 2.4.7 for the p-local stable homotopy category implies the Telescope Conjecture 3.1.5.

3.2 The Smashing Conjecture in commutative algebra

The first triangulated category for which the Smashing Conjecture 2.4.7 was verified is the derived category of a commutative noetherian ring R [Nee92, Hop87]. Neeman classified the thick subcategories in $\mathcal{D}^{\mathrm{per}}(R)$ as well as the localizing and smashing subcategories in $\mathcal{D}(R)$ to deduce the Smashing Conjecture. See [ATJLSS04] for a generalization to schemes.

Let R be a commutative noetherian ring and $\mathfrak{p} \in \operatorname{Spec} R$ be a prime ideal. Define $\kappa(\mathfrak{p})$ to be the residue field, i.e., the quotient of $R_\mathfrak{p}$ by its maximal ideal $\mathfrak{p}R_\mathfrak{p}$. For a complex X let $\operatorname{Supp}(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid X_\mathfrak{p} \neq 0\}$ denote the *support* of X.

Theorem 3.2.1. *[Nee92, Theorem 1.5, Theorem 3.3]*

(i) The following maps are mutually inverse to each other

$$\{\mathcal{C} \subset \mathcal{D}^{\mathrm{per}}(R) \mid \mathcal{C} \text{ thick}\} \underset{g}{\overset{f}{\rightleftarrows}} \{P \subset \operatorname{Spec}(R) \mid P \text{ closed under specialization}\}$$

where $f(\mathcal{C}) = \{\mathfrak{p} \mid \exists X \in \mathcal{C} : \mathfrak{p} \in \operatorname{Supp}(X)\}$ *and* $g(P)$ *is the full subcategory given by* $\{X \in \mathcal{D}^{\mathrm{per}}(R) \mid \operatorname{Supp}(X) \subset P\}$.

(ii) There are mutually inverse bijections

$$\{\mathcal{C} \subset \mathcal{D}(R) \mid \mathcal{C} \text{ smashing}\} \underset{g}{\overset{f}{\rightleftarrows}} \{P \subset \operatorname{Spec}(R) \mid P \text{ closed under specialization}\}$$

where $f(\mathcal{C}) = \{\mathfrak{p} \mid X \otimes k(\mathfrak{p}) \neq 0 \,\forall\, X \in \mathcal{C}\}$ *and* $g(P)$ *is the smallest localizing subcategory in* $\mathcal{D}(R)$ *that contains* $k(\mathfrak{p})$ *for all* $\mathfrak{p} \in P$.

Corollary 3.2.2. *[Nee92, Corollary 3.4] The Smashing Conjecture 2.4.7 is true for the derived category of a commutative noetherian ring.*

3.3 Keller's counterexample

From the perspective of stable homotopy theory commutative *non*-noetherian rings are more interesting because the ring $\pi_*(\mathbb{S})$ of stable homotopy groups of spheres is not noetherian. So it is natural to ask, if the Smashing Conjecture 2.4.7 remains true for the derived category of an arbitrary commutative ring R. Keller gave an example [Kel94b] of a non-noetherian commutative ring for which the Smashing Conjecture is not true.

Let k be a field and $l \geq 2$ be an integer. Define $B := k[t, t^{l^{-1}}, t^{l^{-2}}, \dots]$. Its augmentation ideal J is generated by $t, t^{l^{-1}}, t^{l^{-2}}, \dots$. Let A be the localization of B at the ideal J and I be the Jacobson radical of A.

Theorem 3.3.1. *[Kel94b] The localizing subcategory* $\mathcal{R} \subset \mathcal{D}(A)$ *generated by the ideal* I *is smashing and contains no compact object of* $\mathcal{D}(A)$*. Hence the Smashing Conjecture 2.4.7 for* $\mathcal{D}(A)$ *is* not *true.*

The key property of the pair (A, I) is its homological behavior. Wodzicki showed that $\operatorname{Tor}_i^A(A/I, A/I) = 0$ for $i \geq 1$ [Wod89]. Recall that a *Bézout domain* is a domain for which every finitely generated ideal is principal. If R is a Bézout domain, then we have that for every ideal $\mathfrak{a}$ the equations $\operatorname{Tor}_1^R(R/\mathfrak{a}, R/\mathfrak{a}) = \mathfrak{a}/\mathfrak{a}^2$ and $\operatorname{Tor}_i^R(R/\mathfrak{a}, -) = 0$ for $i > 1$ hold. This forces that the Smashing Conjecture for $\mathcal{D}(R)$ is not true [Kra05, Section 15].

3.4 The Smashing Conjecture for stable module categories

We describe Krause and Solberg's characterization of the Smashing Conjecture for stable module categories in terms of cotorsion pairs [KS03, Conjecture 7.9] and discuss Angeleri-Hügel, Trlifaj and Šaroch's result [AHST06, Corollary 4.10] concerning a generalization of the Smashing Conjecture.

Recall that for a Frobenius ring R there are enough projective and injective modules and that these concepts coincide. Furthermore the stable module categories $\underline{\mathrm{Mod}}(R)$ and $\underline{\mathrm{mod}}(R)$ are triangulated (Example 2.1.4), and the compact objects in $\underline{\mathrm{Mod}}(R)$ are $\underline{\mathrm{mod}}(R)$.

The first step toward the reformulation is to understand the localizing subcategories in $\underline{\mathrm{Mod}}(R)$. In fact, they arise as pairs of subcategories.

A localizing subcategory $\mathcal{C}$ in a triangulated category $\mathcal{T}$ determines another subcategory, the *$\mathcal{C}$-local objects*,

$$\mathcal{C}_{\mathrm{loc}} = \{Y \in \mathcal{T} \mid \mathcal{T}(X,Y) = 0 \ \forall X \in \mathcal{C}\}.$$

A pair $(\mathcal{C}, \mathcal{C}_{\mathrm{loc}})$ of subcategories is called *localizing pair*. It turns out that localizing pairs are related to the following data in the module category.

Definition 3.4.1. Let $\mathcal{A}$ be an abelian category. A pair of subcategories $\mathfrak{X} = (\mathcal{C}, \mathcal{F})$ is called *cotorsion pair*, if the following axioms hold:

(i) $\mathrm{Ext}^1(X,Y) = 0$ for all $X \in \mathcal{C}$, if and only if $Y \in \mathcal{F}$.

(ii) $\mathrm{Ext}^1(X,Y) = 0$ for all $Y \in \mathcal{F}$, if and only if $X \in \mathcal{C}$.

(iii) Every object $A \in \mathcal{A}$ has a *special right $\mathcal{C}$-approximation*, i.e., there is a short exact sequence
$$0 \to Y \to X \to A \to 0$$
with $X \in \mathcal{C}$ and $Y \in \mathcal{F}$.

(vi) Every object $A \in \mathcal{A}$ has a *special left $\mathcal{F}$-approximation*, i.e., there is a short exact sequence
$$0 \to A \to Y \to X \to 0$$
with $X \in \mathcal{C}$ and $Y \in \mathcal{F}$.

A cotorsion pair $(\mathcal{C}, \mathcal{F})$ is called *thick*, if $\mathcal{C} \subset \mathcal{A}$ is closed under kernels of epimorphisms, cokernels of monomorphisms. It is called *hereditary*, if $\mathrm{Ext}^i(X,Y) = 0$ for $i \geq 2$ and all $X \in \mathcal{C}$ and $Y \in \mathcal{F}$. A cotorsion pair $(\mathcal{C}, \mathcal{F})$ is *cogenerated* by a set $\mathcal{G} \subseteq \mathcal{C}$, if the following holds: $\mathrm{Ext}^1(G,Y) = 0$ for all $G \in \mathcal{G}$ if and only if $Y \in \mathcal{F}$. A cotorsion pair in $\mathrm{Mod}(R)$ is of *finite type*, if it is cogenerated by a set $\mathcal{G} \subseteq \mathcal{C} \cap \mathrm{mod}(R)$.

A ring R is by definition *self-injective* if R is injective as a right R-module. Note that an artin algebra A is self-injective, if and only if it is a Frobenius ring. Krause and Solberg translate localizing pairs in the stable category into thick cotorsion pairs in the module category.

Theorem 3.4.2. *[KS03, 6.3] Let A be a self-injective artin algebra. There is a one to one correspondence between the thick cotorsion pairs in* $\mathrm{Mod}(A)$ *and the localizing pairs in* $\underline{\mathrm{Mod}}(A)$ *given by* $(\mathcal{C}, \mathcal{F}) \mapsto (\underline{\mathcal{C}}, \underline{\mathcal{F}})$.

Here $\underline{\mathcal{C}}$ denotes the image of $\mathcal{C}$ under the canonical functor $\mathrm{Mod}(A) \to \underline{\mathrm{Mod}}(A)$. If furthermore $\underline{\mathcal{C}}$ is *smashing*, then the corresponding cotorsion pair is of a special form:

Theorem 3.4.3. *[KS03, 7.6, 7.7] Let A be a self-injective artin algebra and let $(\mathcal{C}, \mathcal{F})$ be a cotorsion pair in* $\mathrm{Mod}(A)$*. The category $\underline{\mathcal{C}}$ is smashing, if and only if $\mathcal{F}$ is closed under filtered colimits. In that case $\mathcal{C}$ is also closed under filtered colimits.*

Let $\mathcal{A}$ be a set of objects in $\mathrm{Mod}(A)$. Then $\mathrm{colim}(\mathcal{A})$ denotes the full subcategory of all A-modules that are filtered colimits of modules in $\mathcal{A}$. If $\underline{\mathcal{C}}$ is of finite type in the sense of Definition 2.4.6, then the cotorsion pair fulfills an additional finiteness property.

Theorem 3.4.4. *[KS03, 7.7] Let A be a self-injective artin algebra and let $(\mathcal{C}, \mathcal{F})$ be a cotorsion pair in* $\mathrm{Mod}(A)$*. Then $\underline{\mathcal{C}}$ is of finite type, if and only if $\mathcal{C} = \mathrm{colim}(\mathcal{C} \cap \mathrm{mod}(A))$, i.e., the modules in $\mathcal{C}$ are precisely the filtered colimits of finitely presented objects in $\mathcal{C}$.*

The preceding considerations motivate the following conjecture.

Conjecture 3.4.5. [KS03, Conjecture 7.9] Let A be an artin algebra and $(\mathcal{C}, \mathcal{F})$ a cotorsion pair in $\mathrm{Mod}(A)$. If $\mathcal{C}$ and $\mathcal{F}$ are closed under filtered colimits, then $\mathcal{C} = \mathrm{colim}(\mathcal{C} \cap \mathrm{mod}(A))$.

Note that if A is self-injective, this conjecture is equivalent to the Smashing Conjecture for $\underline{\mathrm{Mod}}(A)$. Angeleri-Hügel, Šaroch and Trlifaj proved the Conjecture 3.4.5 in the following case:

Theorem 3.4.6. *[AHST06, 3.3,4.10] Let R be a noetherian ring and $(\mathcal{C}, \mathcal{F})$ be a hereditary cotorsion pair in* $\mathrm{Mod}(R)$ *such that $\mathcal{C}$ and $\mathcal{F}$ are closed under filtered colimits. If $\mathcal{C}$ consists of modules of bounded projective dimension* or *$\mathcal{B}$ consists of modules of bounded injective dimension, then $(\mathcal{C}, \mathcal{F})$ is of finite type.*

The idea of the proof is based on the following. If $\mathcal{C}$ consists of modules of bounded projective dimension, then $\mathfrak{X}$ is a tilting cotorsion pair which are known to be of finite type. In the other case $\mathfrak{X}$ is known to be countably generated and by using methods from set theory it is shown to be finitely generated.

3.5 The Smashing Conjecture for arbitrary triangulated categories

The Smashing Conjecture 2.4.7 cannot be true for arbitrary compactly generated triangulated categories with direct sums as shown by Paragraph 3.3. Nevertheless Krause has shown a generalization of the Smashing Conjecture 2.4.7 in [Kra00] and discovered a relation to cohomological quotients [Kra05].

Let $\mathcal{T}$ be a compactly generated triangulated category with small coproducts. A localizing subcategory $\mathcal{C}$ in $\mathcal{T}$ is said to be *generated* by a class I of maps in $\mathcal{T}$, if every map in I factors through an object in $\mathcal{C}$. Note that $\mathcal{C}$ is generated by the set of identity maps $\{\mathrm{id}_{X_i}\}_{i \in J}$, if and only if $\mathcal{C}$ is generated by the objects $\{X_i \mid i \in J\}$. In that the following is a generalization of the Smashing Conjecture 2.4.7:

Theorem 3.5.1. *[Kra00, Corollary A] Every smashing subcategory is generated by a set of maps between compact objects.*

The Smashing Conjecture 2.4.7 is related to cohomological quotients and the theory of rings with several objects [Kra05].

Definition 3.5.2. Let $F\colon \mathcal{S} \to \mathcal{T}$ be an exact functor of triangulated categories. The *annihilator* of F is defined as $\mathrm{Ann}(F) := \{f \in \mathrm{Mor}(\mathcal{S}) \,|\, F(f) = 0\}$. An exact functor $F\colon \mathcal{S} \to \mathcal{T}$ is called a *cohomological quotient functor*, if for any cohomological functor $H\colon \mathcal{S} \to \mathcal{A}$ to an abelian category $\mathcal{A}$ with $\mathrm{Ann}(F) \subset \mathrm{Ann}(H)$, there is a unique cohomological functor making the following triangle commutative

$$\begin{tikzcd} \mathcal{S} \arrow[r, "F"] \arrow[d, "H"'] & \mathcal{T} \arrow[dl, dotted, "\exists!"] \\ \mathcal{A}. & \end{tikzcd}$$

The annihilator $\mathrm{Ann}(F)$ of a cohomological quotient functor $F\colon \mathcal{S} \to \mathcal{T}$ is called *exact ideal*.

The following theorem establishes a connection between the Smashing Conjecture, cohomological quotients and flat epimorphisms between rings with several objects.

Theorem 3.5.3. *[Kra05, 13.4] The Smashing Conjecture 2.4.7 for $\mathcal{S}$ is equivalent to each of the following statements*

(i) Every exact ideal in $\mathcal{S}_c$ is generated by idempotent elements.

(ii) Every cohomological functor $F\colon \mathcal{S}_c \to \mathcal{T}$ induces an equivalence up to direct factors $\mathcal{S}_c/\ker(F) \to \mathcal{T}$.

(iii) Every two-sided flat epimorphism $F\colon \mathcal{S}_c \to \mathcal{T}$ satisfying $\Sigma(\mathrm{Ann}(F)) = \mathrm{Ann}(F)$ is an Ore-localization.

In particular (i) shows that the Smashing Conjecture can be reduced to an assertion on the compact objects in the triangulated category.

3.6 Applications

We discuss two applications that were discovered by Krause. The first deals with the relation of non-commutative localization and the Smashing Conjecture. In the second application the validity of the Smashing Conjecture implies the existence of a long exact sequence in algebraic K-theory for certain rings [Kra05]. Both themes were originally studied in [NR04].

Let R be a ring and $R \to S$ be a ring homomorphism. Consider the problem of lifting a complex of S-modules or a map between complexes of S-modules along the map $R \to S$ to a chain complex/map of R-modules up to homotopy. To be precise, given a complex Y in $K^b(S)$, the homotopy category of bounded complexes of S-modules, we are looking for a complex $X \in K^b(R)$ such that $X \otimes_R S \cong Y$ in the homotopy category. Similarly, if we are given a map $\alpha\colon X \otimes_R S \to X' \otimes_R S$, we seek for maps $\phi\colon Y \to X$ and $\alpha'\colon Y \to X'$ such that $\phi \otimes_R S$ is invertible and $\alpha = (\alpha' \otimes_R S) \circ (\phi \otimes_R S)^{-1}$.

If $R \to S$ is a commutative localization, then both lifting problems can be solved. The question is now, to what extend this is true in a *non*-commutative situation. So let $R \to S$ be a *homological epimorphism* in the sense of [GL91], i.e., $S \otimes_R S \cong S$ and $\mathrm{Tor}_i^R(S,S) = 0$.

Proposition 3.6.1. *[Kra05, Corollary 14.7] The map $R \to S$ is a homological epimorphism, if and only if*

$$- \otimes_R S\colon K^b(R) \to K^b(S)$$

is a cohomological quotient in the sense of Definition 3.5.2.

Since cohomological quotients are related to the Smashing Conjecture by Theorem 3.5.3(ii) this proposition indicates that the Smashing Conjecture should affect the lifting problem for homological epimorphisms:

Theorem 3.6.2. *[Kra05, Corollary 14.7] If the Smashing Conjecture 2.4.7 is true for $\mathcal{D}(R)$, then both lifting problems can be solved, if and only if $R \to S$ is a homological epimorphism.*

For a commutative localization $R \to S$ there is a long exact sequence in algebraic K-groups [NR04]. If the Smashing Conjecture for the derived category of the ring R is true, we obtain a generalization:

Theorem 3.6.3. *[Kra05, Theorem 2] If the Smashing Conjecture 2.4.7 is true for $\mathcal{D}(R)$ and $R \to S$ is a homological epimorphism, then the following maps*

$$K(R, f) \to K(R) \to K(S)$$

form a homotopy fiber sequence, apart from the surjectivity of $K_0(R) \to K_0(S)$. In particular there is a long exact sequence of algebraic K-groups

$$\cdots \to K_n(R) \to K_n(S) \to K_{n-1}(R, f) \to \cdots \to K_0(R) \to K_0(S).$$

Here $K(R, f)$ is the Waldhausen K-theory of a suitable bi-complicial Waldhausen category in the sense of Thomason [TT90].

The preceding two theorems show that the Smashing Conjecture effects non-commutative algebra. Besides its original consequences on the Telescope Conjecture and with it on algebraic topology, these theorems foreshadow applications of the Smashing Conjecture in other areas.

4 Cotorsion pairs, model categories and finite generation

The Smashing Conjecture 2.4.7 for the stable module category of a self-injective artin algebra A is equivalent to the assertion that a class of cotorsion pairs in $\mathrm{Mod}(A)$ is of finite type by Paragraph 3.4. There is a strong connection between cotorsion pairs and model categories as was independently shown by Beligiannis-Reiten [BR02] and Hovey [Hov02]. Here we investigate how the finite type of cotorsion pairs is reflected in model category theory. We continue and deepen Hovey's study and prove a reformulation of the Smashing Conjecture in terms of model categories.

4.1 Model categories

In this paragraph the language of model categories is recalled. In particular we discuss cofibrantly and finitely generated model categories that become important later on. The concepts are illustrated with the model category of modules over a Frobenius ring R.

Definition 4.1.1. Let $\mathcal{C}$ be a cocomplete category. An object X in $\mathcal{C}$ is *finite* if for every sequence $Y_0 \to Y_1 \to \cdots \to Y_n \to \dots$, the canonical map

$$\operatorname{colim}_{n\in\mathbb{N}} \operatorname{Hom}_{\mathcal{C}}(X, Y_n) \to \operatorname{Hom}_{\mathcal{C}}(X, \operatorname{colim}_{n\in\mathbb{N}} Y_n)$$

is bijective.

For example the sets with finitely many elements are finite, and the finitely presented modules over a ring R are finite in the category $\mathrm{Mod}(R)$.

Lemma 4.1.2. *In a cocomplete category pushouts of finite objects are finite.*

Proof. Since filtered colimits commute with finite limits [ML98, IX.2 Theorem 1] the finite objects are closed under finite colimits and in particular under pushouts. □

Definition 4.1.3. A *model category* consists of a complete and cocomplete category $\mathcal{C}$ together with three nonempty classes of morphisms $\mathrm{weq}(\mathcal{C})$, $\mathrm{cof}(\mathcal{C})$ and $\mathrm{fib}(\mathcal{C})$ that are called *weak equivalences*, *cofibrations*, and *fibrations*, respectively. The elements of $\mathrm{weq}(\mathcal{C}) \cap \mathrm{cof}(\mathcal{C})$ are named *acyclic cofibrations* and the morphisms in $\mathrm{weq}(\mathcal{C}) \cap \mathrm{fib}(\mathcal{C})$ are called *acyclic fibrations*. These maps are subject to the following conditions.

MC1 The weak equivalences satisfy the"two out of three axiom", i.e., let f and g be morphisms in $\mathcal{C}$ such that the composition $f \circ g$ exists; if two of the three morphisms f, g and $f \circ g$ are weak equivalences, then so is the third.

MC2 Cofibrations, fibrations and weak equivalences are stable under retracts, that is, if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"'] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

such that both compositions of the horizontal maps are the identity and g is a weak equivalence, cofibration or fibration, then so is f.

MC3 The cofibrations have the left lifting property (LLP) with respect to the acyclic fibrations. That is, for every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"'] & X \arrow[d, two heads, "\sim"', "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

in which i is a cofibration, and p is an acyclic fibration, there is a lift $B \to X$ making the diagram commute.

The fibrations have the right lifting property (RLP) with respect to the acyclic cofibrations, that means, if p is a fibration and i is an acyclic cofibration, that both fit into a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"', "\sim"] & X \arrow[d, two heads, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

then there is a lift $B \to X$ making both triangles commute.

MC4 A morphism $f\colon X \to Y$ can be factored into a cofibration followed by an acyclic fibration

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail] & & Y \\ & Z \arrow[ur, two heads, "\sim"] & \end{tikzcd}$$

and an acyclic cofibration followed by a fibration

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail, "\sim"] & & Y \\ & Z. \arrow[ur, two heads] & \end{tikzcd}$$

Remark 4.1.4. The triple $(\mathrm{weq}(\mathcal{C}), \mathrm{cof}(\mathcal{C}), \mathrm{fib}(\mathcal{C}))$ is called a *model structure* on $\mathcal{C}$. We use $\xrightarrow{\sim}$ as a symbol for a weak equivalence, $\overset{(\sim)}{\rightarrowtail}$ for (acyclic) cofibrations and $\overset{(\sim)}{\twoheadrightarrow}$ for (acyclic) fibrations like in axiom **MC3**. Since $\mathcal{C}$ is complete and cocomplete it has an initial object $\emptyset$ and a terminal object $*$. A model category is called *pointed* if the initial and terminal objects are isomorphic. An object X is called *fibrant* if the unique morphism $X \to *$ is a fibration and *cofibrant* if the unique morphism $\emptyset \to X$ is a cofibration. For an arbitrary object X in $\mathcal{C}$, the morphism $\emptyset \to X$ can be factored by axiom **MC4** as a cofibration $\emptyset \to X^c$ followed by an acyclic fibration $X^c \to X$. So the object X^c is cofibrant. Since the factorizations can be chosen functorially in all examples that arise the assignment $X \mapsto X^c$ is an endofunctor on $\mathcal{C}$. It is called *cofibrant replacement functor* and is equipped with a natural transformation q from the identity on $\mathcal{C}$ to $(-)^c$. Dually, there is a *fibrant replacement functor* $X \mapsto (X)^f$ that is obtained by factoring $X \to *$ into an acyclic cofibration followed by a fibration. An object X in a pointed model category is called *acyclic* if the canonical map $X \to 0$ is a weak equivalence.

Example 4.1.5. Let R be Frobenius ring, that is, $\mathrm{Mod}(R)$ has enough projectives and injectives and they coincide. Two maps $f, g \colon M \to N$ between R-modules are called *stably equivalent* if the difference $f - g$ factors through a projective. A map $f \colon M \to N$ between R-modules is defined to be a *stable equivalence* if there is a map $g \colon N \to M$ such that $f \circ g$ and $g \circ f$ are stably equivalent to id_N and id_M, respectively. By definition a stable equivalence becomes an isomorphism in the stable category $\underline{\mathrm{Mod}}(R)$. The cofibrations are the injective maps and the fibrations are the surjective maps. These three classes of maps specify a model structure on $\mathrm{Mod}(R)$ [Hov99, Theorem 2.2.12]. Originally this structure was discovered by Pirashvili [Pir86]. This model category has the very special property that all objects are fibrant and cofibrant.

By axiom **MC3** fibrations have the RLP with respect to acyclic cofibrations, and acyclic fibrations have the RLP with respect to cofibrations. They are in fact characterized by this property.

Proposition 4.1.6. *[Hov99, Lemma 1.1.10] Let $\mathcal{C}$ be a model category.*

(i) A map is a fibration if and only if it has the RLP with respect to all acyclic cofibrations.

(ii) A map is an acyclic fibration if and only if it has the RLP with respect to all cofibrations.

Corollary 4.1.7. *[Hov99, Corollary 1.1.11] Let $\mathcal{C}$ be a model category. Then pushouts along (acyclic) cofibrations are (acyclic) cofibrations, i.e., let*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C \arrow[d, "g"] \\ B \arrow[r] & D \end{tikzcd}$$

be a pushout diagram. If f is a cofibration or acyclic cofibration, then g is also a cofibration or an acyclic cofibration. Dually, fibrations and acyclic fibrations are closed under pullbacks.

Remark 4.1.8. In a model category $\mathcal{C}$ it is possible to define when two maps are homotopic [Hov99, Definition 1.2.4]. Let $\mathcal{C}_{cf}$ denote the full subcategory of cofibrant and fibrant objects. The homotopy relation on the morphisms of $\mathcal{C}_{cf}$ defines an equivalence relation that is compatible with the composition [Hov99, Corollary 1.2.7]. Then it is possible to define the *homotopy category* $\mathrm{Ho}(\mathcal{C})$. Its objects are the cofibrant fibrant objects of $\mathcal{C}$, and the morphisms between objects X and Y are defined as the set of equivalence classes of morphisms in $\mathcal{C}$ with respect to the homotopy relation.

In Example 4.1.5 the two maps f and g are homotopic if and only if they are stably equivalent. The homotopy category is the stable module category.

In many examples the fibrations are determined by having the RLP with respect to only a subset I of the class of acyclic cofibrations. In the same way the acyclic fibrations are determined by a set J of cofibrations. We illustrate this property with the category of modules over a Frobenius ring R described in Example 4.1.5. Recall that a fibration is a surjective map $M \to N$ between R-modules. The following lemma is immediate.

Lemma 4.1.9. *Let R be an arbitrary ring. A map $M \to N$ in* $\mathrm{Mod}(R)$ *is surjective, if and only if it has the RLP with respect to the trivial map $0 \to R$.*

So the fibrations are determined by having the RLP with respect to a single map or in other words with respect to the set $J := \{0 \to R\}$. An acyclic fibration can be characterized as follows.

Proposition 4.1.10. *[Hov99, Lemma 2.2.7, Theorem 2.2.12] Let R be a Frobenius ring. A map in the module category* $\mathrm{Mod}(R)$ *is an acyclic fibration if and only if it is a surjective map with injective kernel.*

Recall that a module Q is injective if and only if $Q \to 0$ has the RLP with respect to all injective maps $M \to N$ of R-modules. Baer's criterion says that not all these maps are necessary.

Proposition 4.1.11. *[Jac89, Proposition 3.15] Let R be an arbitrary ring. An R- module Q is injective, if and only if $Q \to 0$ has the RLP with respect to all $\mathfrak{a} \hookrightarrow R$ where $\mathfrak{a}$ is an ideal in R.*

This criterion motivates and is the key to the following.

Proposition 4.1.12. *[Hov99, Proposition 2.2.9] Let R be a Frobenius ring. A map in* $\mathrm{Mod}(R)$ *is an acyclic fibration if and only if it has the RLP with respect to the set*

$$I := \{\mathfrak{a} \hookrightarrow R \mid \mathfrak{a} \ \textit{ideal in} \ R\}.$$

If I is a set of morphisms, then let $\mathrm{RLP}(I)$ denote the class of maps which have the RLP with respect to all maps in I. Now we introduce the underlying concept.

Definition 4.1.13. A model category $\mathcal{C}$ is *cofibrantly generated* if there are sets I and J such that

(i) the class of fibrations is the class $\mathrm{RLP}(J)$.

(ii) The class of acyclic fibrations is the class $\mathrm{RLP}(I)$.

The elements in I are called *generating cofibrations* and the elements in J *generating acyclic cofibrations*. A cofibrantly generated model category $\mathcal{C}$ is called *finitely generated* if the domains and codomains of the maps in I and J are finite.

Remark 4.1.14. In the original definition of a cofibrantly generated model category the sources and targets of generating cofibrations and generating acyclic cofibrations are asked to be *small*. The notion of smallness involves cardinal numbers and is quite technical. It is suppressed from the notation since all examples of cofibrantly generated model categories we consider in this section are module categories. In a module category every object is small since it is a Grothendieck category [Hov01b, Proposition A.2].

Note that by Proposition 4.1.6 the elements in I (and J) are cofibrations (and acyclic cofibrations). If R is a Frobenius ring then the model structure specified in Example 4.1.5 is cofibrantly generated:

Theorem 4.1.15. *[Hov99, Theorem 2.2.12] Let R be a Frobenius ring. Then there is a finitely generated model structure on* $\mathrm{Mod}(R)$ *where the cofibrations are the injections, the fibrations are the surjections, the weak equivalences are the stable equivalences, $I := \{0 \to R\}$ is a set of generating cofibrations and $J := \{\mathfrak{a} \hookrightarrow R \mid \mathfrak{a}$ ideal in $R\}$ is a set of generating acyclic cofibrations.*

Remark 4.1.16. Nearly all model categories that occur in nature are cofibrantly generated and most of them are finitely generated. It requires some work to construct examples of non cofibrantly generated model categories. Nevertheless, there are examples [Isa01, AHRT02, CH02, Cho03].

4.2 Cotorsion pairs and model categories

Beligiannis-Reiten and Hovey have shown a remarkable theorem which relates model structures to cotorsion pairs. In this paragraph we recall it and prove a specialization of this result.

Let $\mathcal{A}$ be a bicomplete abelian category and a model category. Furthermore assume that the model structure and the abelian structure are *compatible* in the following sense:

- Every cofibration $i : A \rightarrowtail B$ in $\mathcal{A}$ is a monomorphism.
- A map $p : B \to C$ is a fibration if and only if p is an epimorphism and the kernel $\ker(p)$ in the short exact sequence

$$0 \to \ker(p) \to B \to C \to 0$$

  is fibrant. A fibration is acyclic, if and only if its kernel is acyclic.

We call such a category $\mathcal{A}$ *abelian model category.*

Theorem 4.2.1. *[BR02, Theorem 5.3] , [Hov02, Theorem 2.2] Let $\mathcal{A}$ be a bicomplete abelian category. If $\mathcal{A}$ is equipped with a compatible model structure, let $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ denote the full subcategories of cofibrant, fibrant, and acyclic objects in $\mathcal{A}$, respectively. Then*

(i) $\mathcal{W}$ is thick, i.e., closed under extensions and retracts, and

(ii) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are cotorsion pairs.

Conversely, classes $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ in $\mathcal{A}$ satisfying (i) and (ii) determine a unique model structure on $\mathcal{A}$ that is compatible with the abelian structure.

We now specialize to the case where $\mathcal{W} = \mathcal{A}$.

Theorem 4.2.2. *Let $(\mathcal{C}, \mathcal{F})$ be a cotorsion pair in a bicomplete abelian category $\mathcal{A}$. Then there is an abelian model structure on $\mathcal{A}$ such that all maps are weak equivalences, the fibrations are epimorphisms with kernel in $\mathcal{F}$ and the cofibrations are the monomorphisms with cokernel in $\mathcal{C}$.*

Proof. The proof agrees in most instances with the proof of 4.2.1 applied to the special situation of this theorem. We include it for the convenience of the reader.

The "2 out of 3" property for weak equivalences is evident. For the retract axiom, assume that we are given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"'] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

such that $A \to C \to A$ and $B \to D \to B$ are identity maps. We show that if g is a monomorphism with cokernel in $\mathcal{C}$ then so is f. By a diagram chase one can prove that f is a monomorphism. Passing to cokernels, we obtain maps $\mathrm{coker}(f) \to \mathrm{coker}(g) \to \mathrm{coker}(f)$ whose composition is $\mathrm{id}_{\mathrm{coker}(f)}$. Therefore $\mathrm{coker}(f)$ is a direct summand of $\mathrm{coker}(g)$ and since $\mathrm{Ext}^1_{\mathcal{A}}(X, \mathrm{coker}(g)) = 0$ for all $X \in \mathcal{F}$, we can conclude that $\mathrm{Ext}^1_{\mathcal{A}}(X, \mathrm{coker}(f)) = 0$ for all $X \in \mathcal{F}$. Hence $\mathrm{coker}(f) \in \mathcal{C}$.

Note that the notions of "(co-)fibration" and "acyclic (co-)fibration" coincide because of the choice of the class of weak equivalences. Therefore to prove the lifting axiom, we only have to construct a lift l in the commutative diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ 0 \arrow[d] & F \arrow[d] \\ A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[ur, dotted, "l"] \arrow[d, "j"] & Y \arrow[d] \\ C \arrow[d] & 0 \\ 0 & \end{tikzcd}$$

where the vertical rows are exact and $C \in \mathcal{C}$ and $F \in \mathcal{F}$. The construction of l will be done in two steps following [Hov01a, 4.2]. First we find a map $h : B \to X$ that makes the upper triangle commute and secondly, we subtract a map such that in addition the lower triangle commutes.

Consider the following commutative diagram with exact columns and exact lower row:

$$\begin{tikzcd} & \mathcal{A}(B,X) \arrow[r, "p_*"] \arrow[d, "i^*"] & \mathcal{A}(B,Y) \arrow[d, "i^*"] \\ & \mathcal{A}(A,X) \arrow[r, "p_*"] \arrow[d, "\delta"] & \mathcal{A}(A,Y) \arrow[d, "\delta"] \\ \mathrm{Ext}^1_{\mathcal{A}}(C,F) \arrow[r] & \mathrm{Ext}^1_{\mathcal{A}}(C,X) \arrow[r, "p_*"] & \mathrm{Ext}^1_{\mathcal{A}}(C,Y). \end{tikzcd}$$

The map $f \in \mathcal{A}(A, X)$ is sent to $p \circ f$ in $\mathcal{A}(A, Y)$. Since $i^*(g) = g \circ i = p \circ f$ the morphism $p_*(f)$ is mapped to 0 by δ. Since $\operatorname{Ext}^1(C, F) = 0$ (because $C \in \mathcal{C}$ and $F \in \mathcal{F}$) the map $p_* : \operatorname{Ext}^1(C, X) \to \operatorname{Ext}^1(C, Y)$ is injective. Since $p_*(\delta(f)) = \delta(p_*(f)) = 0$, we can therefore conclude that $\delta(f) = 0$. By the exactness of the left column in the diagram we find a map $h : B \to X$ such that $h \circ i = f$.

For the second step consider the exact sequence

$$\mathcal{A}(C, Y) \xrightarrow{j^*} \mathcal{A}(B, Y) \xrightarrow{i^*} \mathcal{A}(A, Y).$$

The map $p \circ h - g$ is sent to 0 by i^*. Therefore there is a morphism $F \colon C \to Y$ such that $F \circ j = p \circ h - g$. In the exact sequence

$$\mathcal{A}(C, X) \xrightarrow{p_*} \mathcal{A}(C, Y) \to \operatorname{Ext}^1_{\mathcal{A}}(C, F)$$

the end term is 0 and hence we find a map $G \colon C \to X$ such that $p \circ G = F$. The map $l := h - G \circ j$ is the desired lift.

Following [Hov01a, 5.4] the factorization axiom can be proved in two steps. First a map $f \colon A \to B$ can be factored into a monomorphism followed by an epimorphism in the following way:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "{(1_A,0)}"'] & & B \\ & A \oplus B. \arrow[ur, "f \oplus 1_B"'] & \end{tikzcd}$$

We have no reason to expect that the kernel of the epimorphism is in $\mathcal{F}$ or that the cokernel of the monomorphism is in $\mathcal{C}$. But we show in the next step that every monomorphism or epimorphism can be factored into a monomorphism with cokernel in $\mathcal{C}$ followed by an epimorphism with kernel in $\mathcal{F}$. Therefore we can find a monomorphism i_1 with cokernel in $\mathcal{C}$ and an epimorphism p_1 with kernel in $\mathcal{F}$ such that $(1_A, 0) = p_1 \circ i_1$. The composition $(f \oplus 1_B) \circ p_1$ is an epimorphism and can be factored $(f \oplus 1_B) \circ p_1 = p_2 \circ i_2$, where p_2 is an epimorphism with kernel in $\mathcal{F}$ and i_2 is a monomorphism with cokernel in $\mathcal{C}$. Lemma 4.2.3, which is shown after this proof, tells us that the kernel of a composition of two monomorphisms with kernel in $\mathcal{C}$ is again contained in $\mathcal{C}$. Therefore $f = p_2 \circ i_2 \circ i_1$ is the desired factorization.

Modulo the lemma we only have to show that a monomorphism and an epimorphism can be factored into a monomorphism with cokernel in $\mathcal{C}$ followed by an epimorphism with kernel in $\mathcal{F}$. So let $i \colon A \to B$ be a monomorphism with cokernel X. Choose an approximation

$$0 \to F_X \to C_X \to X \to 0$$

such that $F_X \in \mathcal{F}$ and $C_X \in \mathcal{C}$. Consider the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & F_X \arrow[r, equal] \arrow[d] & F_X \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "j"] \arrow[d, equal] & B' \arrow[r] \arrow[d, "q"] & C_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns in which B' is a pullback. Then $i = q \circ j$ is the desired factorization.

Now let $p\colon X \to Y$ be an epimorphism with kernel K. Choose an approximation

$$0 \to K \to F_K \to C_K \to 0$$

with $F_K \in \mathcal{F}$ and $\mathcal{C}_K \in \mathcal{C}$. Consider the diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & X \arrow[r, "p"] \arrow[d, "j"] & Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F_K \arrow[r] \arrow[d] & X' \arrow[r, "q"] \arrow[d] & Y \arrow[r] & 0 \\
 & C_K \arrow[r, equal] \arrow[d] & C_K \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

such that X' is a pushout. Then $p = q \circ j$ is the wanted factorization. With the following lemma we are done. $\square$

Lemma 4.2.3. *Let $(\mathcal{C}, \mathcal{F})$ be a cotorsion pair in $\mathcal{A}$. If $f\colon A \to B$ and $g\colon B \to C$ are epimorphisms with kernel in $\mathcal{F}$ then the kernel of the composition $g \circ f$ is in $\mathcal{F}$. The monomorphisms with cokernel in $\mathcal{C}$ are also stable under composition of maps.*

Proof. We prove the first statement, and the second assertion follows dually.

Let f and g be epimorphisms such that their kernel is in $\mathcal{F}$. Let $F = \ker(f)$, $F' = \ker(g)$

and $F'' = \ker(g \circ f)$. An application of the snake lemma to the following diagram

$$\begin{tikzcd} & 0 \arrow[d] & & & \\ 0 \arrow[r] & F \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & F'' \arrow[r] & A \arrow[r, "g\circ f"] & C \arrow[r] & 0 \end{tikzcd}$$

shows that $F''/F \cong F'$. Therefore the sequence

$$0 \to F \to F'' \to F' \to 0$$

is exact. Furthermore F and F' are in $\mathcal{F}$ by assumption. For any $X \in \mathcal{C}$, we obtain an exact sequence

$$\mathrm{Ext}^1_{\mathcal{A}}(F', X) \to \mathrm{Ext}^1_{\mathcal{A}}(F'', X) \to \mathrm{Ext}^1_{\mathcal{A}}(F, X)$$

in which the first and the last term vanish. Therefore, $\mathrm{Ext}^1_{\mathcal{A}}(F'', X) = 0$ for all $X \in \mathcal{C}$ and hence $F'' \in \mathcal{F}$. □

Definition 4.2.4. Let $\mathcal{A}$ be an abelian category and let $\mathfrak{X} = (\mathcal{C}, \mathcal{F})$ be a cotorsion pair in $\mathcal{A}$. The category $\mathcal{A}$ together with the unique model structure of Theorem 4.2.2 on $\mathcal{A}$ is called *model category associated with* $\mathfrak{X}$ and is abbreviated with $\mathcal{A}_{\mathfrak{X}}$.

Remark 4.2.5. This model category structure is quite unusual from the perspective of model category theory since all maps are weak equivalences. Therefore in the homotopy category all maps get inverted, and the homotopy category is a the trivial additive category.

Note that the concepts of cofibration and acyclic cofibration and the notions of fibrations and acyclic cofibration coincide here.

4.3 Finite generation and the Smashing Conjecture

In this part the relation between cogeneration of a cotorsion pair and cofibrant generation of the associated model structure is studied. As a consequence we obtain a reformulation of the Smashing Conjecture in terms of finite generation of this model structure.

Nearly all model categories that occur in nature are cofibrantly generated [Hov99]. So it is natural to ask how this property is resembled in the theory of cotorsion pairs. Hovey showed in a slightly more general setup the following

Proposition 4.3.1. *[Hov02, Lemma 6.7] Let R be a ring and $\mathfrak{X} = (\mathcal{C}, \mathcal{F})$ be a cotorsion pair in* $\mathrm{Mod}(R)$ *that is cogenerated by a set $\mathcal{G}$ of R-modules. Choose for every G in $\mathcal{G}$ a free module F_G that is of finite rank if G is finitely generated together with an epimorphism $F_G \xrightarrow{p_G} G \to 0$. Then the model category* $(\mathrm{Mod}(R))_{\mathfrak{X}}$ *is cofibrantly generated by the set*

$$I_{\mathcal{G}} := \{\ker(p_G) \to F_G \,|\, G \in \mathcal{G}\} \cup \{0 \to R\}.$$

If furthermore R is noetherian, and $\mathcal{G}$ consists of finitely generated modules then $\mathrm{Mod}(R)_{\mathfrak{X}}$ *is finitely generated.*

Proof. Since every map in $I_{\mathcal{G}}$ is a monomorphism and $\mathcal{G} \subset \mathcal{C}$ by definition we conclude that all maps in $I_{\mathcal{G}}$ are cofibrations. Therefore the class of fibrations of $(\mathrm{Mod}(R))_{\mathfrak{X}}$ is contained in the class $\mathrm{RLP}(I_{\mathcal{G}})$.

For the other inclusion assume $p : X \to Y$ has the $\mathrm{RLP}(I_{\mathcal{G}})$. In particular there is a lift l in the following diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ R \arrow[r] \arrow[ur, dotted, "l"] & Y \end{tikzcd}$$

which is equivalent to the surjectivity of p. The map $\ker(p) \to 0$ has the RLP with respect to all morphisms in $I_{\mathcal{G}}$ as a pullback of the map p which has this lifting property. Therefore, we find for each $G \in \mathcal{G}$ a lift in

$$\begin{tikzcd} M_G \arrow[r] \arrow[d] & \ker(p) \arrow[d] \\ F_G \arrow[r] \arrow[ur, dotted] & 0, \end{tikzcd}$$

where M_G is the kernel of the epimorphism $F_G \to G \to 0$. Hence the map

$$\mathrm{Hom}_R(F_G, \ker(p)) \to \mathrm{Hom}_R(M_G, \ker(p))$$

is surjective for all $G \in \mathcal{G}$. Applying $\mathrm{Hom}(-, \ker(p))$ to

$$0 \to M_G \xrightarrow{i_G} F_G \xrightarrow{p_G} G \to 0$$

yields the exact sequence

$$\mathrm{Hom}_R(F_G, \ker(p)) \xrightarrow{i_G^*} \mathrm{Hom}_R(M_G, \ker(p)) \xrightarrow{\partial} \mathrm{Ext}^1_R(G, \ker(p)) \to \mathrm{Ext}^1_R(F_G, \ker(p))$$

in which i_G^* is surjective and $\mathrm{Ext}^1(F_G, \ker(p)) = 0$ because F_G is free. Therefore the connecting homomorphism ∂ is surjective and trivial, and hence $\mathrm{Ext}^1_R(G, \ker(p)) = 0$ for all $G \in \mathcal{G}$. As $\mathcal{G}$ cogenerates $\mathfrak{X}$ the module $\ker(p)$ is in $\mathcal{F}$ and $p : X \to Y$ is a fibration.

If R is noetherian, then submodules of finitely generated modules are finitely generated. Therefore if G and F_G are finitely generated then so is $\ker(p_G)$ as a submodule of F_G. $\square$

Now assume conversely that we are given a finitely generated, abelian model category. What can we say about the associated cotorsion pairs $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ of Theorem 4.2.1? In [Hov02] Hovey did not find an answer but proclaimed later on [Hov07] that the converse of Proposition 4.3.1 is easy to show. Here you find the precise statement and a proof.

Proposition 4.3.2. *Let $\mathcal{A}$ be an abelian model category that is cofibrantly generated. Then the cotorsion pairs $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are cogenerated by a set. If furthermore $\mathcal{A}$ is finitely generated then $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are of finite type.*

Proof. Let I be the set of generating cofibrations and J be the set of generating acyclic cofibrations. Then $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is cogenerated by $\mathcal{G} := \{\mathrm{coker}(i) \mid i \in I\}$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is cogenerated by $\{\mathrm{coker}(j) \mid j \in J\}$. We show the first statement. The second assertion

can be proved similarly. We have to show that $\mathrm{Ext}^1_{\mathcal{A}}(G,X)=0$ for all $G\in\mathcal{G}$ if and only if X is fibrant and acyclic.
Assume that $\mathrm{Ext}^1_{\mathcal{A}}(G,X)=0$ for all $G\in\mathcal{G}$. By definition for every $i\colon A\to B$ in I the object $G_i:=\mathrm{coker}(i)$ is in $\mathcal{G}$, and the following sequence is exact:

$$0\to A\xrightarrow{i}B\to G_i\to 0.$$

The map $i^*\colon \mathcal{A}(B,X)\to\mathcal{A}(A,X)$ is surjective for all $i\in I$ since $\mathrm{Ext}^1_{\mathcal{A}}(G_i,X)$ vanishes in the long exact Ext-sequence. But this means that for all generating cofibrations $i\in I$ there is a lift l in

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dotted, "l"] & 0. \end{tikzcd}$$

Therefore $X\to 0$ is an acyclic fibration.
Conversely, let X be acyclic and fibrant. Let G be an arbitrary element in $\mathcal{G}$. An element in $\mathrm{Ext}^1_{\mathcal{A}}(G,X)$ is represented by a short exact sequence

$$0\to X\to M\xrightarrow{p}G\to 0. \tag{1}$$

Since $p\colon M\to G$ is an epimorphism with acyclic, fibrant kernel, it is an acyclic fibration. By definition, there is a map $i\colon A\to B$ such that G is the cokernel of i, or in other words the pushout in the following pushout diagram:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & G. \end{tikzcd}$$

Since pushouts along cofibrations are again cofibrations by Corollary 4.1.7, the object G is cofibrant. Therefore we find a lift s in the following diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, tail] & M \arrow[d, two heads, "p"', "\sim"] \\ G \arrow[r, equal] \arrow[ur, dotted, "s"] & G \end{tikzcd}$$

that splits the sequence (1). Hence $\mathrm{Ext}^1_{\mathcal{A}}(G,X)=0$.

If $\mathcal{A}$ is finitely generated, then the domains and codomains of the generating cofibrations and generating acyclic cofibrations are finite. The set of cogenerators is obtained by taking pushouts of generating cofibrations and generating acyclic cofibrations with the zero map, respectively. Since pushouts of finite objects are finite by Lemma 4.1.2, we know that all elements in the set of cogenerators are finite. Therefore $(\mathcal{C},\mathcal{F}\cap\mathcal{W})$ and $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ are of finite type. □

Corollary 4.3.3. *Let $\mathfrak{X}=(\mathcal{C},\mathcal{F})$ be a cotorsion pair in $\mathcal{A}$. If the associated model category $\mathcal{A}_{\mathfrak{X}}$ is finitely generated then the cotorsion pair is of finite type.*

Note that the concepts of the set of generating acyclic cofibrations and generating cofibrations coincide for a model category associated with a cotorsion pair.

Example 4.3.4. Let R be a ring, and let $\mathrm{Proj}(R)$ be the class of projective modules. The model structure associated with the cotorsion pair $(\mathrm{Proj}(R), \mathrm{Mod}(R))$ is finitely generated by $J := \{0 \to R\}$ by [Hov99, 2.2.5, 2.2.11]. In more detail, the cofibrations are the injections with projective cokernel and the surjections are the fibrations. Indeed a map of R-modules is a surjection, if it has the right lifting property (RLP) with respect to $0 \to R$.

Example 4.3.5. Let R be a noetherian ring and let $\mathrm{Inj}(R)$ be the class of injective modules. Dually, the model structure associated with $(\mathrm{Mod}(R), \mathrm{Inj}(R))$ is finitely generated by the set

$$I := \{\mathfrak{a} \hookrightarrow R \mid \mathfrak{a} \text{ ideal in } R\}$$

because of [Hov99, 2.2.9, 2.2.10]. In this model structure the cofibrations are the injective maps and the fibrations are the surjective maps with injective kernel. The fibrations are the maps with the RLP with respect to I.

It is quite interesting that I and J are the generating cofibrations respectively the generating acyclic cofibrations of the model structure of the module category over a Frobenius ring described in Paragraph 4.2.

Theorem 4.3.6. *Let R be a self-injective artin algebra. Then the Smashing Conjecture 2.4.7 is equivalent to the statement that for all cotorsion pairs $\mathfrak{X} = (\mathcal{C}, \mathcal{F})$ such that $\mathcal{C}$ and $\mathcal{F}$ are closed under filtered colimits the associated model category $\mathrm{Mod}(R)_{\mathfrak{X}}$ is finitely generated.*

Proof. By Theorem 3.4.4 and Theorem 3.4.3 the Smashing Conjecture 2.4.7 is equivalent to Conjecture 3.4.5 which says that all cotorsion pairs $(\mathcal{C}, \mathcal{F})$ such that $\mathcal{C}$ and $\mathcal{F}$ are closed under filtered colimits are of finite type.

Fix a cotorsion pair $\mathfrak{X} = (\mathcal{C}, \mathcal{F})$ with the property that $\mathcal{C}$ and $\mathcal{F}$ are closed under filtered colimits. Since an artin ring is also noetherian, we can use Proposition 4.3.1 to conclude that Conjecture 3.4.5 implies that the model category $\mathrm{Mod}(R)_{\mathfrak{X}}$ is finitely generated. Conversely, if the model category $\mathrm{Mod}(R)_{\mathfrak{X}}$ is finitely generated, then Proposition 4.3.2 implies that the cotorsion pair $\mathfrak{X}$ is of finite type. □

Philosophically, this result is satisfactory since nearly all model categories are finitely generated and it is quite hard to find examples that are not. The same is commonly believed for the Smashing Conjecture: it is hard to find examples of smashing subcategories (or cotorsion pairs) that are not of finite type.

Given an arbitrary cotorsion pair $\mathfrak{X} = (\mathcal{C}, \mathcal{F})$ in a module category over a ring R, and assume we have a set of generating cofibrations $I_{\mathfrak{X}}$ for the associated model structure. Since every fibration is surjective it is natural to ask that the map $0 \to R$ must belong to $I_{\mathfrak{X}}$. The condition $I_{\mathfrak{X}} \subset \{\mathfrak{a} \hookrightarrow R \mid \mathfrak{a} \text{ ideal in } R\}$ forces that the surjections with injective kernel are a subset of the fibrations. Since all injective R-modules are in $\mathcal{F}$ this is a sensible requirement.

Conjecture 4.3.7. Let R be a noetherian ring and $\mathfrak{X} = (\mathcal{C}, \mathcal{F})$ be a cotorsion pair such that $\mathcal{C}$ and $\mathcal{F}$ are closed under filtered colimits. The model structure associated with the cotorsion pair $\mathfrak{X}$ is cofibrantly generated by the set

$$\{\mathfrak{a} \hookrightarrow R \mid \mathfrak{a} \text{ ideal in } R \text{ such that } R/\mathfrak{a} \in \mathcal{C}\}.$$

Note that in the examples 4.3.4 and 4.3.5 the conjecture is fulfilled. Therefore if R is self-injective and artinian, then Conjecture 4.3.7 implies the Smashing Conjecture 2.4.7 for the stable module category $\underline{\mathrm{Mod}}(R)$.

5 Realizing smashing localizations of differential graded algebras

The results in this chapter are joint work with Birgit Huber [BH07] which have been achieved with substantial contributions of Bernhard Keller. We show that every smashing localization of a derived category of a differential graded algebra (or shortly dg algebra) can be realized by a morphism of dg algebras. More precisely if A is a dg algebra and $L\colon \mathcal{D}(A) \to \mathcal{D}(A)$ a smashing localization, we prove the existence of a dg algebra A_L with the property that $\mathcal{D}(A)/\ker L \simeq \mathcal{D}(A_L)$. Furthermore we show that there is a dg algebra A' quasi-isomorphic to A and a zigzag of dg algebra morphisms

$$A \xleftarrow{\sim} A' \to A_L$$

which identifies in cohomology with the algebra map $L\colon \mathcal{D}(A)(A,A)^* \to \mathcal{D}(A)(LA,LA)^*$. If the dg algebra A is cofibrant, then the algebra map $\mathcal{D}(A)(A,A)^* \to \mathcal{D}(A)(LA,LA)^*$ is induced by a morphism $A \to A_L$, and the quotient functor is naturally isomorphic to the left derived functor $- \otimes^{\mathbf{L}}_A A_L$.

As a direct consequence we are able to show that every smashing localization functor $L\colon \mathcal{T} \to \mathcal{T}$ on an algebraic triangulated category that is generated by one compact object is induced by a morphism of dg algebras.

As an application, in Section 5.6 we consider dg algebras with graded-commutative cohomology ring. For such a dg algebra A, we introduce the *localization of A at a prime $\mathfrak{p}$ in cohomology* and denote this dg algebra by $A_{\mathfrak{p}}$. It has the property $H^*(A_{\mathfrak{p}}) \cong (H^*A)_{\mathfrak{p}}$. Moreover we show that with this identification of graded algebras, the canonical morphism $H^*A \to (H^*A)_{\mathfrak{p}}$ is induced by a zigzag of dg algebra morphisms.

5.1 Differential graded algebras

In this section we review dg algebras and the derived category of a dg algebra. We refer the reader to [Kel94a, Sch04, Kra04] for more details.

Fix a commutative ring k.

Definition 5.1.1. A *dg algebra A* over k is a $\mathbb{Z}$-graded k-algebra together with a k-linear differential $d : A^n \to A^{n+1}$ that interacts with the multiplication according to the Leibniz rule

$$d(xa) = d(x)a + (-1)^n x d(a) \tag{2}$$

for all $x \in A^n$ and all $a \in A$. A *(right) dg A-module* is a $\mathbb{Z}$-graded module M with a differential $d\colon M^n \to M^{n+1}$ such that equation (2) holds for all $x \in M^n$ and all $a \in A$. A morphism of a dg algebra or of a dg module is a map of the underlying graded algebra or module that commutes with the differential.

Cohomology of a dg A-module, the notion of a quasi-isomorphism and the shift are defined on the underlying chain complexes. The cohomology of a dg algebra A is a graded ring and the cohomology of every dg A-module becomes a graded module over $H^*(A)$. Denote by $\mathrm{Mod}_{\mathrm{dg}}\, A$ the category of dg A-modules and by dga$/k$ the category of dg algebras over k. A homotopy between morphisms of dg modules is a map of the underlying graded modules that is also a chain homotopy. The homotopy category $\mathcal{K}(A)$ is the quotient of $\mathrm{Mod}_{\mathrm{dg}}\, A$ by the ideal of null-homotopic maps.

Example 5.1.2. If X, Y are dg A-modules, then the homomorphism complex $\mathcal{H}\mathrm{om}_A(X, Y)$ in degree n is defined by:

$$\mathcal{H}\mathrm{om}_A(X, Y)^n = \mathrm{Hom}_A(X, Y[n]).$$

The differential $d^n\colon \mathrm{Hom}_A(X, Y[n]) \to \mathrm{Hom}_A(X, Y[n+1])$ is defined to be

$$d^n(f) = d_Y \circ f - (-1)^n f \circ d_X.$$

There is an isomorphism

$$H^n\, \mathcal{H}\mathrm{om}_A(X, Y) \cong \mathrm{Hom}_{\mathcal{K}(A)}(X, Y[n]).$$

The endomorphism ring $\mathcal{E}\mathrm{nd}_A(X) = \mathcal{H}\mathrm{om}_A(X, X)$ is a dg algebra and $\mathcal{H}\mathrm{om}_A(X, Y)$ becomes a dg module over $\mathcal{E}\mathrm{nd}_A(X)$ by composition of graded maps.

Let A and B be dg algebras over k. A dg A-B-bimodule X is a graded (A, B)-bimodule which carries in addition a k-linear differential d of degree $+1$ satisfying

$$d(axb) = (da)xb + (-1)^p a(dx)b + (-1)^{p+q} ax(db)$$

for all $a \in A^p, x \in X^q, b \in B$. Fix an A-B-bimodule X. There is an internal Hom-functor and a tensor product that form an adjoint pair

$$\mathrm{Mod}_{\mathrm{dg}}\, A \underset{-\otimes_A X}{\overset{\mathcal{H}\mathrm{om}_B(X,-)}{\rightleftarrows}} \mathrm{Mod}_{\mathrm{dg}}\, B.$$

The *derived category* $\mathcal{D}(A)$ of A can then be defined as the localization of the homotopy category with respect to the quasi-isomorphisms. The homotopy category and the derived category are triangulated and the canonical functor $\mathcal{K}(A) \to \mathcal{D}(A)$ is exact.

The following well-known lemma collects basic results on the derived category of a dg algebra

Lemma 5.1.3. *[Kel94a, 3.1, 5.3] Let A be a dg algebra over k and M be a dg A-module.*

(i) The evaluation map

$$\mathcal{D}(A)(A, M)^* \to H^* M, \quad f \mapsto f(1),$$

is a natural isomorphism of graded H^*A-modules, where $\mathcal{D}(A)(A, X)^*$ becomes a graded H^*A-module via the isomorphism $\mathcal{D}(A)(A, A)^* \cong H^*A$.

(ii) A dg module is compact in $\mathcal{D}(A)$, if and only if it is contained in the full subcategory of perfect complexes $\mathcal{D}^{\mathrm{per}}(A)$, i.e., the smallest thick subcategory of $\mathcal{D}(A)$ containing A.

A consequence of this lemma is that $\mathcal{D}(A)$ is compactly generated by the dg algebra A.

Definition 5.1.4. A dg A-module M is *homotopically projective* if $\mathcal{K}(A)(M, N) = 0$ for all acyclic dg A-modules N. *Homotopically injective* modules are defined dually.

Up to homotopy the homotopically projective modules are the cofibrant objects in the model category $\mathrm{Mod}_{\mathrm{dg}} A$ described in [SS00, Theorem 4.1].

For every dg A-module M there is a homotopically projective module pM and a quasi-isomorphism $pM \to M$. Dually, there is a homotopically injective module iM and a quasi isomorphism $M \to iM$, see [Kel98, 8.2.4].

Let A and B be dg algebras over a commutative ring k and X be a dg A-B-bimodule. The internal Hom-functor and the tensor product descend to an adjoint pair in the derived category [Kel98, 8.2.6]

$$\mathcal{D}(A) \underset{-\otimes_A^{\mathbf{L}} X}{\overset{\mathbf{R}\,\mathcal{H}\mathrm{om}_{\mathbf{B}}(\mathbf{X},-)}{\rightleftarrows}} \mathcal{D}(B).$$

Here the derived tensor product maps a dg A-module M to $pM \otimes_A^{\mathbf{L}} X$.

5.2 Cofibrant differential graded algebras

The category of dg algebras dga/k over a commutative ring k admits a model category structure [SS00] in which the fibrations are the degree-wise surjective dg algebra morphisms and the weak equivalences equal the quasi-isomorphisms. Recall that a dg algebra is cofibrant if for any morphism of dg algebras $f\colon A \to C$ and every surjective quasi-isomorphism of dg algebras $g\colon B \to C$, there exists a lift $h\colon A \to B$. That is, we have a commutative diagram

$$\begin{tikzcd} & B \arrow[d, two heads, "g", "\sim"'] \\ A \arrow[ur, dotted, "h"] \arrow[r, "f"] & C. \end{tikzcd}$$

There is a cofibrant replacement functor $\mathrm{dga}/k \to \mathrm{dga}/k$ that sends a dg algebra A to A^c. Furthermore there is a surjective quasi-isomorphism $A^c \to A$.

Example 5.2.1. A class of cofibrant dg algebras arises from the tensor algebra functor $T\colon \mathrm{Ch}(k) \to \mathrm{Mod}_{\mathrm{dg}} A$ which is left adjoint to the functor that forgets the multiplicative structure and only remembers the chain complex over k. If C is a cofibrant chain complex, i.e., a homotopically projective dg module over the ground ring k, then the algebra $T(C)$ is cofibrant for all n.

We are interested in dg algebras with graded-commutative cohomology. The following example provides a class of such dg algebras that are in addition cofibrant.

Example 5.2.2. Let k be a field and V a positively graded k-vectorspace. The *tensor algebra* TV is defined by $TV = \bigoplus_{q=0}^{\infty} T^q V$, where $T^q V$ is the tensor product of q copies of V. It becomes an algebra via the multiplication $ab := a \otimes b$. Note that q is *not* the degree. The degree of $v_1 \otimes \cdots \otimes v_q \in T^q V$ is the sum of the degrees of the elements v_i. Let I be the ideal in TV generated by the elements $v \otimes w - (-1)^{nm} w \otimes v$, where $v \in V^n$ and $w \in V^m$. The *free graded-commutative algebra* ΛV is defined as TV/I.

A *Sullivan algebra* is a dg algebra $(\Lambda V, d)$, whose underlying graded algebra is the free graded-commutative algebra of a vectorspace V that is graded in positive degrees, and such that there is an increasing exhaustive filtration

$$V(0) \subset V(1) \subset \cdots \subset V(k) \subset \cdots \subset V$$

of graded subspaces such that $d|_{V(0)} = 0$ and $\operatorname{im}(d|_{V(k)}) \subset \Lambda V(k-1)$ for $k \geq 1$. The Sullivan algebras are cofibrant [FHT01, Lemma 12.4].

5.3 Cohomological $\mathfrak{p}$-Localization

Assume that A is a dg algebra such that H^*A is graded-commutative. Let $\mathfrak{p}$ be a graded prime ideal of H^*A, i.e., a prime ideal which is generated by homogeneous elements. Let $\mathcal{C}_\mathfrak{p}$ denote the full subcategory of objects X in $\mathcal{D}(A)$ such that $(H^*X)_\mathfrak{p} = 0$. In other words $\mathcal{C}_\mathfrak{p}$ is the kernel of the cohomological functor

$$(- \otimes_{H^*A} (H^*A)_\mathfrak{p}) \circ \mathcal{D}(A)(A, -)^*.$$

From Theorem 2.3.13, Remark 2.3.14 and Proposition 2.3.15 we deduce:

Corollary 5.3.1. *The localization*

$$\begin{tikzcd} \mathcal{D}(A) \arrow[r, shift left, "Q"'] & \mathcal{D}(A)/\mathcal{C}_\mathfrak{p} \arrow[l, shift left, "R"'] \end{tikzcd}$$

is smashing, and there is an isomorphism $r\colon \mathcal{D}(A)(A,A)^*_\mathfrak{p} \xrightarrow{\cong} \mathcal{D}(A)/\mathcal{C}_\mathfrak{p}(QA,QA)^*$ *of graded rings making the diagram*

$$\begin{tikzcd} \mathcal{D}(A)(A,A)^* \arrow[r, "\mathrm{can}"] \arrow[d, "Q"'] & \mathcal{D}(A)(A,A)^*_\mathfrak{p} \arrow[dl, "\cong", "r"'] \\ \mathcal{D}(A)/\mathcal{C}_\mathfrak{p}(QA,QA)^* & \end{tikzcd}$$

commutative. Furthermore the squares

$$\begin{tikzcd} \mathcal{D}(A) \arrow[r, "\mathcal{D}(A)(A,-)^*"] \arrow[d, "Q"'] & \mathrm{Mod}_{\mathrm{gr}}\, H^*A \arrow[d, "-\otimes_{H^*A}(H^*A)_\mathfrak{p}"] \\ \mathcal{D}(A)/\mathcal{C}_\mathfrak{p} \arrow[r, "\mathcal{D}(A)/\mathcal{C}_\mathfrak{p}(QA,-)^*"] & \mathrm{Mod}_{\mathrm{gr}}\, H^*A_\mathfrak{p} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}(A) \arrow[r, "\mathcal{D}(A)(A,-)^*"] & \mathrm{Mod}_{\mathrm{gr}}\, H^*A \\ \mathcal{D}(A)/\mathcal{C}_\mathfrak{p} \arrow[u, "R"'] \arrow[r, "\mathcal{D}(A)/\mathcal{C}_\mathfrak{p}(QA,-)^*"] & \mathrm{Mod}_{\mathrm{gr}}\, H^*A_\mathfrak{p} \arrow[u, "inc"'] \end{tikzcd}$$

commute up to natural isomorphism. □

5.4 Smashing localizations of the derived category of a dg algebra

Let A be a differential graded algebra over some commutative ring k and let

$$L\colon \mathcal{D}(A) \to \mathcal{D}(A)$$

be a smashing localization. If $\mathcal{C}$ denotes the category of L-acyclic objects, then we have an adjoint pair of functors

$$\begin{tikzcd} \mathcal{D}(A) \arrow[r, shift left, "Q"'] & \mathcal{D}(A)/\mathcal{C} \arrow[l, shift left, "R"'] \end{tikzcd}$$

satisfying $R \circ Q = L$. The right adjoint R is fully faithful and commutes with arbitrary direct sums.

Our first aim is to write the quotient category $\mathcal{D}(A)/\mathcal{C}$ as derived category of a differential graded algebra A_L. Then we construct a zigzag of dg algebra morphisms $A \xleftarrow{\sim} A' \to A_L$ which induces the algebra morphism

$$\mathcal{D}(A)(A,A)^* \to \mathcal{D}(A)(LA,LA)^*, \quad f \mapsto Lf,$$

in cohomology. For this purpose we identify the functors

$$H^*\colon \mathcal{D}(A) \;\to\; \operatorname{Mod}_{\mathrm{gr}} H^*A$$

and

$$\mathcal{D}(A)(A,-)^*\colon \mathcal{D}(A) \to \operatorname{Mod}_{\mathrm{gr}} H^*A$$

by the natural evaluation isomorphism $\mathcal{D}(A)(A,X)^* \to H^*X,\ f \mapsto f(1)$.

The following lemma which we learned from Dave Benson is the key to our construction.

Lemma 5.4.1. *Let A, B be dg algebras and M be a dg (B,A)-bimodule. Let $\alpha\colon A \to M$ and $\beta\colon B \to M$ be maps of dg modules which satisfy $\alpha(1) = \beta(1)$. Then*

$$X = \{(a,b) \in A \times B \mid \alpha(a) = \beta(b)\}$$

is a dg algebra with differential $d_X = (d_A, d_B)$ and the projections p_1, p_2 in the pullback diagram

$$\begin{tikzcd} X \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & M \end{tikzcd}$$

are dg algebra morphisms. If β is a surjective quasi-isomorphism, then the diagram induces a pullback diagram in cohomology.

Proof. The first assertions are immediately checked. For the last one we show that $H^*X = \{(\overline{a},\overline{b}) \in H^*A \times H^*B \mid H^*\alpha(\overline{a}) = H^*\beta(\overline{b})\}$.

A pair $(\overline{a},\overline{b}) \in H^*X$ trivially satisfies the property $H^*\alpha(\overline{a}) = H^*\beta(\overline{b})$ and consequently, the inclusion $\subseteq$ is always fulfilled. For the other inclusion we need to assume that β is a surjective quasi-isomorphism. Let $(\overline{a},\overline{b}) \in H^*A \times H^*B$ such that $H^*\alpha(\overline{a}) = H^*\beta(\overline{b})$. We choose representing cocycles a of $\overline{a}$ and b of $\overline{b}$. Then $\alpha(a) - \beta(b) = m$ for some coboundary $m \in M$. Since β is a surjective quasi-isomorphism, there is a coboundary $b' \in B$ such that $\beta(b') = m$. Hence the pair $(a, b+b')$ satisfies $\alpha(a) = \beta(b+b')$ and thus $(\overline{a},\overline{b}) = (\overline{a},\overline{b+b'}) \in H^*X$. □

The following lemmas ensure that the cohomology of the dg algebra A_L which we construct below is independent of all choices that we will make.

Recall that $\operatorname{Mod}_{\mathrm{dg}} A$ is an exact category (in the sense of Quillen, see [Qui73]) with respect to the exact sequences of dg A-modules

$$0 \to X \to Y \to Z \to 0$$

which are split considered as sequences of graded A-module maps. Furthermore $\operatorname{Mod}_{\mathrm{dg}} A$ is a Frobenius category. That is, there are enough projective and injective modules in $\operatorname{Mod}_{\mathrm{dg}} A$ and the projective and injective modules coincide [Kel98, 8.3.3]. Since the maps factoring through an injective object are precisely the null-homotopic maps, the associated stable category coincides with the homotopy category $\mathcal{K}(A)$. We refer to [Kel98, Sect. 8.2.3] and [Kel94a, Sect. 2.2] for more details.

For a given map $X \to Y$ there is an injective envelope I of X such that the canonical map $X \to Y \oplus I$ represents the same map in $\mathcal{K}(A)$ and is a split monomorphism. Hence we obtain

Lemma 5.4.2. *Let $\varphi\colon X \to Y$ be any morphism in $\mathcal{K}(A)$. Then φ can be represented by a morphism in $\operatorname{Mod}_{\mathrm{dg}} A$ which is a split monomorphism in the category of graded A-modules.* □

Lemma 5.4.3. *Let X, Y be dg A-modules and let $\nu\colon X \to Y$ be an isomorphism in $\mathcal{K}(A)$.*

(1) Let $X \to I(X)$ denote the injective hull of X in the Frobenius category $\operatorname{Mod}_{\mathrm{dg}} A$. There exists a dg algebra S and a zigzag of quasi-isomorphisms of dg algebras

$$\mathcal{E}\mathrm{nd}_A(X) \xleftarrow{\sim} S \xrightarrow{\sim} \mathcal{E}\mathrm{nd}_A(Y \oplus I(X)).$$

(2) Let I be any injective module in the Frobenius category $\operatorname{Mod}_{\mathrm{dg}} A$. There is a dg algebra T and a zigzag of quasi-isomorphisms of dg algebras

$$\mathcal{E}\mathrm{nd}_A(Y) \xleftarrow{\sim} T \xrightarrow{\sim} \mathcal{E}\mathrm{nd}_A(Y \oplus I).$$

(3) There exists a zigzag of quasi-isomorphisms of dg algebras from $\mathcal{E}\mathrm{nd}_A(X)$ to $\mathcal{E}\mathrm{nd}_A(Y)$.

Proof. (1) By Lemma 5.4.2 we can choose a representing dg A-module map

$$\bar{\nu}\colon X \to Y \oplus I(X)$$

of $\nu \in \mathcal{K}(A)(X, Y)$ which is split as map of graded A-modules. Hence the map

$$\bar{\nu}^*\colon \mathcal{E}\mathrm{nd}_A(Y \oplus I(X)) \to \mathcal{H}\mathrm{om}_A(X, Y \oplus I(X)), \quad f \mapsto f \circ \bar{\nu},$$

is surjective. Applying Lemma 5.4.1, the pullback diagram

$$\begin{tikzcd} S \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{E}\mathrm{nd}_A(Y \oplus I(X)) \arrow[d, two heads, "\bar{\nu}^*"', "\sim"] \\ \mathcal{E}\mathrm{nd}_A(X) \arrow[r, "\bar{\nu}_*", "\sim"'] & \mathcal{H}\mathrm{om}_A(X, Y \oplus I(X)) \end{tikzcd}$$

yields the claim.

(2) The dg A-module map $\iota\colon Y \xrightarrow{[\mathrm{id}\ 0]} Y \oplus I$ is obviously a split monomorphism inducing id_Y in the homotopy category. Hence we obtain a pullback diagram

$$\begin{tikzcd} T \arrow[r, "p_2", "\sim"'] \arrow[d, "p_1"', "\sim"] & \mathcal{E}\mathrm{nd}_A(Y \oplus I) \arrow[d, two heads, "\iota^*"', "\sim"] \\ \mathcal{E}\mathrm{nd}_A(Y) \arrow[r, "\iota_*", "\sim"'] & \mathcal{H}\mathrm{om}_A(Y, Y \oplus I) \end{tikzcd}$$

yielding the claim.

(3) is a trivial consequence of (1) and (2). □

The proof of the following lemma is immediate.

Lemma 5.4.4. *The object QA is a compact generator of $\mathcal{D}(A)/\mathcal{C}$.* $\square$

Fix a homotopically projective replacement of $RQA \in \mathcal{D}(A)$. By abuse of notation we denote the replacement also by RQA.

Proposition 5.4.5. *The functor $\mathbf{R}\mathcal{H}\mathrm{om}(RQA, R-)\colon \mathcal{D}(A)/\mathcal{C} \to \mathcal{D}(\mathcal{E}\mathrm{nd}_A(RQA))$ is an equivalence of triangulated categories.*

Proof. Note that $\mathbf{R}\mathcal{H}\mathrm{om}(RQA, R-)$ preserves arbitrary direct sums because for any family $(X_i)_{i\in I}$ in $\mathcal{D}(A)/\mathcal{C}$, the map

$$\coprod_{i\in I} \mathbf{R}\mathcal{H}\mathrm{om}(RQA, RX_i) \to \mathbf{R}\mathcal{H}\mathrm{om}(RQA, R\coprod_{i\in I} X_i)$$

is a quasi-isomorphism.

Moreover the functor $\mathbf{R}\mathcal{H}\mathrm{om}(RQA, R-)$ maps the compact generator QA of $\mathcal{D}(A)/\mathcal{C}$ to $\mathcal{E}\mathrm{nd}_A(RQA)$ which compactly generates $\mathcal{D}(\mathcal{E}\mathrm{nd}_A(RQA))$. Finally the map

$$\mathcal{D}(A)/\mathcal{C}(QA, QA[n]) \xrightarrow{\mathbf{R}\mathcal{H}\mathrm{om}(RQA,R-)} \mathcal{D}(\mathcal{E}\mathrm{nd}_A(RQA))(\mathcal{E}\mathrm{nd}_A(RQA), \mathcal{E}\mathrm{nd}_A(RQA)[n])$$

is an isomorphism for all $n \in \mathbb{Z}$ since RQA being homotopically projective implies that the diagram

$$\begin{tikzcd}
\mathcal{D}(A)/\mathcal{C}(QA, QA[n]) \arrow[r, "\mathbf{R}\mathcal{H}\mathrm{om}(RQA,R-)"] \arrow[d, "R"', "\cong"] & \mathcal{D}(\mathcal{E}\mathrm{nd}_A(RQA))(\mathcal{E}\mathrm{nd}_A(RQA), \mathcal{E}\mathrm{nd}_A(RQA)[n]) \\
\mathcal{D}(A)(RQA, RQA[n]) \arrow[r, "\cong"'] & H^n(\mathcal{E}\mathrm{nd}_A(RQA)) \arrow[u, "\cong"]
\end{tikzcd}$$

is commutative. The claim now follows from a version of 'Beilinson's Lemma' [Bei78] which is stated in [Sch04, Prop. 3.10]. $\square$

Hence we have shown that the quotient category $\mathcal{D}(A)/\mathcal{C}$ is equivalent to the derived category of the dg algebra $\mathcal{E}\mathrm{nd}_A(LA)$, where LA was chosen to be homotopically projective. Note that Lemma 5.4.3 provides a zigzag of quasi-isomorphisms between the endomorphism dg algebras of two different homotopically projective replacements of an object in $\mathcal{D}(A)$.

In order to construct a zigzag $A \xleftarrow{\sim} A' \to \mathcal{E}\mathrm{nd}_A(LA)$ of dg algebra morphisms inducing

$$\mathcal{D}(A)(A, A)^* \to \mathcal{D}(A)(LA, LA)^*$$

in cohomology, we need to make another choice for the dg A-module representing LA.

Let $\eta\colon \mathrm{id} \to RQ$ be the unit and $\varepsilon\colon QR \to \mathrm{id}$ the counit of the adjunction

$$\mathcal{D}(A) \underset{Q}{\overset{R}{\leftrightarrows}} \mathcal{D}(A)/\mathcal{C}.$$

Since A is homotopically projective, we can view η_A as an element in $\mathcal{K}(A)(A, RQA)$.

Lemma 5.4.6. *For any map* $\bar{\eta}_A$ *in* $\operatorname{Mod}_{\mathrm{dg}} A$ *that represents* $\eta_A \in \mathcal{K}(A)(A, RQA)$ *and any dg* A*-module* M*, the map*

$$\bar{\eta}_A^*\colon \mathcal{H}\mathrm{om}_A(RQA, RQM) \to \mathcal{H}\mathrm{om}_A(A, RQM), \quad f \mapsto f \circ \bar{\eta}_A,$$

is a quasi-isomorphism.

Proof. Since R is fully faithful, the usual adjunction isomorphism (see [ML98, Ch. IV.1]) gives rise to the mutually inverse maps

$$H^n(\bar{\eta}_A^*)\colon \mathcal{D}(A)(RQA, RQM[n]) \to \mathcal{D}(A)(A, RQM[n]), \quad f \mapsto f \circ \eta_A,$$

and

$$\mathcal{D}(A)(A, RQM[n]) \to \mathcal{D}(A)(RQA, RQM[n]), \quad g \mapsto R(\varepsilon_{QA}) \circ RQ(g). \qquad \square$$

Remark 5.4.7. By Lemma 5.4.2 we may represent $\eta_A\colon A \to LA$ by a monomorphism of dg A-modules

$$\bar{\eta}_A\colon A \to \widehat{LA},$$

which is split as map of graded A-modules. Remember that $\widehat{LA} = LA \oplus I(A)$, where $A \to I(A)$ is the injective hull of A in the Frobenius category $\operatorname{Mod}_{\mathrm{dg}} A$, and that LA was already chosen to be homotopically projective. By Lemma 5.4.3 we have a zigzag of quasi-isomorphisms

$$\mathcal{E}\mathrm{nd}_A(LA) \xleftarrow{\sim} T \xrightarrow{\sim} \mathcal{E}\mathrm{nd}_A(\widehat{LA}).$$

We define the dg algebra A_L to be $\mathcal{E}\mathrm{nd}_A(\widehat{LA})$. By abuse of notation we write $A_L = \mathcal{E}\mathrm{nd}_A(LA)$. Note that from Lemma 5.4.3 and Proposition 5.4.5 it follows that

$$\mathcal{D}(A_L) \simeq \mathcal{D}(A)/\mathcal{C}.$$

Theorem 5.4.8. *The algebra map*

$$\mathcal{D}(A)(A, A)^* \to \mathcal{D}(A)(LA, LA)^*, \quad f \mapsto L(f),$$

is induced by a zigzag of dg algebra maps

$$A \xleftarrow{\sim} A' \xrightarrow{\varphi} A_L.$$

That is, there exists a dg algebra A' *quasi-isomorphic to* A *and a morphism of dg algebras* $\varphi\colon A' \to A_L$ *such that we have the commutative diagram*

$$\begin{tikzcd} H^*A' \arrow[d, "\cong"'] \arrow[dr, "H^*\varphi"] & \\ \mathcal{D}(A)(A, A)^* \arrow[r, "L"] & \mathcal{D}(A)(LA, LA)^* \end{tikzcd}$$

in cohomology.

Proof. We identify the dg algebras $\mathcal{E}\mathrm{nd}_A(A)$ and A through the isomorphism given by evaluation at 1. Let

$$\begin{tikzcd} A' \arrow[r, "p_2"] \arrow[d, "p_1"', "\sim"] & A_L \arrow[d, two heads, "\sim"', "\bar{\eta}_A^*"] \\ \mathcal{E}\mathrm{nd}_A(A) \arrow[r, "\bar{\eta}_{A*}"] & \mathcal{H}\mathrm{om}_A(A, LA) \end{tikzcd}$$

be a pullback diagram.

The map $\bar{\eta}_{A*}$ is a quasi-isomorphism (Lemma 5.4.6) and surjective since $\bar{\eta}_A$ is a split monomorphism of graded A-modules (Remark 5.4.7). We infer from Lemma 5.4.1 that A' is a dg algebra quasi-isomorphic to A, and we set $\varphi = p_2$.

In cohomology we obtain the commutative diagram

$$\begin{tikzcd} H^* A' \arrow[r, "H^*(p_2)"] \arrow[d, "H^*(p_1)"', "\cong"] & H^* \mathcal{E}\mathrm{nd}_A(LA) \arrow[d, "\cong"', "H^*(\bar{\eta}_A^*)"] \\ H^* \mathcal{E}\mathrm{nd}_A(A) \arrow[r, "H^*(\bar{\eta}_{A*})"] & H^*(\mathcal{H}\mathrm{om}_A(A, LA)) \end{tikzcd}$$

and thus it only remains to show that the composition

$$H^*(\bar{\eta}_A^*)^{-1} \circ H^*(\bar{\eta}_{A*})$$

identifies with the map

$$\mathcal{D}(A)(A, A)^* \to \mathcal{D}(A)(LA, LA)^*, \quad f \mapsto L(f).$$

In fact, for $f \in \mathcal{D}(A)(A, A)^*$ we have

$$\big(H^*(\bar{\eta}_A^*)^{-1} \circ H^*(\bar{\eta}_{A*})\big)(f) = R(\varepsilon_{QA}) \circ RQ(\eta_A) \circ RQ(f) \in \mathcal{D}(A)(RQA, RQA).$$

As it is well-known that $\varepsilon_{QA} \circ Q(\eta_A) \cong \mathrm{id}_{QA}$ (see [ML98, Ch. IV.1]), the claim follows. $\square$

If we assume in addition that A is a cofibrant dg algebra (see Section 5.2), then the map $p_1 \colon A' \to A$ in the pullback diagram above splits. In particular the algebra map $L \colon \mathcal{D}(A)(A, A)^* \to \mathcal{D}(A)(LA, LA)^*$ is not only induced by a zigzag of dg algebra maps, but by a *morphism* $A \to A_L$.

Corollary 5.4.9. *Let A be a cofibrant dg algebra. The algebra morphism*

$$\mathcal{D}(A)(A, A)^* \to \mathcal{D}(A)(LA, LA)^*, \quad f \mapsto L(f),$$

lifts to a dg algebra morphism $\psi \colon A \to A_L$. $\square$

Now our aim is to show that if A is cofibrant, then we can identify the functors $Q \colon \mathcal{D}(A) \to \mathcal{D}(A)/\mathcal{C} \simeq \mathcal{D}(A_L)$ and $- \otimes_A^{\mathbf{L}} A_L \colon \mathcal{D}(A) \to \mathcal{D}(A_L)$, where A_L is a dg (A, A_L)-bimodule through the morphism $\psi \colon A \to A_L$.

Lemma 5.4.10. *There exists a natural transformation*

$$\lambda \colon \mathbf{R}\mathcal{H}\mathrm{om}_A(A, -) \to \mathbf{R}\mathcal{H}\mathrm{om}_A(LA, L-)$$

in $\mathcal{D}(A)$ which commutes with the suspension functor. For every $M \in \mathcal{D}(A)$, λ_M induces the map

$$\mathcal{D}(A)(A, M) \to \mathcal{D}(A)(LA, LM), \quad f \mapsto Lf,$$

in cohomology.

Proof. By Lemma 5.4.6 the adjunction unit $\eta_A\colon A \to LA$ induces a natural isomorphism $\mathbf{R}\mathcal{H}\mathrm{om}_A(\eta_A, LM)$. Therefore we can define the morphism λ_M to be the composition

$$\mathbf{R}\mathcal{H}\mathrm{om}_A(A, M) \xrightarrow{\mathbf{R}\mathcal{H}\mathrm{om}(A,\eta_M)} \mathbf{R}\mathcal{H}\mathrm{om}_A(A, LM) \xrightarrow[\cong]{\mathbf{R}\mathcal{H}\mathrm{om}(\eta_A,LM)^{-1}} \mathbf{R}\mathcal{H}\mathrm{om}_A(LA, LM),$$

which obviously induces $L\colon \mathcal{D}(A)(A, M)^* \to \mathcal{D}(A)(LA, LM)$ in cohomology. The naturality of λ_M follows from the naturality of $\mathbf{R}\mathcal{H}\mathrm{om}(A, \eta_M)$ and $\mathbf{R}\mathcal{H}\mathrm{om}(\eta_A, LM)$.

The unit η of the adjoint pair (Q, R) commutes with the suspension functor [1], hence so does $\mathbf{R}\mathcal{H}\mathrm{om}(A, \eta_M)$. Since $\mathbf{R}\mathcal{H}\mathrm{om}(\eta_A, LM)$ commutes with [1], we conclude that $\lambda \circ [1] \cong [1] \circ \lambda$. □

Note that if A is cofibrant, then λ_A equals the dg algebra morphism $\psi\colon A \to A_L$ constructed in Corollary 5.4.9. In addition $\mathbf{R}\mathcal{H}\mathrm{om}_A(LA, LM)$ becomes an object in $\mathcal{D}(A)$ through the dg algebra morphism ψ.

Proposition 5.4.11. *Suppose that A is a cofibrant dg algebra. Then the diagram*

$$\begin{tikzcd} \mathcal{D}(A) \arrow[r, "-\otimes^{\mathbf{L}}_A A_L"] \arrow[d, "Q"'] & \mathcal{D}(A_L) \\ \mathcal{D}(A)/\mathcal{C} \arrow[ur, "\simeq", "\mathbf{R}\mathcal{H}\mathrm{om}_A(RQA,R-)"'] & \end{tikzcd}$$

commutes up to natural isomorphism.

Proof. We show that the functors $\mathbf{R}\mathcal{H}\mathrm{om}_A(LA, L-)$ and $-\otimes^{\mathbf{L}}_A A_L$ are naturally isomorphic. A natural transformation

$$\tau\colon\ -\otimes^{\mathbf{L}}_A A_L \longrightarrow \mathbf{R}\mathcal{H}\mathrm{om}_A(LA, L-)$$

is given as the composition of the three natural maps in the diagram

$$\begin{tikzcd} M \otimes^{\mathbf{L}}_A A_L \arrow[r, dotted, "\tau_M"] \arrow[d, "\mathrm{can}_M \otimes^{\mathbf{L}}_A A_L"', "\cong"] & \mathbf{R}\mathcal{H}\mathrm{om}_A(LA, LM) \\ \mathbf{R}\mathcal{H}\mathrm{om}_A(A, M) \otimes^{\mathbf{L}}_A A_L \arrow[r, "\lambda_M \otimes^{\mathbf{L}}_A A_L"] & \mathbf{R}\mathcal{H}\mathrm{om}_A(LA, LM) \otimes^{\mathbf{L}}_A A_L, \arrow[u, "\nu_M"'] \end{tikzcd}$$

where can_M is the canonical identification and ν_M is defined by

$$\begin{aligned} \nu_M\colon \mathbf{R}\mathcal{H}\mathrm{om}_A(LA, LM) \otimes^{\mathbf{L}}_A \mathcal{E}\mathrm{nd}_A(LA) &\to \mathbf{R}\mathcal{H}\mathrm{om}_A(LA, LM). \\ f \otimes g &\mapsto f \circ g \end{aligned}$$

In order to prove that τ is an isomorphism, one checks that the full subcategory $\{M \in \mathcal{D}(A) \mid \tau_M \text{ is an isomorphism}\}$ of $\mathcal{D}(A)$ is a localizing subcategory containing A. Note that τ commutes with the suspension functor since this holds for λ by Lemma 5.4.10. □

5.5 Smashing subcategories of algebraic triangulated categories

In this paragraph we will use a result of Keller to expand Theorem 5.4.8 to a broader class of triangulated categories.

Definition 5.5.1. A triangulated category is *algebraic* if it is triangle equivalent to the stable category of a Frobenius category.

The derived category of a ring or a dg algebra, the stable module category of a Frobenius ring and the homotopy category of a ring or a dg algebra are algebraic. Triangulated categories that are non-algebraic arise from topology, for instance the stable homotopy category of Example 2.1.5.

Theorem 5.5.2. *[Kel94a, 4.3] If $\mathcal{T}$ is an algebraic triangulated category with small coproducts that is generated by a compact object, then there is a dg algebra A such that $\mathcal{T}$ is triangle equivalent to $\mathcal{D}(A)$.*

Using this theorem we obtain:

Corollary 5.5.3. *Let $\mathcal{T}$ be as in Theorem 5.5.2. Let $L\colon \mathcal{T} \to \mathcal{T}$ be a smashing localization and let $Q\colon \mathcal{T} \to \mathcal{T}/\ker(L)$ be the corresponding quotient functor. There is a dg algebra A and a morphism of dg algebras $A \to A_L$ such that Q is induced by $A \to A_L$.*

Proof. By Theorem 5.5.2 there is a dg algebra A such that $\mathcal{T} \simeq \mathcal{D}(A)$. Without loss of generality we may assume that A is cofibrant since there is a quasi isomorphism from the cofibrant replacement of A to A. Therefore by Corollary 5.4.9 we obtain a morphism $A \to A_L$. By Proposition 5.4.11 we know that $Q \cong - \otimes^{\mathbf{L}}_A A_L$. Therefore, the morphism $A \to A_L$ induces Q. □

5.6 The $\mathfrak{p}$-localization of a dg algebra

Let A be a dg algebra over a commutative ring k and assume throughout this paragraph that the cohomology algebra H^*A is graded-commutative. We fix a prime ideal $\mathfrak{p}$ of H^*A that is generated by homogeneous elements. By $\mathcal{C}_\mathfrak{p}$ we denote the full subcategory of objects M in $\mathcal{D}(A)$ such that $(H^*M)_\mathfrak{p} = 0$. The localization $L_\mathfrak{p}\colon \mathcal{D}(A) \to \mathcal{D}(A)$, given by the adjoint pair

$$\mathcal{D}(A) \underset{Q}{\overset{R}{\rightleftarrows}} \mathcal{D}(A)/\mathcal{C}_\mathfrak{p},$$

is smashing by Corollary 5.3.1. Now we apply the results of Section 5.4 to this special case. We define

$$A_\mathfrak{p} = A_{L_\mathfrak{p}},$$

and we call $A_\mathfrak{p}$ *localization of A at a prime $\mathfrak{p}$ in cohomology.*

From Lemma 5.4.3 and Proposition 5.4.5 we infer that $\mathcal{D}(A)/\mathcal{C}_\mathfrak{p} \simeq \mathcal{D}(A_\mathfrak{p})$. For this special smashing localization we have

Theorem 5.6.1. *Let A be a dg algebra over a commutative ring k such that H^*A is graded-commutative and let $\mathfrak{p}$ be a graded prime ideal of H^*A. The dg algebra $A_\mathfrak{p}$ has the property $H^*(A_\mathfrak{p}) \cong (H^*A)_\mathfrak{p}$. Moreover with this identification of graded algebras, the canonical map*

$$\mathrm{can}\colon H^*A \to (H^*A)_\mathfrak{p}$$

is induced by a zigzag of dg algebra maps

$$A \xleftarrow{\sim} A' \xrightarrow{\varphi} A_{\mathfrak{p}}.$$

That is, we have a commutative diagram

$$\begin{tikzcd} H^*A' \arrow[d, "\cong"'] \arrow[drr, "H^*\varphi"] & & \\ H^*A \arrow[r, "\mathrm{can}"'] & (H^*A)_{\mathfrak{p}} \arrow[r, "\cong"'] & H^*(A_{\mathfrak{p}}). \end{tikzcd}$$

Proof. Since $\mathcal{D}(A)(A,A)_{\mathfrak{p}} \cong \mathcal{D}(A)(L_{\mathfrak{p}}A, L_{\mathfrak{p}}A)$ by Corollary 5.3.1, the dg algebra $A_{\mathfrak{p}}$ satisfies $H^*(A_{\mathfrak{p}}) \cong (H^*A)_{\mathfrak{p}}$. Theorem 5.4.8 shows that the zigzag $A \xleftarrow{\sim} A' \xrightarrow{\varphi} A_{\mathfrak{p}}$ induces the map

$$L_{\mathfrak{p}} \colon \mathcal{D}(A)(A,A)^* \to \mathcal{D}(A)(L_{\mathfrak{p}}A, L_{\mathfrak{p}}A)^*, \quad f \mapsto L_{\mathfrak{p}}(f)$$

in cohomology. But we may identify the algebra maps $\mathrm{can} \colon H^*A \to H^*(A_{\mathfrak{p}})$ and $L_{\mathfrak{p}}$ by Corollary 5.3.1. □

The following result is an immediate consequence of Corollary 5.4.9 and Theorem 5.6.1.

Corollary 5.6.2. *Let A be a cofibrant dg algebra such that H^*A is graded-commutative and let $\mathfrak{p}$ be a graded prime ideal of H^*A. Then the canonical algebra morphism* $\mathrm{can} \colon H^*A \to (H^*A)_{\mathfrak{p}}$ *lifts to a dg algebra morphism*

$$\psi \colon A \longrightarrow A_{\mathfrak{p}}.$$ □

A class of cofibrant dg algebras with graded-commutative cohomology are the Sullivan algebras introduced in Example 5.2.2.

Let A be a dg algebra over a commutative ring k such that H^*A is graded-commutative and $\mathfrak{p}$ be a prime ideal in H^*A that is generated by homogeneous elements.

The cohomology of the dg algebra $A_{\mathfrak{p}}$ satisfies a universal property since $H^*(A_{\mathfrak{p}})$ is isomorphic to the ring of fractions $S^{-1}(H^*A) = (H^*A)_{\mathfrak{p}}$, where S is the subset of homogeneous elements in $H^*A \setminus \mathfrak{p}$. If $\beta \colon A \to B$ is a morphism of dg algebras such that $H^*\beta$ makes S invertible, then $H^*\beta$ factors uniquely over the canonical morphism $\mathrm{can} \colon H^*A \to (H^*A)_{\mathfrak{p}}$.

Without loss of generality we assume from now on that A is cofibrant. Then the map $\mathrm{can} \colon H^*A \to (H^*A)_{\mathfrak{p}}$ is induced by a morphism of dg algebras $\psi \colon A \to A_{\mathfrak{p}}$, and the universal property yields a unique algebra morphism $g \colon H^*(A_{\mathfrak{p}}) \to H^*B$ making the following diagram commute:

$$\begin{tikzcd} H^*A \arrow[r, "H^*\beta"] \arrow[d, "H^*\psi"'] & H^*B \\ H^*(A_{\mathfrak{p}}) \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

The dg algebra morphisms $\beta \colon A \to B$ and $\psi \colon A \to A_{\mathfrak{p}}$ give rise to functors

$$F_\beta \colon \mathcal{D}(A) \xrightarrow{-\otimes^{\mathbf{L}}_A B} \mathcal{D}(B) \quad \text{and} \quad F_\psi \colon \mathcal{D}(A) \xrightarrow{-\otimes^{\mathbf{L}}_A A_{\mathfrak{p}}} \mathcal{D}(A_{\mathfrak{p}}).$$

Now we prove a universal property on the level of derived categories.

Proposition 5.6.3. *There is a unique functor $G\colon \mathcal{D}(A_{\mathfrak{p}}) \to \mathcal{D}(B)$ making the following diagram commute:*

$$\begin{tikzcd} \mathcal{D}(A) \arrow[r, "F_\beta"] \arrow[d, "F_\psi"'] & \mathcal{D}(B) \\ \mathcal{D}(A_{\mathfrak{p}}) \arrow[ur, dotted, "G"'] & \end{tikzcd}$$

Proof. We first note that by Proposition 5.4.11 the functor F_ψ is nothing but the quotient functor $Q\colon \mathcal{D}(A) \to \mathcal{D}(A)/\mathcal{C}_{\mathfrak{p}}$ composed with the equivalence $\mathcal{D}(A)/\mathcal{C}_{\mathfrak{p}} \simeq \mathcal{D}(A_{\mathfrak{p}})$. Thus we can use the universal property of Q and only need to show that $F_\beta(\mathcal{C}_{\mathfrak{p}}) = 0$.

In Proposition 2.3.16 we have shown that

$$\mathcal{M} = \{\mathrm{cone}(\sigma) \mid \sigma\colon A \to A[n] \in S\}$$

is a set of compact generators of $\mathcal{C}_{\mathfrak{p}}$ and thus it suffices to check that F_β vanishes on $\mathcal{M}$.

Any element of $\mathcal{M}$ fits into an exact triangle

$$A \xrightarrow{x\cdot} A[n] \to \mathrm{cone}(x\cdot) \to A[1]$$

in $\mathcal{D}(A)$, where $x\cdot$ denotes multiplication with an element $x \in A$ whose cohomology H^*x belongs to S. Applying the functor F_β to this triangle we obtain a triangle in $\mathcal{D}(B)$ naturally isomorphic to

$$B \xrightarrow{\beta(x)\cdot} B[n] \to F_\beta(\mathrm{cone}(x\cdot)) \to B[1].$$

Since $H^*\beta(x)$ is invertible, we infer that $F_\beta(\mathrm{cone}(x\cdot))$ is contractible and consequently, the object $F_\beta(\mathrm{cone}(x\cdot))$ is zero in $\mathcal{D}(B)$. □

Since $\mathcal{C}_{\mathfrak{p}}$ is generated by compact elements, the quotient functor $Q\colon \mathcal{D}(A) \to \mathcal{D}(A)/\mathcal{C}_{\mathfrak{p}}$ gives rise to a quotient functor $\mathcal{D}^{\mathrm{per}}(A) \to \mathcal{D}^{\mathrm{per}}(A)/\mathcal{C}_{\mathfrak{p}}^{\mathrm{per}}$, where $\mathcal{C}_{\mathfrak{p}}^{\mathrm{per}} = \mathcal{C}_{\mathfrak{p}} \cap \mathcal{D}^{\mathrm{per}}(A)$. Furthermore this quotient functor identifies with the functor

$$\mathcal{D}^{\mathrm{per}}(A) \xrightarrow{-\otimes^{\mathbf{L}}_A A_{\mathfrak{p}}} \mathcal{D}^{\mathrm{per}}(A_{\mathfrak{p}}).$$

This proves

Corollary 5.6.4. *There is a unique functor $G\colon \mathcal{D}^{\mathrm{per}}(A_{\mathfrak{p}}) \to \mathcal{D}^{\mathrm{per}}(B)$ which makes the following diagram commute:*

$$\begin{tikzcd} \mathcal{D}^{\mathrm{per}}(A) \arrow[r, "F_\beta"] \arrow[d, "F_\psi"'] & \mathcal{D}^{\mathrm{per}}(B) \\ \mathcal{D}^{\mathrm{per}}(A_{\mathfrak{p}}) \arrow[ur, dotted, "G"'] & \end{tikzcd}$$

□

Remark 5.6.5. The discussion above raises the question whether the functor $G\colon \mathcal{D}(A_{\mathfrak{p}}) \to \mathcal{D}(B)$ and with it the algebra map $g\colon H^*(A_{\mathfrak{p}}) \to H^*B$ can be lifted to a zigzag of dg algebra morphisms. Our construction in Section 5.4 does not apply since in general, we cannot expect that G is a smashing localization. It remains to enlighten the relation of our construction with DG quotients, which have a universal property and were introduced by Drinfeld [Dri04].

There is also a construction by Toën [Toe] (see also [Kel06]) which seems to be related: Let dgcat_k be the category of small dg categories over a commutative ring k. The localization of dgcat_k with respect to the quasi-equivalences is denoted by Hqe. If $\mathcal{A}$ is a small dg category and if S is a set of morphisms in $H^0(\mathcal{A})$, then a morphism $F\colon \mathcal{A} \to \mathcal{B}$ in Hqe is said to make S invertible if the induced functor $H^0(\mathcal{A}) \to H^0(\mathcal{B})$ takes each $s \in S$ to an isomorphism. Toën constructs a morphism $Q\colon \mathcal{A} \to \mathcal{A}[S^{-1}]$ in Hqe which makes S invertible. This morphism has a universal property: Each morphism in Hqe making S invertible factors uniquely through Q.

However if $\mathcal{A}$ is a dg algebra, viewed as dg category with a single object, then the object $\mathcal{A}[S^{-1}]$ is in general not a dg algebra, but a dg category with more than one object.

6 Thick subcategories of the derived category of a hereditary algebra

A full subcategory of a triangulated category is *thick* if it is closed under forming suspensions, triangles and retracts. Thick subcategories were studied in stable homotopy theory, commutative algebra and representation theory of groups. The first classification theorem was obtained by Hopkins and Smith for the p-local finite stable homotopy category [HS98]. They showed that any thick subcategory is equivalent to the $K(n)_*$-acyclics of the cohomology theory represented by a Morava K-theory spectrum $K(n)$. Hopkins and Neeman showed that the thick subcategories in the category of perfect complexes $\mathcal{D}^{\mathrm{per}}(R)$ of a commutative noetherian ring R correspond to the specialization closed subsets of the prime ideal spectrum of R [Hop87, Nee92]. There also exists a generalization of this result to schemes [Tho97]. Benson, Carlson and Rickard classified the thick subcategories of the stable module category of the group algebra kG of a p-group G in terms of closed subvarieties of the maximal ideal spectrum of the group cohomology ring $H^*(G;k)$ [BCR97].

In the main theorem of this section, we classify the thick subcategories of the bounded derived category $\mathcal{D}^b(\mathcal{A})$ of a hereditary abelian category $\mathcal{A}$. This result includes for instance the bounded derived category of finitely presented right modules $\mathcal{D}^b(\mathrm{mod}A)$ for a finite dimensional algebra over a field k and enhances therefore the study of thick subcategories to the field of representation theory of algebras. We determine the thick subcategories explicitly in two examples. Furthermore we classify the localizing subcategories in the full derived category $\mathcal{D}(\mathcal{A})$ in a similar way. At the end we show that the Smashing Conjecture holds for the derived category of a hereditary artin algebra of finite representation type.

6.1 Representation theory of hereditary algebras of finite representation type

In this paragraph we describe the structure theory for the module category of a hereditary finite dimensional algebra of finite representation type via Auslander-Reiten theory.

Fix an algebraically closed field k. All our quivers are finite and acyclic.

Definition 6.1.1. A k-algebra A is *hereditary* if every ideal of A is projective as an A-module.

There are several characterizations of the notion a hereditary algebra. Recall that the global dimension of an algebra is the supremum of the projective dimensions of all A-modules.

Theorem 6.1.2. *[ASS06, VII.Theorem 1.4] Let A be a k-algebra. The following assertions are equivalent:*

(i) A is hereditary.

(ii) Submodules of projective A-modules are projective.

(iii) The global dimension of A is at most one.

(iv) $\mathrm{Ext}^i_A(M,N)=0$ *for all A-modules M and N and for all $i\geq 2$.*

Example 6.1.3. Let Q be a quiver. The *path algebra kQ* is generated as a k-vector space by the paths in Q and the multiplication is defined by concatenation of paths. The path algebra of any quiver is hereditary [ARS97, III.Proposition 1.4].

In certain cases the module category of a hereditary algebra is determined by a path algebra.

Theorem 6.1.4. *[ARS97, III Corollary 1.10, Proposition 1.13] Let A be a finite dimensional hereditary k-algebra. Then there is a quiver Q_A such that kQ_A and A are Morita-equivalent, i.e.,* $\mathrm{mod}(A)$ *and* $\mathrm{mod}(kQ_A)$ *are equivalent.*

Recall that a module M is *indecomposable*, if the existence of a decomposition $M = N \oplus L$ implies that N or L are trivial.

Definition 6.1.5. A k-algebra A is *representation finite*, if there are only a finite number of non-isomorphic, finitely generated indecomposable A-modules.

The representation finite path algebras are classified. Therefor recall the notion of a Dynkin graph [ASS06, VII.2].

Theorem 6.1.6 (Gabriel). *Let Q be a connected quiver. Then the path algebra kQ is representation finite, if and only if the underlying graph of Q is Dynkin of type $\mathbb{A}_n$ for $n \geq 1$, $\mathbb{D}_n$ for $n \geq 4$ or of type $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$.*

Now we turn to the structure theory of the module category. Fix a finite dimensional hereditary k-algebra A which is representation finite.

The following fundamental theorem reduces the study of modules to indecomposable modules.

Theorem 6.1.7 (Krull-Remak-Schmidt). *Let A be a finite dimensional k-algebra. For every finitely generated A-module M there are indecomposable A-modules $M_1, \dots, M_n$ such that $M \cong \bigoplus_{i=1}^n M_i$. Furthermore the modules $M_1, \dots, M_n$ are unique up to permutation.*

The following notion is central in the classification of morphisms.

Definition 6.1.8. Let A be a finite dimensional k-algebra. A morphism of A-modules $f\colon M \to N$ is an *irreducible morphism*, if

(i) f is neither a section nor a retraction and

(ii) if $f = f_1 \circ f_2$, then either f_1 is a retraction or f_2 is a section.

Denote by $\mathrm{Irr}(M, N)$ the k-vectorspace of irreducible morphisms from M to N.

As for objects the study of morphisms is reduced to the study of irreducible ones.

Theorem 6.1.9. *[ARS97, V.Theorem 7.8] Let A be a finite dimensional k-algebra of finite representation type. Every morphism between finitely presented indecomposable A-modules that is not invertible is a finite sum of finite compositions of irreducible maps.*

The information of $\mathrm{mod}(A)$ can be collected in a combinatorial object.

Definition 6.1.10. The *Auslander-Reiten quiver (AR-quiver)* $\Gamma(A)$ of the algebra A has as vertices the isomorphism classes of indecomposable modules. The arrows from $[M]$ to $[N]$ correspond bijectively to a k-basis of the vector space of irreducible maps $\operatorname{Irr}(M, N)$.

The quiver $\Gamma(A)$ is locally finite in the sense that every vertex has only finitely many neighbors. The Auslander-Reiten quiver is equipped with an extra structure: the *translate*. It is a bijective map

$$\tau\colon \Gamma(A) \setminus \operatorname{Proj}(A) \to \Gamma(A) \setminus \operatorname{Inj}(A),$$

where $\operatorname{Proj}(A)$ and $\operatorname{Inj}(A)$ denote the sets of isomorphism classes of indecomposable projective and injective modules, respectively. The following notion is central for the structure of the AR-quiver.

Definition 6.1.11. A short exact sequence

$$0 \to L \to M \to N \to 0$$

is called *almost split* or an *Auslander-Reiten sequence (AR-sequence)*, if L and N are indecomposable and the maps $L \to M$ and $M \to N$ are irreducible.

The following theorem describes the relation between an indecomposable module N and its translate $\tau(N)$.

Theorem 6.1.12. *[ASS06, IV.Theorem 4.4, Theorem 3.1] Let A be a finite dimensional k-algebra. For every indecomposable non-projective A-module N there is an AR-sequence*

$$0 \to \tau N \to \bigoplus_{i=1}^{n} M_i^{n_i} \to N \to 0$$

in which $n_i \geq 0$ and the modules M_i are pairwise non-isomorphic indecomposable. Furthermore $n_i = \dim_k \operatorname{Irr}(M_i, N) = \dim_k \operatorname{Irr}(\tau N, M_i)$.

If N is an indecomposable non-projective module, then the theorem tells us that there is the same number of arrows in $\Gamma(A)$ ending in $[N]$ as the number of arrows starting in $\tau[N]$. Therefore a typical part of the AR-quiver can be visualized as follows:

$$\begin{tikzcd} & {[M_1]} \arrow[dr] \arrow[dd, dotted, no head] & \\ {[\tau N]} \arrow[ur] \arrow[dr] & & {[N]} \arrow[ll, dashed, "\tau"'] \\ & {[M_n].} \arrow[ur] & \end{tikzcd}$$

Here we assume that $n_i = 1$ for all $i = 1, \ldots, n$. The subquiver starting at a vertex $[\tau N]$ and ending in $[N]$ is called *mesh*. The following example illustrates the structure of the AR-quiver.

Example 6.1.13. Let Q be the A_3-quiver $1 \longleftarrow 2 \longleftarrow 3$. The Auslander-Reiten quiver of kQ looks as follows:

$$\begin{tikzcd}
 & & P_3 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & & \frac{P_3}{P_1} \arrow[ll, dashed, "\tau"'] \arrow[dr] & \\
P_1 \arrow[ur] & & \frac{P_2}{P_1} \arrow[ll, dashed, "\tau"'] \arrow[ur] & & \frac{P_3}{P_2} \arrow[ll, dashed, "\tau"'] .
\end{tikzcd}$$

The indecomposable projective modules are P_1, P_2 and P_3.

Every finite dimensional hereditary k-algebra A of finite representation type is Morita-equivalent to the path algebra of a quiver Q. The quiver Q can be obtained from the AR-quiver $\Gamma(A)$ as the opposite of the full subquiver with the indecomposable projectives as vertices. Since the category $\mathrm{mod}(A)$ is equivalent to $\mathrm{mod}(kQ)$, the AR-quiver determines the module category combinatorially.

The whole module category $\mathrm{Mod}(A)$ is determined by the category of finitely generated modules.

Theorem 6.1.14. *[Aus74, RT74] Let A be an artin algebra of finite representation type. Then every module is a direct sum of finitely generated indecomposable modules.*

Therefore Auslander-Reiten theory provides a complete picture of both $\mathrm{mod}(A)$ and $\mathrm{Mod}(A)$.

6.2 The derived category of hereditary abelian categories

Here we describe the structure of the derived category of a hereditary abelian category which serves as the main tool to obtain the classification result in Paragraph 6.4.

Definition 6.2.1. An abelian category $\mathcal{A}$ is called hereditary if $\mathrm{Ext}^i_{\mathcal{A}}(M, N)$ vanishes for all $M, N \in \mathcal{A}$ and all $i \geq 2$.

Throughout this paragraph let $\mathcal{A}$ be a hereditary abelian category.

Example 6.2.2. A module category over a hereditary ring A is hereditary.

There is a canonical embedding $i \colon \mathcal{A} \hookrightarrow \mathcal{D}(\mathcal{A})$ which sends an object M to the *stalk complex* $\cdots \to 0 \to M \to 0 \to \ldots$ which is concentrated in degree zero. By abuse of notation we do not distinguish between objects in $\mathcal{A}$ and $\mathrm{im}(i)$.

The derived category $\mathcal{D}(\mathcal{A})$ of a hereditary abelian category $\mathcal{A}$ is closely related to $\mathcal{A}$ itself since every complex of $\mathcal{D}(\mathcal{A})$ is isomorphic to a direct sum (and direct product) of stalk complexes:

Lemma 6.2.3. *For every $X \in \mathcal{D}(\mathcal{A})$ there are isomorphisms in $\mathcal{D}(\mathcal{A})$*

$$\prod_{n \in \mathbb{Z}} H^n X[-n] \cong X \cong \bigoplus_{m \in \mathbb{Z}} H^m X[-m].$$

A proof of this well known lemma can be found in [Kra04]. The homomorphisms in $\mathcal{D}(\mathcal{A})$ therefore reduce to

$$\mathrm{Hom}^*_{\mathcal{D}(\mathcal{A})}(M,N) \cong \mathrm{Hom}_{\mathcal{A}}(M,N) \oplus \mathrm{Ext}^1_{\mathcal{A}}(M,N)$$

for $M, N \in \mathcal{A}$. So the derived category consists of shifted copies of $\mathcal{A}$, and the morphisms are given by extensions and homomorphisms in $\mathcal{A}$. This structure is visualized in Figure 1.

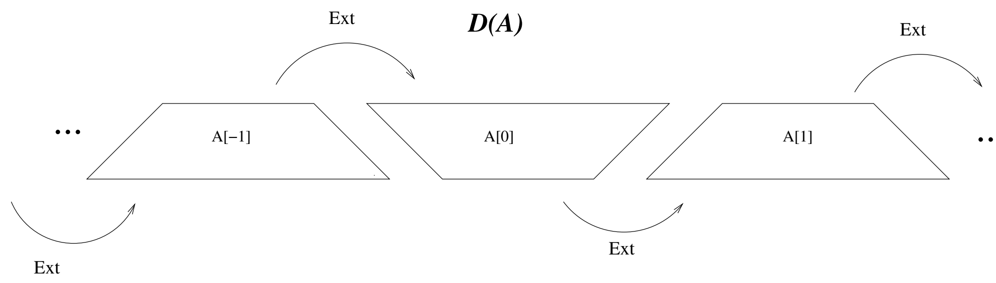


FIGURE 1

Non-equivalent hereditary abelian categories can give rise to the same derived category:

Theorem 6.2.4. *[Hap88, I.5.5, 5.6] Let k be an algebraically closed field. If Q and Q' are Dynkin quivers of the same type but of different orientation then $\mathcal{D}(\mathrm{mod}(kQ))$ and $\mathcal{D}(\mathrm{mod}(kQ'))$ are equivalent as triangulated categories.*

This result suggests that for representation finite path algebras the Dynkin type plays an essential role.

The structure of the derived category motivates why the thick subcategories in $\mathcal{D}(\mathcal{A})$ should be determined by data in $\mathcal{A}$. If in addition $\mathcal{A} = \mathrm{mod}(kQ)$ is the module category of a path algebra of a Dynkin quiver then we should be able to describe the thick subcategories combinatorially.

6.3 Thick subcategories of abelian categories

We define and investigate thick subcategories of an abelian category $\mathcal{A}$ and discuss Hovey's classification of the thick subcategories in the category of modules over a regular coherent commutative ring.

Throughout this paragraph let $\mathcal{A}$ be an abelian category.

Definition 6.3.1. A full subcategory $\mathcal{M}$ of $\mathcal{A}$ is called *thick* if for every exact sequence

$$M_1 \to M_2 \to M_3 \to M_4 \to M_5$$

the object M_3 is in $\mathcal{M}$ if the objects M_1, M_2, M_4, M_5 are in $\mathcal{M}$.

Hovey calls these subcategories "wide" [Hov01a]. In the following two lemmas some easy properties of thick subcategories are deduced. For the convenience of the reader the proof [Hov01a] is reproduced here.

Lemma 6.3.2. *A full subcategory $\mathcal{M}$ in $\mathcal{A}$ is thick, if and only if it is closed under forming of extensions, kernels and cokernels.*

Proof. Let $\mathcal{M} \subset \mathcal{A}$ be thick and

$$M_1 \to M_2 \to M_3 \to M_4 \to M_5$$

be exact in $\mathcal{A}$. If in the exact sequence above $M_1 = M_5 = 0$ and M_2 and M_4 are in $\mathcal{M}$, then M_3 is in $\mathcal{M}$ since $\mathcal{M}$ is thick. Therefore $\mathcal{M}$ is closed under extensions. If we set $M_1 = M_2 = 0$, respectively $M_4 = M_5 = 0$ it follows that $\mathcal{M}$ is closed under kernels, respectively cokernels.

Conversely, let $\mathcal{M} \subset \mathcal{A}$ be closed under extensions, kernels and cokernels and let

$$M_1 \to M_2 \to M_3 \to M_4 \to M_5$$

be exact with $M_1, M_2, M_4, M_5 \in \mathcal{M}$. Since $\mathcal{M}$ is closed under cokernels and kernels, $C := \operatorname{coker}(M_1 \to M_2)$ and $K := \ker(M_4 \to M_5)$ are in $\mathcal{M}$. Hence we obtain a diagram:

$$\begin{tikzcd}
M_1 \arrow[r] & M_2 \arrow[rr] \arrow[dr] & & M_3 \arrow[rr] \arrow[dr] & & M_4 \arrow[r] & M_5 \\
& & C \arrow[ur] \arrow[dr] & & K \arrow[ur] \arrow[dr] & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0. &
\end{tikzcd}$$

Therefore M_3 is an extension of C and K and hence it is in $\mathcal{M}$. □

As an additional property we have

Lemma 6.3.3. *A thick category in $\mathcal{A}$ is closed under direct summands.*

Proof. Let $M \oplus N$ be in the thick category $\mathcal{M}$. The kernel of the map $M \oplus N \to M \oplus N$ which sends (m, n) to $(0, n)$ is M. □

So a thick subcategory in $\mathcal{A}$ is an abelian subcategory in $\mathcal{A}$ that is closed under retracts such that the inclusion functor is exact. This property motivates its name.

There are geometric examples of thick subcategories.

Example 6.3.4. The category of coherent modules over the structure sheaf $\mathcal{O}_{\mathbb{X}}$ of a scheme $\mathbb{X}$ is thick [Gro60, 5.3.5].

Other examples of thick subcategories arise from the category $\operatorname{add}(M)$ of direct sums of direct summands of M.

Lemma 6.3.5. *Let k be a field.*

(i) Let A be an arbitrary k-algebra. If M is an indecomposable finitely presented A-module with $\operatorname{Hom}_A(M, M) = k$ *and* $\operatorname{Ext}^1_A(M, M) = 0$, *then* $\operatorname{add}(M)$ *is thick.*

(ii) If A is a finite dimensional hereditary k-algebra of finite representation type and M is an indecomposable A-module, then $\operatorname{add}(M)$ *is thick.*

Proof. Since M is indecomposable the equation $\mathrm{add}(M) = \{\bigoplus_{i=1}^{n} M \mid n \geq 0\}$ holds. The functor $\mathrm{Ext}^1_A(-,-)$ is additive in both variables. Therefore $\mathrm{add}(M)$ is closed under extensions because M has no non-trivial self-extensions. For positive integers n and m let $f\colon M^n \to M^m$ be A-linear. Every non-trivial component of f is of the form $x \cdot \mathrm{id}_M$ for some $x \in k \setminus \{0\}$ because $\mathrm{Hom}_A(M, M) = k$. Since k is a field every element $x \in k \setminus \{0\}$ is invertible. Therefore the kernel and the cokernel of $x \cdot \mathrm{id}_M$ are trivial and the kernel and the cokernel of f are in $\mathrm{add}(M)$. Therefore (i) follows.
If M is an indecomposable module over a finite dimensional hereditary k-algebra A of finite type, then by [ASS06, VII 5.14] $\mathrm{Hom}_A(M, M) = k$ and $\mathrm{Ext}^1_A(M, M) = 0$. Hence (ii) follows from (i). □

Hovey proved a classification result in a commutative situation. Note that if the global dimension of a ring coherent R is finite, then $\mathcal{D}^b(\mathrm{mod}(R))$ is equivalent to the category of perfect complexes $\mathcal{D}^{\mathrm{per}}(R)$.

Theorem 6.3.6. *[Hov01a, Theorem 3.6] Let R be a commutative regular coherent ring. There is a one-to-one correspondence between the thick subcategories in $\mathcal{D}^b(\mathrm{mod}(R))$ and the thick subcategories of $\mathrm{mod}(R)$.*

If R is regular noetherian, then a thick subcategory is also closed under subobjects, quotient-objects and extensions [Hov01a, 3.7] and is therefore a *Serre subcategory*. Garkusha and Prest generalized Theorem 6.3.6 in the following way: if R is a commutative coherent ring, then the thick subcategories in $\mathcal{D}^{\mathrm{per}}(R)$ correspond bijectively to the Serre subcategories in $\mathrm{mod}(R)$ [GP07, Theorem C].

In these theorems the classifications [Nee92, Tho97] of the thick subcategories of $\mathcal{D}^{\mathrm{per}}(R)$ are used to determine the thick subcategories of $\mathrm{mod}(R)$. We go the other way around and describe thick subcategories of the triangulated category in terms of the abelian category.

6.4 Classification of thick subcategories

In this section we prove the classification result and determine all thick subcategories in two examples combinatorially.

Theorem 6.4.1. *Let $\mathcal{A}$ be a hereditary abelian category. The assignments*

$$f\colon \mathcal{C} \mapsto \{H^0C \mid C \in \mathcal{C}\} \qquad \text{and} \qquad g\colon \mathcal{M} \mapsto \{C \in \mathcal{D}^b(\mathcal{A}) \mid H^nC \in \mathcal{M}\,\forall\, n \in \mathbb{Z}\}$$

induce mutually inverse bijections between

- *the class of thick subcategories in $\mathcal{D}^b(\mathcal{A})$ and*
- *the class of thick subcategories in $\mathcal{A}$.*

Proof. The proof mainly uses Lemma 6.2.3. First note that g is well defined because $\mathcal{M}$ is thick and closed under direct summands by Lemma 6.3.3. The map f is well defined because of the following lemma:

Lemma 6.4.2. *Let $\mathcal{C} \subset \mathcal{D}^b(\mathcal{A})$ be thick. The full subcategory $f(\mathcal{C}) \subset \mathcal{A}$ is thick.*

It remains to show that f and g are mutually inverse. The inclusion $f(g(\mathcal{M})) \subset \mathcal{M}$ is obvious. Any object $M \in \mathcal{M}$ is in $f(g(\mathcal{M}))$ since the stalk complex $\cdots \to 0 \to M \to 0 \to \ldots$ is in $g(\mathcal{M})$. Since a complex is determined by its homology (Lemma 6.2.3) the equality $g(f(\mathcal{C})) = \mathcal{C}$ holds. □

In order to prove Lemma 6.4.2 we need the following

Lemma 6.4.3. *If $g : C \to D$ is a map of complexes such that the differentials of C and D are zero and $g_m = 0$ for all $m \neq n$, then $\ker(g)$ and $\operatorname{coker}(g)$ are retracts of $H^*(\operatorname{cone}(g))$.*

Proof. The only non-zero differential in $\operatorname{cone}(g)$ is $\operatorname{cone}(g)^{n-1} \to \operatorname{cone}(g)^n$:

$$\begin{tikzcd}
\operatorname{cone}(g)^{n-2} \arrow[d, "d"] & C^{n-1} \arrow[d, "0"] \arrow[drr, "0"] & \oplus & D^{n-2} \arrow[d, "0"] \\
\operatorname{cone}(g)^{n-1} \arrow[d, "d"] & C^{n} \arrow[d, "0"] \arrow[drr, "g"] & \oplus & D^{n-1} \arrow[d, "0"] \\
\operatorname{cone}(g)^{n} \arrow[d, "d"] & C^{n+1} \arrow[d, "0"] \arrow[drr, "0"] & \oplus & D^{n} \arrow[d, "0"] \\
\operatorname{cone}(g)^{n+1} & C^{n+2} & \oplus & D^{n+1}.
\end{tikzcd}$$

Thus we can compute the homology:

$$H^m(\operatorname{cone}(g)) = \begin{cases} C^{m+1} \oplus D^m & m \leq n-2 \text{ or } m \geq n+1 \\ \ker(g) \oplus D^{n-1} & m = n-1 \\ C^{n+1} \oplus \operatorname{coker}(g) & m = n. \end{cases}$$

□

Proof of Lemma 6.4.2. We show that $f(\mathcal{C})$ is closed under extensions, kernels and cokernels. So let C_1, C_2 be in $\mathcal{C}$ and $M \in \mathcal{A}$ such that there is a short exact sequence

$$0 \to H^0C_1 \to M \to H^0C_2 \to 0.$$

This sequence corresponds to a triangle

$$H^0C_1 \to M \to H^0C_2 \to \Sigma H^0C_1$$

in $\mathcal{D}^b(\mathcal{A})$. Here we consider an object of $\mathcal{A}$ as a complex in $\mathcal{D}^b(\mathcal{A})$ by means of the inclusion $\mathcal{A} \to \mathcal{D}^b(\mathcal{A})$ which sends the object M to the complex concentrated in degree 0. By abuse of notation we call it again M. Each homology group of a complex $C \in \mathcal{C}$ is again contained in $\mathcal{C}$ since by Lemma 6.2.3 H^nC is a retract of C up to isomorphism and $\mathcal{C}$ is thick. Therefore H^0C_1 and H^0C_2 are in $\mathcal{C}$, and because $\mathcal{C}$ is closed under suspensions, $\Sigma H^0C_1 \in \mathcal{C}$. Since $\mathcal{C}$ is closed under extensions, we conclude that M is in $\mathcal{C}$. Hence $M \in f(\mathcal{C})$ because the zeroth homology of $\cdots \to 0 \to M \to 0 \to \ldots$ is M.
So it only remains to show that $f(\mathcal{C})$ is closed under kernels and cokernels. Let C_1, C_2 be in $\mathcal{C}$ and f be a morphism in the exact sequence in $\mathcal{A}$:

$$0 \to \ker(f) \to H^0C_1 \xrightarrow{f} H^0C_2 \to \operatorname{coker}(f) \to 0.$$

Now extend f to a map of complexes

$$\bigoplus_{n\in\mathbb{Z}} H^n C_1[-n] \to \bigoplus_{m\in\mathbb{Z}} H^m C_2[-m]$$

which is f in degree 0 and zero in all other degrees. We call it again f. Since $C_i \cong \bigoplus_{n\in\mathbb{Z}} H^n C_i[-n]$ for $i = 1, 2$, the map f belongs to $\mathcal{C}$. The cone of f is in $\mathcal{C}$. By Lemma 6.4.3 $\ker(f)$ and $\operatorname{coker}(f)$ are retracts of $H^0(\operatorname{cone}(f))$ and are hence (considered as stalk complexes) in $\mathcal{C}$. Therefore the kernel and cokernel of f, considered as objects in $\mathcal{A}$, are in $f(\mathcal{C})$. □

Corollary 6.4.4. *Let $\mathcal{A}$ and $\mathcal{A}'$ be hereditary abelian categories. If $\mathcal{D}^b(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A}')$ are triangle equivalent, then there is a one-to-one correspondence between the thick subcategories in $\mathcal{A}$ and the thick subcategories in $\mathcal{A}'$.*

With this theorem we have reduced the classification of thick subcategories in the triangulated category $\mathcal{D}^b(\mathcal{A})$ to the task of understanding thick subcategories in $\mathcal{A}$. In easy examples it is possible to determine them combinatorially. Let k be a field and A be a representation finite hereditary k-algebra. As a consequence of Lemma 6.3.5 there are examples of thick subcategories of the category of finitely presented modules $\operatorname{mod}(A)$. As an immediate consequence we are able to determine the thick subcategories of finite dimensional representations of an A_2- and an A_3-quiver. For the two examples let k be an algebraically closed field, Q the respective quiver, $A = kQ$ the path algebra and $\mathcal{A} = \operatorname{mod}(kQ)$ the category of finitely presented modules over A. We use the Auslander-Reiten quiver to describe the category $\mathcal{A}$ combinatorially.

Example 6.4.5. Let Q be the quiver $1 \longleftarrow 2$. The Auslander-Reiten quiver Γ_{kQ} is the following graph:

$$\begin{tikzcd}
& P_2 \arrow[dr] & \\
P_1 \arrow[ur] \arrow[rr, dotted, no head] & & \frac{P_2}{P_1}.
\end{tikzcd}$$

There are exactly four non-trivial thick subcategories: $\operatorname{add}(P_1)$, $\operatorname{add}(P_2)$, $\operatorname{add}(\frac{P_2}{P_1})$ and $\operatorname{mod}(kQ)$.

Example 6.4.6. Let Q be the quiver $1 \longleftarrow 2 \longleftarrow 3$. The Auslander-Reiten quiver has the following shape:

$$\begin{tikzcd}
& & P_3 \arrow[dr] & & \\
& P_2 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \frac{P_3}{P_1} \arrow[dr] & \\
P_1 \arrow[ur] \arrow[rr, dotted, no head] & & \frac{P_2}{P_1} \arrow[ur] \arrow[rr, dotted, no head] & & \frac{P_3}{P_2}.
\end{tikzcd}$$

Lemma 6.3.5 tells us that there are six thick subcategories containing exactly one indecomposable. Furthermore there are two thick subcategories that contain two indecomposable modules, four with three indecomposables and the whole module category with six indecomposables.

The left column of Table 1 shows the thick subcategories in terms of the contained indecomposable modules. E.g. $\langle P_1, P_3\rangle$ is the smallest thick subcategory containing P_1 and P_3. The right column displays the part of the corresponding Auslander-Reiten quiver that is contained in the thick subcategory $\mathcal{C}$. Modules in $\mathcal{C}$ are labelled with fat bullets and morphisms in $\mathcal{C}$ with full arrows.

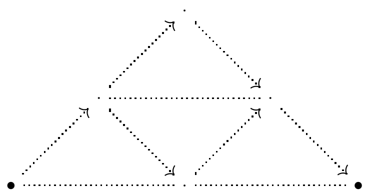
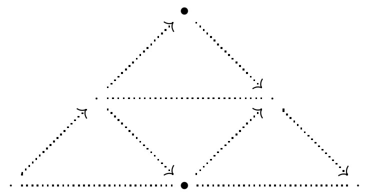
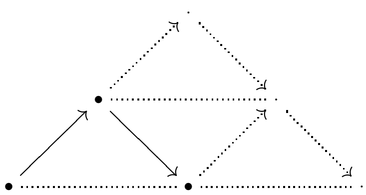
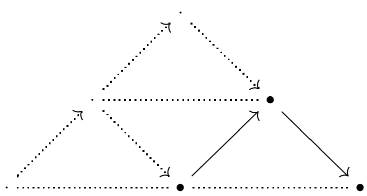
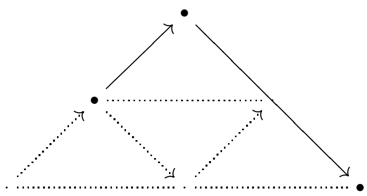
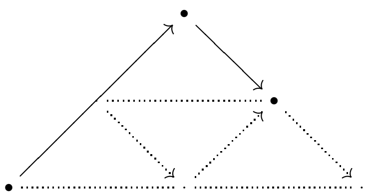
$\langle P_1\rangle, \ldots, \langle P_3/P_1\rangle$	$\bullet$
$\langle P_1, P_3/P_2\rangle$	
$\langle P_3, P_2/P_1\rangle$	
$\langle P_1, P_2, P_2/P_1\rangle$	
$\langle P_2/P_1, P_3/P_1, P_3/P_2\rangle$	
$\langle P_2, P_3, P_3/P_2\rangle$	
$\langle P_1, P_3, P_3/P_1\rangle$	
$\langle P_1, P_2, P_3\rangle$	$\mathrm{mod}(A)$

TABLE 1

The thick subcategories are symmetric with respect to reflection at the axis going through P_3 and $\frac{P_2}{P_1}$ in the Auslander-Reiten quiver. The categories $\mathrm{add}(P_3)$, $\mathrm{add}(\frac{P_2}{P_1})$, $\langle P_3, \frac{P_2}{P_1}\rangle$, $\langle P_1, \frac{P_3}{P_2}\rangle$ and $\mathrm{mod}(kQ)$ are invariant under the reflection. Under the reflection $\mathrm{add}(P_3)$ corresponds to $\mathrm{add}(\frac{P_3}{P_1})$, $\mathrm{add}(P_1)$ corresponds to $\mathrm{add}(\frac{P_3}{P_2})$, $\langle P_1, P_2, \frac{P_2}{P_1}\rangle$ corresponds to $\langle \frac{P_3}{P_1}, \frac{P_2}{P_1}, \frac{P_3}{P_2}\rangle$ and $\langle P_1, P_3, \frac{P_3}{P_1}\rangle$ corresponds to $\langle P_2, P_3, \frac{P_3}{P_2}\rangle$.

It would be interesting to work out all thick subcategories for all representation finite algebras.

In fact, the orientation of the quiver does not play a role.

Corollary 6.4.7. *Let k be an algebraically closed field. Let Q and Q' be quivers whose underlying graph is Dynkin of the same type but whose orientation is different. Then* $\mathrm{mod}(kQ)$ *contains the same number of thick subcategories as* $\mathrm{mod}(kQ')$.

Proof. By Theorem 6.2.4 $\mathrm{mod}(kQ)$ and $\mathrm{mod}(kQ')$ are derived equivalent and Corollary 6.4.4 yields the assertion. □

If the algebra A is not of global dimension one, then Lemma 6.2.3 does not remain true. But if the global dimension of A is finite the Happel functor $\mathcal{D}^b(\mathrm{mod}(A)) \to \underline{\mathrm{mod}}(\hat{A})$ is an equivalence [Hap88, II.4.9]. Here, $\hat{A}$ denotes the repetitive algebra of A. A generalization of the classification Theorem 6.4.1 may possibly be achieved by characterizing the thick subcategories of $\underline{\mathrm{mod}}(\hat{A})$ in terms of the thick subcategories of $\mathrm{mod}(A)$.

6.5 Classification of localizing subcategories

In this section we use the strategy of Theorem 6.4.1 to classify the localizing subcategories of the full derived category of a hereditary Grothendieck category. As an application we prove that the Smashing Conjecture is true for $\mathcal{D}(A)$ for a hereditary artin algebra A of finite representation type.

Recall that a full subcategory of a triangulated category with arbitrary direct sums is called *localizing* if it is thick and closed under arbitrary direct sums. These categories are the unbounded analogues of the thick subcategories. Recall from Example 2.1.3 that for a Grothendieck category $\mathcal{A}$ the unbounded derived category exists.

Theorem 6.5.1. *Let $\mathcal{A}$ be a hereditary Grothendieck category. The assignments*

$$f\colon \mathcal{C} \mapsto \{H^0C \mid C \in \mathcal{C}\} \qquad \text{and} \qquad g\colon \mathcal{M} \mapsto \{C \in \mathcal{D}(\mathcal{A}) \mid H^nC \in \mathcal{M}\, \forall\, n \in \mathbb{Z}\}$$

induce mutually inverse bijections between

- *the class of localizing subcategories in $\mathcal{D}(\mathcal{A})$ and*
- *the class of thick subcategories in $\mathcal{A}$ that are closed under small coproducts.*

Proof. Adding the following comments the proof of Theorem 6.4.1 applies. Lemma 6.2.3 is not limited to the bounded derived category, and hence can be used here. The map g is well-defined, since the homology functor commutes with infinite direct sums. And finally if $\mathcal{C}$ is localizing, then $f(\mathcal{C})$ is closed under direct sums for the same reason. □

Since the module category of a representation finite algebra is determined by the finitely generated modules by Theorem 6.1.14, we can show the following

Corollary 6.5.2. *Let A be a hereditary artin algebra of finite representation type.*

(i) Every thick subcategory $\mathcal{M} \subset \mathrm{Mod}(A)$ that is closed under direct sums is the smallest thick subcategory that contains $\mathcal{M} \cap \mathrm{mod}(A)$ and is closed under direct sums.

(ii) Every localizing subcategory $\mathcal{C} \subset \mathcal{D}(A)$ is determined by its intersection with the perfect complexes: $\mathcal{C} = \langle \mathcal{C} \cap \mathcal{D}^{\mathrm{per}}(A)\rangle_{\mathrm{loc}}$.

Proof. By Theorem 6.1.14, (i) is true. For the assertion (ii) let $\mathcal{C} \subset \mathcal{D}(A)$ be localizing and $C \in \mathcal{C}$ be an object. By Lemma 6.2.3 it suffices to show that H^0C is contained in $\langle \mathcal{C} \cap \mathcal{D}^{\mathrm{per}}(A)\rangle_{loc}$. Because of Theorem 6.1.14 there are a set I and finitely generated modules $\{M_i \mid i \in I\}$ such that $H^0C \cong \bigoplus_{i\in I} M_i$. Since $\mathcal{C}$ is thick, it follows that $M_i \in \mathcal{C}$. For every M_i choose a projective resolution

$$0 \to P_i^0 \to P_i^1 \to M_i \to 0$$

such that P_i^0, P_i^1 are finitely generated. The complex $P_i: \; 0 \to P_i^0 \to P_i^1 \to 0$ is perfect and hence in $\mathcal{D}^{\mathrm{perf}}(A)$. Since $P_i \to M_i$ is a quasi isomorphism and $M_i \in \mathcal{C}$ we can conclude that $P_i \in \mathcal{C} \cap \mathcal{D}^{\mathrm{per}}(A)$. Hence H^0C is a direct sum of perfect complexes in $\mathcal{C}$. ☐

Since the perfect complexes form precisely the compact objects in $\mathcal{D}(A)$, Corollary 6.5.2(ii) shows:

Corollary 6.5.3. *The Smashing Conjecture 2.4.7 is true for the derived category of a hereditary artin algebra of finite representation type.*

In fact, even all localizing subcategories are determined by the intersection with the compact objects.

If A is not of finite type the Smashing Conjecture is possibly also true since every module over A is a filtered colimit of finitely presented modules. Choosing a clever indexing category may lead to a proof of the Smashing Conjecture for arbitrary hereditary algebras.

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