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Espaces duaux de certains produits semi-directs et noyaux associés aux orbites plates

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À mes parents

Résumé

Le premier problème abordé dans cette thèse est la description de la topologie du dual unitaire des groupes de Lie à radical nilpotent co-compact, en particulier les produits semi-directs $\mathrm{G} = K \ltimes N$ des groupes compacts K avec les groupes de Lie nilpotents N. L'espace dual $\hat{G}$ de G a été déterminé par la théorie de Mackey et la paramétrisation géométrique donnée par R. L. Lipsmann qui ont prouvé l'existence d'une bijection entre $\hat{G}$ et l'espace des orbites coadjointes admissibles de G. Notre objectif est de comparer la topologie de Fell du dual unitaire avec la topologie quotient de l'espace des orbites coadjointes admissibles. Le premier exemple traité dans ce travail est le cas des groupes de déplacement $M_n = SO(n) \ltimes \mathbb{R}^n$. Nous avons prouvé que l'espace dual de M_n est homéomorphe à son espace des orbites coadjointes admissibles. Ce résultat peut être vrai aussi pour les groupes $G_n = U(n) \ltimes \mathbb{H}_n$, où $\mathbb{H}_n$ est le groupe de Heisenberg de dimension $2n + 1$ (il est uniquement prouvé pour le groupe G_1). Le deuxième problème considéré dans cette thèse est la déterminaton des représentations unitaires irréductibles π d'un groupe G, dont le noyau de π dans $L^1(G)$ est donné par les fonctions dont la transformée de Fourrier s'annule sur l'orbite $\mathcal{O}_\pi$ de π. Ce problème a été résolu dans le cas de groupes de Lie nilpotents par J. Ludwig, qui a montré que $\ker(\pi) = \{f \in L^1(G); \hat{f}(\mathcal{O}_\pi) = \{0\}\}$ si et seulement si l'orbite coadjointe $\mathcal{O}_\pi$ est plate. Le travail consiste à prouver qu'on a un résultat équivalent pour les groupes de Lie complètement résolubles.

Abstract

The first problem treated in this thesis is the description of the dual topology of Lie groups with co-compact nilpotent radical, in particular the semi direct products $G = K \ltimes N$ of compacts groups K with nilpotent Lie groups N, The dual space $\hat{G}$ of G had been determined via Mackey's theory and the geometric parametrization given by R. L. Lipsmann who had proved that there is a bijection between $\hat{G}$ and the admissible coadjoint orbit space of G. Our object is to compare the Fell topology of the dual space with the natural topology of the quotient space of admissible coadjoint orbits. The first example treated in this work is the case of the motion groups $M_n =$

$SO(n) \ltimes \mathbb{R}^n$. We have shown that the dual space of M_n is homeomorphic with its admissible coadjoint orbit space. This result may be true also for the groups $G_n = U(n) \ltimes \mathbb{H}_n$, where $\mathbb{H}_n$ is the $2n+1$ dimensional Heisenberg Lie group (it is only proved for the group G_1). The second issue regarded in this thesis is the determinaton of the irreducible unitary representation π of a group G, for which the kernel of π in $L^1(G)$ is given by the functions whose the Fourrier transform annihilates on the orbit $\mathcal{O}$ of π. This problem was solved for the case of nilpotent groups by J. Ludwig who had shown that $\ker(\pi) = \{f \in L^1(G); \hat{f}(\mathcal{O}_\pi) = \{0\}\}$ if and only if $\mathcal{O}_\pi$ is a flat orbit. The work is to prove that this result remains true for completely solvable Lie groups.

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Introduction

Les groupes de Lie s'introduisent naturellement dans de nombreuses questions de mathématiques pures et appliquées. Créée à l'origine au XIXe siècle par le mathématicien norvégien Sophus Lie, la théorie a été développée tout au long du XXe siècle en parallèle avec les progrès de l'algèbre, de la topologie et de la géométrie différentielle et aussi sous l'impulsion des recherches en physique et en mécanique théorique. Elle englobe plusieurs théories comme : la mesure de Haar, la théorie du produit de composition, les séries de Fourrier, les fonctions presque-périodiques, les groupes d'opérateurs unitaires, et en partie, la théorie de potentiel, la théorie ergodique et la topologie algébrique.

L'un des problèmes essentiels dans l'analyse harmonique est la détermination de l'espace dual $\hat{G}$ d'un groupe localement compact G, c'est-à-dire, l'ensemble des classes d'équivalence de représentations unitaires irréductibles de G. Pour certains groupes G, la théorie de Mackey des représentations induites nous permet d'identifier les éléments de $\hat{G}$. On désire si possible, donner pour chaque classe de telles représentations une réalisation concrète de l'une d'entre elles, en terme d'un objet géométrique lié au groupe. Une réponse complète à cette question a été apportée dans un premier lieu par A. A. Kirillov qui a établi, dans le cadre des groupes nilpotents, une bijection naturelle entre l'espace des orbites de la représentation coadjointe du groupe G et son dual unitaire $\hat{G}$. Étant donnée une orbite de la représentation coadjointe de G, à toute polarisation invariante de cette orbite, Kirillov fait correspondre une réalisation de l'élément de $\hat{G}$ correspondant à l'orbite. Ces résultats ont été généralisés, en partie aux groupes de Lie résolubles (voir les travaux de P . Bernat, L . Pukanszky, ...), et aux groupes de Lie à radical nilpotent co-compact par Lipsmann qui a prouvé dans [Lip] l'existence d'une correspendance entre $\hat{G}$ et l'espace quotient des orbites coadjointes admissibles.

Un autre axe de recherche assez important dans la théorie des représentations est celui de l'étude de la topologie du dual unitaire. Soient G un groupe abélien localement compact, et $\hat{G}$ le groupe dual, ensemble des caractères continus sur G, depuis Pontrjagin on munit classiquement $\hat{G}$ de la topologie de la convergence uniforme sur tout compact de G. Cette topologie a été généralisée par J. M. G. Fell ([Fe1], [Fe2], [Fe3]) comme suit. Soit G un groupe localement compact quelconque et Γ l'ensemble des (classes de) représentations unitaires continues π de G. Si $\pi \in \Gamma$ et $Y \subset \Gamma$, on dit que π est faiblement contenue dans Y si toute fonction de type positif associée à π est une limite uniforme sur tout compact de G de sommes finies de fonctions de type positif associées à des représentations appartenant à Y. Si $\pi \in \hat{G}$, on peut supprimer les mots "sommes finies de" dans la définition précédente. Pour $Y \subset \hat{G}$, on appelle fermeture de Y l'ensemble $\overline{Y}$ des $\pi \in \hat{G}$ qui sont faiblement contenues dans Y. On dit que Y est fermée dans $\hat{G}$ si et seulement si $Y = \overline{Y}$. Cette notion d'ensemble fermé définit sur $\hat{G}$ une topologie, appelée topologie de Fell. Il arrive souvent qu'elle ne soit pas séparée au sens de Hausdorff. L'étude de la topologie de l'espace dual des groupes localement compacts a été developpée à travers les travaux de L. W. Baggett qui a donné dans [Ba] une description de la convergence dans le dual unitaire des produits semi-directs $K \ltimes N$, avec N nilpotent, et K abélien ou compact. On trouve aussi les travaux de I. Schochetman qui a étudié le cas des groupes des extensions ([Sch]).

Le problème fondamental lié à la paramétrisation géométrique de l'espace dual $\hat{G}$ d'un groupe de Lie G et à la description de sa topologie est d'étudier la continuité de la bijection entre $\hat{G}$ et l'espace des orbites coadjointes. Pour un groupe de Lie connexe, simplement connexe, et nilpotent, le fait que cette bijection soit un homéomorphisme a été conjecturé par Kirillov dans [Kirillov] en 1962, et prouvé pour la première fois par Brown dans [Br] en 1974. Par une approche fondamentalement différente de celle de Brown, une autre preuve, moins retentissante, fut donnée par Joy dans [Joy] en 1984. En 1994, H. Leptin et J. Ludwig ont démontré que ce résultat est aussi vrai pour les groupes de Lie exponentiels résolubles (pour les details, voir [Lep-Lud]).

La première partie de ma thèse est une contribution à l'étude de ce type de problèmes en analyse harmonique. J'ai essayé, en collaboration avec le Professeur J. Ludwig, de traiter le cas des produits semi-direct $G = K \ltimes N$ de groupes compacts K et nilpotents N. L'espace dual de ces groupes a été déterminé à l'aide de la théorie des petits groupes de Mackey et de la théorie des orbites de Kirillov par R. L. Lipsmann. Le problème auquel nous nous étions consacrés fût de comparer la topologie de Fell de l'espace dual de ces groupes à la topologie naturelle de l'espace des orbites co-adjointes admissibles. Même le cas le plus simple, celui du groupe $M_n := SO(n) \ltimes \mathbb{R}^n$ n'avait pas encore été élucidé. La topologie de l'espace dual de M_n avait été décrite par L. W. Baggett dans [Ba]. Un premier résultat obtenu en 2007 montre que cette topologie coïncide avec celle de l'espace des orbites co-adjointes admissibles. Pour obtenir ce résultat, nous avons du faire des calculs très précis sur la structure de ces orbites coadjointes, étudier en détail le comportement de suites convergentes dans l'espace des orbites et comparer cette convergence à celle des représentations irréductibles correspondantes. Ces résultats ont donné naissance à l' article "Dual topology of the motion groups $SO(n) \ltimes \mathbb{R}^n$" qui a été accepté pour publication dans Forum Mathematicum. On a étudié ensuite le cas des groupes $D_n := U(n) \ltimes \mathbb{C}^n$. Ici la démarche est analogue à celle des groupes M_n. Par la suite, nous avons travaillé sur le problème beaucoup plus difficile des groupes $G_n = U(n) \ltimes \mathbb{H}_n$, où $\mathbb{H}_n$ désigne le groupe de Heisenberg de dimension $2n+1$. La topologie de l'espace dual de ces groupes n'étant pas encore connue, il fallait donc comprendre la topologie de l'espace des orbites co-adjointes admissibles et en même temps que celle de l'espace dual de ces groupes. On a réussi à décrire la topologie de l'espace des orbites co-adjointes en explicitant pour les suites fortement convergentes l'ensemble des point limites de ces suites. Les espaces qu'on regarde ici sont non séparés, ce qui entraîne un comportement souvent inattendu de celles-ci. On a aussi étudié la convergence dans l'espace dual $\widehat{G}_n$ et montré dans le cas particulier du groupe G_1 que la topologie de l'espace des orbites admissibles coïncide avec celle de l'espace dual.

Le deuxième problème abordé dans cette thèse est celui de la détermination des représentations unitaires irréductibles π du groupe, pour lesquelles le noyau de π dans l'algèbre $L^1(G)$ est donné par les fonctions, dont la transformée de Fourier s'annule sur l'orbite $\mathcal{O}$ de π. Ce problème a été résolu dans le cas nilpotent par J. Ludwig dans [Lud], où il a été démontré que c'est uniquement vrai pour les orbites plates. Le travail consiste à prouver que le résultat pour les groupes nilpotents reste vrai dans le cas résoluble exponentiel.

Plan de la thèse. Cette thèse est constituée de quatre chapitres :

– Dans le premier chapitre, on rappelle les principales définitions et propriétés liées à la théorie des représentations des groupes localement compacts, en particulier, les groupes de Lie nilpotents, les groupes de Lie exponentiels et les produits semi-directs compacts nilpotents. On y rappelle aussi les notions suivantes : la théorie des orbites établie par Lipsmann, et la topologie de l'espace dual en se reportant au livre de J. Dixmier [Dix] sur les C^*-algèbres.

- Le deuxième chapitre est consacré à la preuve du premier résultat de cette thèse, à savoir l'existence d'un homéomorphisme entre l'espace dual du groupe de déplacement euclidien $M_n := SO(n) \ltimes \mathbb{R}^n$, $n \geq 2$, et l'espace quotient des orbites coadjointes admissibles.
- Au troisième chapitre, nous montrons que, pour les produits semi-directs $G_n = U(n) \ltimes \mathbb{H}_n$, l'application

$$\begin{array}{ccc} \hat{G}_n & \longrightarrow & \mathfrak{g}_n^{\ddagger}/G_n \\ \pi_\ell & \mapsto & \mathcal{O}_\ell \end{array}$$

  est continue, où $\mathfrak{g}_n^{\ddagger}/G_n$ désigne l'espace des orbites coadjointes admissibles de G_n. En particulier pour le groupe G_1, cette bijection est un homéomorphisme.
- Nous donnons, dans le quatrième chapitre, une caractérisation des représentations unitaires irréductibles d'un groupe de Lie complètement résoluble G. Nous montrons que si $\pi \in \hat{G}$ et si l'orbite coadjointe correspondante $\mathcal{O}_\pi$ est fermée, alors

$$\ker(\pi) = \{f \in L^1(G) : [(f \circ \exp)j]\widehat{}(\mathcal{O}_\pi) = 0\} \Leftrightarrow \text{ l'obite } \mathcal{O}_\pi \text{ est affine,}$$

  où $j(X)$ est le jacobien de la translation à gauche par le vecteur X de l'algèbre de Lie $\mathfrak{g} = Lie(G)$ sur $\mathfrak{g}$.

Chapitre 1

Généralités

Nous donnons dans cette section le matériel nécessaire pour la compréhension de cette thèse. Nous revenons sur la structure des produits semi-directs compacts nilpotents ainsi que leurs duaux unitaires, via la théorie de Mackey. Nous rappelons aussi quelques propriétés sur la topologie du dual unitaire d'un groupe localement compact.

1.1 Représentations unitaires

Soient G un groupe topologique et $\mathcal{H}$ un espace de Hilbert. On note par $\mathcal{L}(\mathcal{H})$ l'espace des opérateurs continus sur $\mathcal{H}$. C'est une algèbre involutive unitaire, l'unité étant l'opérateur identité de $\mathcal{H}$, noté $I_{\mathcal{H}}$. Une représentation de G dans $\mathcal{H}$ est un homomorphisme de groupe de G dans $\mathcal{L}(\mathcal{H})$, vérifiant :

i) $\pi(e) = I_{\mathcal{H}}$ avec e l'élément neutre de G,
ii) $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$, $\forall g_1, g_2 \in G$,
iii) pour tout $v \in \mathcal{H}$, l'application

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{H} \\ g & \mapsto & \pi(g)v \end{array}$$

est continue.

La représentation π est dite irréductible si les seuls sous espaces invariants fermés sont $\{0\}$ ou $\mathcal{H}$. On peut remarquer que, par définition, une représentation de dimension un est irréductible.

La représentation π est dite unitaire, si pour tout $g \in G$, $\pi(g)$ est un opérateur unitaire, i.e.,

$$\forall g \in G,\ \forall v \in \mathcal{H},\ \|\pi(g)v\| = \|v\|.$$

Deux représentations $(\pi_1, \mathcal{H}_1)$ et $(\pi_2, \mathcal{H}_2)$ de G sont dites équivalentes s'il existe une application linéaire A de $\mathcal{H}_1$ dans $\mathcal{H}_2$ telle que

$$A\pi_1(g) = \pi_2(g)A,\ \forall g \in G.$$

On dit que A est un opérateur d'entrelacement.

Dans toute la suite, G désigne un groupe compact et dg une mesure de Haar sur G.

Proposition 1.
i) Toute représentation unitaire de G contient une sous-représentation de dimension finie.
ii) Toute représentation unitaire irréductible de G est de dimension finie.

Théorème 1. *Soit π une représentation $\mathbb{C}$-linéaire de G dans un espace hilbertien $\mathcal{H}$ de dimension d_π. Alors pour tout u, $v \in \mathcal{H}$,*

$$\int_G |\langle \pi(g)u, v\rangle|^2 dg = \frac{1}{d_\pi}\|u\|^2\|v\|^2,$$

et, par polarisation, pour u, v, u', $v' \in \mathcal{H}$,

$$\int_G \langle \pi(g)u, v\rangle \overline{\langle \pi(g)u', v'\rangle} dg = \frac{1}{d_\pi}\langle u, u'\rangle \overline{\langle v, v'\rangle}.$$

On désigne par $L^2_\pi(G)$ le sous-espace de $L^2(G)$ engendré par les coefficients de la représentation π, i.e., les fonctions de la forme

$$g \mapsto \langle \pi(g)u, v\rangle\ \ (u,\ v \in \mathcal{H}).$$

Théorème 2. *Soient $(\pi, \mathcal{H})$ et $(\pi', \mathcal{H}')$ deux représentations unitaires irréductibles d'un groupe compact G qui ne sont pas équivalentes. Alors $L^2_\pi(G)$ et $L^2_{\pi'}(G)$ sont deux sous espaces orthogonaux de $L^2(G)$:*

$$\int_G \langle \pi(g)u, v\rangle \overline{\langle \pi'(g)u', v'\rangle} dg = 0$$

(u, $v \in \mathcal{H}$, u', $v' \in \mathcal{H}'$).

On en déduit que deux représentation irréductibles π_1 et π_2 d'un groupe compact G sont équivalentes si et seulement si les espaces $L^2_{\pi_1}(G)$ et $L^2_{\pi_2}(G)$ sont égaux.

Théorème 3. *(Théorème de Peter-Weyl) Soit $\hat{G}$ l'ensemble des classes d'équivalences de représentations unitaires irréductibles de G. Alors :*

$$L^2(G) = \overline{\bigoplus_{\pi \in \widehat{G}} L^2_\pi(G)}.$$

1.2 Orbites coadjointes

Soit G un groupe de Lie d'algèbre de Lie $(\mathfrak{g}, [.,.])$. Le groupe G agit sur $\mathfrak{g}$ par la représentation adjointe Ad et sur $\mathfrak{g}^*$, l'espace vectoriel dual de $\mathfrak{g}$, par la représentation coadjointe Ad^* définie par

$$\langle Ad^*(g)l, X\rangle = \langle g.l, X\rangle = \langle l, Ad(g^{-1})X\rangle,\ g \in G, l \in \mathfrak{g}^*, X \in \mathfrak{g}.$$

Pour $l \in \mathfrak{g}^*$, on note par

$$\mathfrak{g}(l) := \{X \in \mathfrak{g} |\ \langle l, [X, \mathfrak{g}]\rangle = \{0\}\}$$

le stabilisateur de l dans $\mathfrak{g}$, et par

$$G_l := \{g \in G |\ g.l = l\}$$

le stabilisateur de l dans G. L'ensemble

$$G.l := \{g.l |\ g \in G\} =: \mathcal{O}(l) \subset \mathfrak{g}^*$$

est appelé G-orbite coadjointe de l. On désigne par $\mathfrak{g}^*/G$ l'espace des orbites coadjointes muni de la topologie quotient, i.e., U est un ouvert de $\mathfrak{g}^*/G$ si et seulement si $p_G^{-1}(U)$ est un ouvert de $\mathfrak{g}^*$, où p_G est la projection canonique de $\mathfrak{g}^*$ dans $\mathfrak{g}^*/G$.

Proposition 2. *Soit $(\mathcal{O}_k)_{k\in\mathbb{N}}$ une suite d'éléments dans $\mathfrak{g}^*/G$. Alors $(\mathcal{O}_k)_k$ converge vers une orbite $\mathcal{O}$ dans $\mathfrak{g}^*/G$ si et seulement si pour tout $l \in \mathcal{O}$, il existe une suite $l_k \in \mathcal{O}_k$, $k \in \mathbb{N}$ telle que $(l_k)_k$ converge vers l.*

Démonstration. Si pour tout $k \in \mathbb{N}$, il existe $l_k \in \mathcal{O}_k$ tel que $\lim_{k\to\infty} l_k = l$, alors pour chaque voisinage G-invariant U de $\mathcal{O}$ dans $\mathfrak{g}^*$, il existe $k_U \in \mathbb{N}$ tel que $l_k \in U$, $\forall k \geq k_U$. D'où

$$\mathcal{O}_k \subset U,\ \forall k \geq k_U.$$

Inversement, supposons que $(\mathcal{O}_k)_k$ converge vers une orbite $\mathcal{O}$ dans l'espace des orbites $\mathfrak{g}^*/G$. Alors pour tout $l \in \mathcal{O}$, on peut trouver une famille décroissante de voisinages ouverts relativement compacts $(V_n)_n$ de l telle que

$$\overline{V}_{n+1} \subset V_n \text{ et } \bigcap V_n = \{l\}.$$

Les ensembles

$$U_n := \mathrm{A}d(G)V_n$$

sont des voisinages ouverts G-invariants de $\mathcal{O}$. Donc, il existe $k_n \in \mathbb{N}$ tel que $\mathcal{O}_k \subset U_n$ pour tout $k \geq k_n$. On peut supposer que la suite $(k_n)_n$ est croissante et que $\lim\limits_{n\to\infty} k_n = +\infty$. Pour $k_n \leq k \leq k_{n+1}$, on choisit un élément

$$l_k \in \mathcal{O}_k \cap V_n.$$

Si V est un voisinage de l alors V contient V_n pour certain $n \in \mathbb{N}$ et par suite $l_k \in V$ pour tout $k \geq k_n$. Ceci prouve que $\lim\limits_{k\to\infty} l_k = l$. □

1.3 Représentations induites

Dans ce paragraphe, G désigne un groupe de Lie d'algèbre de Lie $\mathfrak{g}$. Soient dg une mesure invariante à gauche sur G et Δ_G la fonction module de G, qui est définit par la relation :

$$\int_G f(gx^{-1})dg = \Delta_G(x) \int_G f(g)dg,$$

pour tout $x \in G$, et $f \in C_c(G)$, l'espace des fonctions continues sur G à support compact.
Soit H un sous-groupe fermé de G d'algèbre de Lie $\mathfrak{h}$. On note par $\Delta_{H,G}$ le caractère positif de H défini par

$$\Delta_{H,G}(h) = \frac{\Delta_H(h)}{\Delta_G(h)},$$

pour tout $h \in H$. Comme

$$\Delta_G(x) = |\det(\mathrm{A}d(x))|^{-1} \ (x \in G),$$

on a

$$\Delta_{H,G}(\exp(X)) = e^{\mathrm{tr}_{\mathfrak{g}/\mathfrak{h}}(\mathrm{ad}X)} \ (X \in \mathfrak{h}),$$

où exp est l'application exponentielle de $\mathfrak{g}$ dans G, et ad est la représentation adjointe de l'algèbre de Lie $\mathfrak{g}$ sur $\mathfrak{g}$. Il est clair que si H est un sous-groupe distingué de G alors $\Delta_{H,G} = 1$.

Désignons par $\mathcal{E}(G,H)$ l'espace des fonctions continues φ sur G, à valeurs dans $\mathbb{C}$, à support compact modulo H vérifiant la relation de covariance

$$\varphi(gh) = \Delta_{H,G}(h)\varphi(g) \ (g \in G, h \in H).$$

Le groupe G opère sur cet espace par translation à gauche. D'autre part, il existe sur $\mathcal{E}(G,H)$ une forme linéaire positive unique (à un scalaire multiplicatif près) G-invariante (pour les détails voir [B-A]). On la note généralement par $\nu_{G,H}$ et on a ainsi

$$\nu_{G,H}(\varphi) = \oint_{G/H} \varphi(g)d\nu_{G,H}(g).$$

Il est bien connu que si $\Delta_G = \Delta_H$ sur H, alors $\nu_{G,H}$ est une mesure G-invariante sur l'espace homogène G/H et $\mathcal{E}(G,H) = C_c(G/H)$.
On se donne maintenant une représentation unitaire ρ de H dans un espace de Hilbert $\mathcal{H}_\rho$. On considère
l'espace suivant

$$\begin{aligned}\mathcal{E}_\rho(G,H) &= \{\varphi : G \longrightarrow \mathcal{H}_\rho, \text{ continue à support compact modulo } H, \\ &\quad \text{telle que } \varphi(gh) = \Delta_{H,G}(h)^{\frac{1}{2}}\rho(h)^{-1}\varphi(g), \forall g \in G, \forall h \in H\}.\end{aligned}$$

Comme

$$\|\varphi(gh)\|^2_{\mathcal{H}_\rho} = \Delta_{H,G}(h)\|\varphi(g)\|^2_{\mathcal{H}_\rho},$$

la fonction

$$\|\varphi\|^2_{\mathcal{H}_\rho} : g \mapsto \|\varphi(g)\|^2_{\mathcal{H}_\rho}$$

est un élément de l'espace $\mathcal{E}(G,H)$. Ceci nous permet de munir $\mathcal{E}_\rho(G,H)$ de la norme L^2 définie par

$$\|\varphi\|_2 = \Big(\oint_{G/H} \|\varphi(g)\|^2_{\mathcal{H}_\rho} d\nu_{G,H}(g)\Big)^{\frac{1}{2}}.$$

La représentation induite $\pi = \mathrm{ind}_H^G\rho$ de G est la représentation régulière à gauche sur le complété $L^2(G/H,\rho)$ de l'espace $\mathcal{E}_\rho(G,H)$ par rapport à la norme définie ci-dessus, i.e.

$$(\pi(x)\varphi)(y) = \varphi(x^{-1}y), \ \forall x, y \in G, \varphi \in L^2(G/H,\rho).$$

Cette méthode est fréquement utilisée pour la construction des représentations unitaires à partir d'un sous-groupe. En particulier, pour les représentations unitaires dites monomiales qui sont les représentations induites par un caractère unitaire d'un sous-groupe fermé. Il est connu ([B-A], [Bo]) que les groupes exponentiels qu'on va introduire ultérieurement sont monomiales, i.e., toute représentation unitaire irréductible est équivalente à une représentation monomiale.

1.4 Groupes de Lie nilpotents et exponentiels

1.4.1 Définitions

Soit $(\mathfrak{g}, [,])$ une algèbre de Lie réelle de dimension finie.

On considère la suite décroissante de sous-ensembles $(\mathfrak{g}^k)$ définie par $\mathfrak{g}^1 = \mathfrak{g}$, $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}]$ et par récurence

$$\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}], \forall k \in \mathbb{N}$$

L'algèbre $\mathfrak{g}$ est dite nilpotente si $\mathfrak{g}^k = \{0\}$ pour un certain $k \in \mathbb{N}$.

Un groupe de Lie G est dit nilpotent si son algèbre de Lie $\mathfrak{g}$ est nilpotente.

On considère maintenant une deuxième catégorie de suite décroissante de sous-ensembles $(\mathfrak{g}^{(k)})$ définie par $\mathfrak{g}^{(1)} = \mathfrak{g}$, $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$ et par récurence

$$\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}], \forall k \in \mathbb{N}$$

L'algèbre $\mathfrak{g}$ est dite résoluble si $\mathfrak{g}^{(k)} = \{0\}$ pour un certain $k \in \mathbb{N}$.

Un groupe de Lie G connexe simplement connexe et son algèbre de Lie $\mathfrak{g}$ sont dits résolubles exponentiels ou plus simplement exponentiels, si l'application exponentielle :

$$exp : \mathfrak{g} \longrightarrow G$$

est un difféomorphisme de classe C^∞. Désignons par log son application réciproque.

Dans la suite G désignera un groupe de Lie exponentiel connexe simplement connexe, dont l'algèbre de Lie sera notée $\mathfrak{g}$. Soit $\mathfrak{g}^*$ l'espace vectoriel des formes linéaires sur $\mathfrak{g}$.

Soit $l \in \mathfrak{g}^*$. On définit une forme bilinéaire alternée sur $\mathfrak{g} \times \mathfrak{g}$ par

$$B_l(X, Y) = \langle l, [X, Y] \rangle, \ \forall X, Y \in \mathfrak{g}.$$

On appelle polarisation pour l dans $\mathfrak{g}$ toute sous algèbre $\mathfrak{p}_l$ de $\mathfrak{g}$ vérifiant :

(i) $\mathfrak{p}_l$ est isotrope pour B_l, i.e., $\langle l, [\mathfrak{p}_l, \mathfrak{p}_l] \rangle = 0$,

(ii) $\dim(p_l) = \frac{1}{2}(\dim(\mathfrak{g}) + \dim(\mathfrak{g}(l)))$.

La polarisation p_l est dite une polarisation de Pukanszky si

$$Ad^*(P_l)l = l + \mathfrak{p}_l^{\perp}, \text{ où } P_l = \exp(\mathfrak{p}_l).$$

Si G est un groupe de Lie nilpotent, toute polarisation satisfait la condition de Pukanszky.

Le caractère unitaire χ_l de P_l associé à l est donné par l'expression suivante

$$\chi_l(expX) = e^{-i\langle l,X\rangle}, \forall X \in \mathfrak{p}_l.$$

On dit que la G-orbite $G.l$ de $l \in \mathfrak{g}^*$ est saturée par rapport à un idéal de codimension 1 $\mathfrak{g}_0 = Lie(G_0)$ dans $\mathfrak{g}$, si $\mathfrak{g}(l) \subset \mathfrak{g}_0$. On a ainsi $G.l = G.l + \mathfrak{g}_0^{\perp}$ et

$$\dim(G_0.l_0) = \dim(G.l) - 2, \; l_0 = l_{|\mathfrak{g}_0}.$$

1.4.2 Méthode des orbites

Le dual unitaire $\widehat{G}$ de G peut être paramétrisé via la méthode des orbites de Kirillov-Bernat-Vergne.
Soient $l \in \mathfrak{g}^*$ et $\mathfrak{p}_l$ une polarisation de Pukanszky en l. On définit la représentation $\pi_{l,\mathfrak{p}_l}$ par :

$$\pi_{l,\mathfrak{p}_l} = \mathrm{ind}_{P_l}^G \chi_l,$$

avec $P_l = \exp(\mathfrak{p}_l)$.

Théorème 4. *$\pi_{l,\mathfrak{p}_l}$ est une représentation irréductible de G et sa classe d'équivalence $[\pi_{l,\mathfrak{p}_l}]$ ne dépend que de l'orbite coadjointe de l. Chaque représentation irréductible π est équivalente à une représentation $\pi_{l,\mathfrak{p}_l}$ induite d'un caractère χ_l d'une polarisation de Pukanszky. De plus l'application*

$$\begin{array}{rcl} \Theta : \mathfrak{g}^*/G & \longrightarrow & \widehat{G} \\ G.l & \longmapsto & [\pi_{l,\mathfrak{p}_l}] =: \pi_{G.l}, \end{array}$$

appelée l'application de Kirillov, est un homéomorphisme.

Pour les détails, voir [Lep-Lud].

1.5 Produit semi-direct compact nilpotent

Soient N un groupe de Lie nilpotent d'algèbre de Lie $\mathfrak{n}$ et K un sous groupe compact du groupe d'automorphismes de N, noté $Aut(N)$. On peut définir alors le produit semi-direct $G = K \ltimes N$ par la loi de groupe suivante :

$$(k_1, x_1)(k_2, x_2) = (k_1k_2, x_1k_1.x_2), \; (k_1, k_2 \in K, \; x_1, x_2 \in N).$$

Soit $\pi \in \widehat{N}$, le dual unitaire de N. Pour tout $k \in K$, on définit la représentation π_k par

$$\pi_k(x) := \pi(k.x).$$

Le stabilisateur de π sous cette action est $K_\pi := \{k \in K, \pi_k \simeq \pi\}$. Notons pour $l \in \mathfrak{n}^*$, l'espace vectoriel dual de $\mathfrak{n}$, et pour $k \in K$

$$l_k(X) := \langle l, k.X \rangle,\ \ X \in \mathfrak{n}.$$

Alors pour $k, k' \in K$, on a $\pi_{kk'} = (\pi_k)_{k'}$ et $(l_k)_{k'} = l_{kk'}$.
On désigne par $\mathcal{O}_\pi$ l'orbite coadjointe associée à π dans $\mathfrak{n}^*$, on a alors pour tout $k \in K$

$$\mathcal{O}_{\pi_k} = (\mathcal{O}_\pi)_k.$$

En effet, pour tout $k \in K$ et $f \in \mathcal{S}(N)$, l'espace des fonctions de Schwartz définies sur N, on a

$$\pi_k(f) = \int_N \pi(k.x) f(x) dx = \int_N \pi(x) f(k^{-1}.x) dx = \pi(f^k),$$

où $f^k(x) := f(k^{-1} \cdot x),\ x \in G$. Donc

$$tr(\pi_k(f)) = \int_{\mathcal{O}_\pi} \widehat{f^k \circ \exp}(q) d\mu_{\mathcal{O}_\pi}(q).$$

Or

$$\begin{aligned}\widehat{f^k \circ \exp}(q) &= \int_{\mathfrak{n}} f^k \circ \exp(y) e^{-i<q,y>} dy = \int_{\mathfrak{n}} f \circ \exp(k^{-1} \cdot y) e^{-i<q,y>} dy \\ &= \int_n f \circ \exp(y) e^{-i<q,k\cdot y>} dy = \widehat{f \circ \exp}(q_k).\end{aligned}$$

Il s'ensuit que

$$tr(\pi_k(f)) = \int_{(\mathcal{O}_\pi)_k} \widehat{f \circ exp}(q) d\mu_{(\mathcal{O}_\pi)_k}(q).$$

On en déduit alors que K_π est le stabilisteur de $\mathcal{O}_\pi$.

Il est bien connu qu'il existe une représentation projective de K_π, notée W_π, telle que, pour tout $k \in K_\pi$, $W_\pi(k)$ est un opérateur d'entrelacement avec

$$\pi_k(x) = W_\pi(k) \pi(x) W_\pi(k)^{-1}, \forall x \in N.$$

De plus, les deux opérateurs $W_\pi(k_1 k_2)$ et $W_\pi(k_1) \circ W_\pi(k_2)$ entrelacent π et $\pi_{k_1 k_2}$ $\forall k_1, k_2 \in K_\pi$. Cette relation nous permet de définir l'application

$$\sigma(= \sigma_\pi) : K_\pi \times K_\pi \longrightarrow T = \{z \in \mathbb{C}, |z| = 1\}$$

vérifiant $W_\pi(k_1 k_2) = \sigma(k_1, k_2) W_\pi(k_1) W_\pi(k_2)$. On dit que W_π est une σ-représentation de K_π.

Théorème 5. *Soit $\pi \in \widehat{N}$, et on suppose que W_π est une σ-représentation de K_π. Soit T une $\overline{\sigma}$-représentation de K_π. Alors $\rho := T \otimes \pi W_\pi$ est une représentation irréductible de $K_\pi \ltimes N$. Soit $\widetilde{\rho} = ind_{K_\pi \ltimes N}^{K \ltimes N}(\rho)$ la représentation de $K \ltimes N$ induite de ρ sur l'espace $L^2(K \ltimes N/K_\pi \ltimes N, \rho)$. Alors $\widetilde{\rho} \in \widehat{K \ltimes N}$, et toute représentation irréductible de $K \ltimes N$ est obtenue de cette façon. On a de plus*

$$ind_{K_\pi \ltimes N}^{K \ltimes N}(\rho)_{|K} \simeq ind_{K_\pi}^{K}(\rho_{|K_\pi}) = ind_{K_\pi}^{K}(T \otimes W_\pi),$$

et

$$L^2(K \ltimes N/K_\pi \ltimes N, \rho) \cong L^2(K/K_\pi, T \otimes W_\pi).$$

Pour les détails voir [Mackey1].

1.6 Théorie des orbites pour les groupes de Lie à nilradical co-compact

Soit $G = HN$ un groupe de Lie à nilradical co-compact d'algèbre de Lie $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$.

Définition 1. *Une forme linéaire l sur $\mathfrak{g}$ est dite admissible s'il existe un caractère unitaire χ_l de la composante neutre G_l^0 du stabilisateur G_l de l dans G tel que $d\chi_l = il_{|\mathfrak{g}(l)}$.*

Définition 2. *Une forme linéaire l sur $\mathfrak{g}$ est dite alignée si elle vérifie*

$$G_l = H_l N_l \text{ et } G_\theta = H_\theta N_\theta,$$

où $\theta = l_{|\mathfrak{n}}$.

Soit l une forme linéaire admissible alignée sur $\mathfrak{g}$. La restriction ξ de l sur $\mathfrak{h}(\theta)$ est admissible et indépendante de l'alignement de l. De plus, on a $(H_\theta)_\xi = H_l$. On considère l'espace des sections holomorphes

$$\Gamma(\chi_\xi) = \{f : H_\theta^0/(H_\theta^0)_\xi \to E_{\chi_\xi}, \text{ holomorphe telle que } p \circ f = 1\}.$$

où

$$\begin{aligned} E_{\chi_\xi} &= (H_\theta^0 \times_{\chi_\xi} \mathbb{C})/(H_\theta^0)_\xi \\ &= \{[h, z] = [hh_\xi, \chi_\xi(h_\xi)^{-1}z] : h \in H_\theta^0, h_\xi \in (H_\theta^0)_\xi, z \in \mathbb{C}\}, \end{aligned}$$

et p est la projection canonique, i.e. $p[h, z] = h.(H_\theta^0)_\xi$.
D'après le théorème de Borel-Weil, la représentation ν_ξ définie par

$$\nu_\xi(h)f(x) = h.f(h^{-1}.x)$$

est une représentation unitaire irréductible de H_θ^0 sur $\Gamma(\chi_\xi)$.
Lipsman a prouvé qu'il existe $\tau \in \widehat{(H_\theta)_\xi}$ telle que $\tau_{|H_\varphi^0}$ est un multiple du caractère χ_ξ (car $(H_\theta)_\xi^0$ est distingué). Notons par V_τ l'espace vectoriel complexe de τ. On considère le fibré vectoriel holomorphe

$$\begin{aligned} E_\tau &= (H_\theta \times V)/(H_\theta)_\xi \\ &= \{[h,v] = [hh_\xi, \tau(h_\xi)^{-1}v] : h \in H_\theta, h_\xi \in (H_\theta)_\xi, v \in V_\tau\}. \end{aligned}$$

H_θ agit par translation à gauche sur E_τ. On construit l'espace des sections holomorphes

$$\Gamma(\tau) = \{f : H_\theta/(H_\theta)_\xi \to E_\tau, \text{ holomorphe telle que } p \circ f = 1\}$$

où $p[h,v] = h.(H_\theta)_\xi$. La représentation $\sigma_{\xi,\tau}$ définie par

$$\sigma_{\xi,\tau}(h)f(x) = h.f(h^{-1}.x)$$

est une représentation irréductible de H_θ sur $\Gamma(\tau)$ et toutes les représentations irréductibles de H_θ sont obtenues de cette façon.
D'après [Lip], il existe une bijection entre $\check{H}_l$, l'ensemble des représentations unitaires irréductibles de dimension finie τ de $H_l = (H_\theta)_\xi$ telles que $\tau_{|H_l^0}$ est un multiple de χ_ξ, et l'ensemble des représentations unitaires irréductibles de dimension finie $\sigma_{\xi,\tau}$ de H_θ dont la restriction sur H_θ^0 est un multiple de $\sum^{\oplus}_{H_\theta^0/(H_\theta^0)_\xi} h.\nu_\xi$.
Soit $\gamma \in \hat{N}$ induite de θ et $\tilde{\gamma}$ l'extension canonique de γ sur $H_\theta N$, alors la représentation $\pi_{l,\tau} = \underset{hol}{ind}{}^G_{H_\theta N}\, \sigma_{\xi,\tau} \otimes \tilde{\gamma}$ est une représentation unitaire irréductible de G et tous les éléments de $\hat{G}$ sont obtenus de cette façon.

1.7 Topologie sur le dual unitaire d'un groupe localement compact

Dans ce paragraphe, G désigne un groupe localement compact, et $\widehat{G}$ l'ensemble des classes d'équivalence de représentations unitaires irréductibles de G. On se donne $(\pi, \mathcal{H}_\pi)$ une représentation unitaire irréductible de G sur l'espace de Hilbert $\mathcal{H}_\pi$. Soit $f \in L^1(G)$, on lui associe sa transformée de Fourier en π définie par l'opérateur

$$\pi(f) := \int_G f(g)\pi(g)dg.$$

Cette représentation de $L^1(G)$, appelée représentation intégrée, est définie sur $\mathcal{H}_\pi$. Elle vérifie que

$$\|\pi(f)\|_{op} := \sup_{\|\xi\|_{\mathcal{H}_\pi \leq 1}} \|\pi(f)\xi\|_{\mathcal{H}_\pi} \leq \|f\|_1$$

et que

$$\pi(f)^* = \pi(f^*)$$

où $f^*(x) = \Delta_G(x^{-1})\overline{f(x^{-1})}$ pour tout $x \in G$.

On considère sur $L^1(G)$ la norme $\|.\|_{C^*}$ définie par

$$\|f\|_{C^*} := \sup_{\pi \in \widehat{G}} \|\pi(f)\|_{op}.$$

Définition 3. *La C^*-algèbre de G, noté $C^*(G)$, est définie comme le complété de $L^1(G)$ pour la norme $\|.\|_{C^*}$.*

Proposition 3. *Le dual unitaire de $C^*(G)$ est en bijection avec $\hat{G}$.*

Notons par $Prim(C^*(G))$ l'ensemble des idéaux primitifs de la C^*-algèbre de G, muni de la topologie de Jacobson. I est un fermé dans $Prim(C^*(G))$ si et seulement si I est un idéal primitif maximal. L'espace dual $\hat{G}$ est muni de la topologie image réciproque de la topologie de Jacobson de l'espace des idéaux primitifs $Prim(C^*(G))$ par la surjection canonique

$$\begin{array}{ccc} \hat{G} & \longrightarrow & Prim(C^*(G)) \\ \pi & \longmapsto & ker_{C^*(G)}(\pi) \end{array}$$

Autrement dit, si $\pi \in \hat{G}$ et $Y \subset \hat{G}$, alors π est dans $\overline{Y}$, la fermeture de Y, si et seulement si

$$\bigcap_{\sigma \in Y} ker(\sigma) \subset ker(\pi).$$

On dit que π est faiblement contenue dans Y.

L'espace $\hat{G}$ est un espace de Baire localement quasi-compact. Si G est discret, $C^*(G)$ admet un élément unité, donc $\hat{G}$ est quasi compact. Si G est séparable, $\hat{G}$ est séparable. Si $\hat{G}$ est un espace de Hausdorff, alors pour tout $x \in G$, l'application $\pi \mapsto \pi(x)$ est continue.

Soit maintenant $\pi \in \hat{G}$, les fonctions de type positif associées à π sont, par définition, définies sur G par $x \mapsto \langle \pi(x)\xi, \xi \rangle$, où ξ est un vecteur totaliseur de π. Ce sont effectivement des fonctions continues "de type positif", c'est-à-dire des fonctions φ telles que, pour tous $x_1, ..., x_n \in G$ et $c_1, ..., c_n$ complexes,

$$\sum c_i \overline{c_j} \varphi(x_i x_j^{-1}) \geq 0.$$

Théorème 6. *Soient $\pi \in \widehat{G}$ et $(\pi_k)_{k\in\mathbb{N}}$ une famille de représentations unitaires irréductibles de G. Alors $(\pi_k)_k$ converge vers π dans $\hat{G}$ si, et seulement si, pour un vecteur unitaire ξ de $\mathcal{H}_\pi$ il existe ξ_k dans $\mathcal{H}_{\pi_k}$ tels que $\|\xi_k\|_{\mathcal{H}_{\pi_k}} = 1$ et $\langle \pi_k(.)\xi_k, \xi_k\rangle$ converge uniformément sur tout compact de G vers $\langle \pi(.)\xi, \xi\rangle$.*

La topologie faible $\sigma(L^\infty(G), L^1(G))$ sur l'ensemble des fonctions continues de type positif φ de G telles que $\varphi(e) = 1$ coïncide avec la topologie de la convergence uniforme sur tout compact de G.

Théorème 7. *Soit $(\pi_k, \mathcal{H}_{\pi_k})_{k\in\mathbb{N}}$ une famille de représentations unitaires irreducibles de G. Alors $(\pi_k)_k$ converge vers π dans $\widehat{G}$, si et seulement si, pour un (resp. pour chaque) vecteur non nul ξ dans $\mathcal{H}_\pi$, il existe $\xi_k \in \mathcal{H}_{\pi_k}$ telle que la suite des formes linéaires $(\langle \pi_k(.)\xi_k, \xi_k\rangle)_k \subset C^*(G)'$ converge faiblement sur un sous espace dense dans la C^*-algèbre$C^*(G)$ de G vers la forme linéaire $\langle \pi(.)\xi, \xi\rangle$.*

Si G est un groupe de Lie, alors on désigne respectivement par $\mathfrak{g}$ l'agèbre de Lie de G et par $\mathcal{U}(\mathfrak{g})$ l'algèbre enveloppante de $\mathfrak{g}$. pour une représentation unitaire $(\pi, \mathcal{H}_\pi)$ de G, on se donne $\mathcal{H}_\pi^\infty$ le sous espace de $\mathcal{H}_\pi$ constitué des vecteurs C^∞ associés à π.

Corollaire 1. *Soit π une représentation unitaire irréductible de G sur l'espace hilbertien $\mathcal{H}_\pi$. Soit $(\pi_k)_{k\in\mathbb{N}}$ une famille de $\widehat{G}$. Si $(\pi_k)_k$ converge vers π dans $\widehat{G}$, alors pour un vecteur unitaire ξ de $\mathcal{H}_\pi^\infty$, il existe ξ_k dans $\mathcal{H}_{\pi_k}^\infty$ $(k \in \mathbb{N})$, telle que $\|\xi_k\|_{\mathcal{H}_{\pi_k}} = 1$ et $\langle \pi_k(D)\xi_k, \xi_k\rangle$ converge vers $\langle \pi(D)\xi, \xi\rangle$, pour tout D dans $\mathcal{U}(\mathfrak{g})$.*

Exemple 1. On va considérer maintenant le groupe abélien $G = \mathbb{R}^n$. Donc $\widehat{G} := \{\chi_l, l$ forme linéaire sur $\mathbb{R}^n\}$ où le caractère unitaire χ_l est défini par $\chi_l(x) := e^{-i\langle l,x\rangle}$, $\forall x \in \mathbb{R}^n$.

Théorème 8. *Soit $(l_k)_{k\in\mathbb{N}}$ une suite de formes linéaires sur $\mathbb{R}^n$. Alors $(\chi_{l_k})_k$ converge localement uniformément vers χ_l si, et seulement si, $(l_k)_k$ converge vers l.*

Démonstration. " $\Leftarrow$ " Soit $(l_k)_k$ une suite de formes linéaires sur $\mathbb{R}^n$ converge l. Montrons que $\forall r > 0$, $\chi_{l_k}(u)$ tend vers $\chi_l(u)$, $\forall u \in B(0, r)$. Or

$$|\chi_{l_k}(u) - \chi_l(u)| = |e^{-i\langle l_k - l, u\rangle} - 1|.$$

Si on pose $f_k(u) = e^{-i\langle l_k - l,u\rangle}$ alors la différentielle de cette fonction est $df_k(u) = -i\langle l_k - l, u\rangle e^{-i\langle l_k - l,u\rangle}$. Et par la suite, d'après le théorème d'inégalité des accroissements finis on obtient

$$|\chi_{l_k}(u) - \chi_l(u)| \leq \|l_k - l\|\|u\| \leq r\|l_k - l\|, \forall u \in B(0, r).$$

D'où si $l_k \overset{k+\infty}{\longrightarrow} l$ alors $|\chi_{l_k}(u) - \chi_l(u)| \overset{k+\infty}{\longrightarrow} 0$.
" $\Rightarrow$ " hypothèse : $(\chi_{l_k})_n$ converge vers χ_l localement et uniformément. Pour montrer que l_k converge vers l, il suffit de prouver que $(\langle l_k, e_j\rangle)_k$ tend vers $\langle l, e_j\rangle$ $\forall j = 1, .., n$ où $(e_1, e_2, ..., e_n)$ est une base orthonormale de $\mathbb{R}^n$. On a par hypothèse $\forall t \in \mathbb{R}$, $(\chi_{l_k}(te_j))_k$ converge localement et unifomément vers $\chi_l(te_j)$. On note par D_j la derivée partielle dans la direction de e_j. On prend $\varphi \in C_c^\infty(\mathbb{R}^n)$ telle que support$(\varphi)\subset B(0, r)$ et $\widehat{\varphi}(l) = 1$. Donc $\langle \chi_{l_k}, D_j\varphi\rangle \overset{k+\infty}{\longrightarrow} \langle \chi_l, D_j\varphi\rangle$. Or pour tout $k \in \mathbb{N}$

$$\begin{aligned}
\langle \chi_{l_k}, D_j\varphi\rangle &:= \int_{B(0,r)} e^{-i\langle l_k,u\rangle} D_j\varphi(u)du \\
&= -\int_{B(0,r)} D_j(e^{-i\langle l_k,u\rangle})\varphi(u)du \\
&= \int_{B(0,r)} i\langle l_k, e_j\rangle e^{-i\langle l_k,u\rangle}\varphi(u)du \\
&= i\langle l_k, e_j\rangle\widehat{\varphi}(l_k).
\end{aligned}$$

Ce qui implique que $\langle l_k, e_j\rangle\widehat{\varphi}(l_k) \overset{k+\infty}{\longrightarrow} \langle l, e_j\rangle\widehat{\varphi}(l)$. D'où $\langle l_k, e_j\rangle$ converge vers $\langle l, e_j\rangle$ pour tout $j \in \{1, ..., n\}$. □

On a alors $\widehat{\mathbb{R}^n}$ est homéomorphe à $\mathbb{R}^n$. Ceci peut être vu par le théorème de Kirillov puisque $(\mathbb{R}^n, +)$ est un groupe de Lie connexe simplement connexe nilpotent de pas 1.

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Chapitre 2

Dual topology of the motion groups $SO(n) \ltimes \mathbb{R}^n$

Résumé : Dans ce chapitre, on étudie la topologie de l'espace dual du produit semi-direct $M_n = SO(n) \ltimes \mathbb{R}^n$, $n \in \mathbb{N}^*$, et en identifiant $\hat{M}_n$ à l'espace quotient des orbites coadjointes admissibles $\mathfrak{m}_n^\ddagger / M_n$, on montre que cette identification est un homéomorphisme.

Abstract : Let $n \in \mathbb{N}^*$ and let $M_n = SO(n) \ltimes \mathbb{R}^n$ be the corresponding motion group. In this paper, we describe the topology of the dual space $\hat{M}_n$ and identifying $\hat{M}_n$ with the subspace of admissible co-adjoint orbits $\mathfrak{m}_n^\ddagger / M_n$, we show that this identification is a homeomorphism.

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Keywords : Semi-direct product, dual topology, admissible coadjoint orbit space.

2.1 Introduction.

It is known that for a simply connected nilpotent Lie group and more generally for an exponential solvable Lie group $G = \exp\mathfrak{g}$, its dual space $\widehat{G}$ is homeomorphic to the space of co-adjoint orbits $\mathfrak{g}^*/G$ through the Kirillov mapping (see [Lep-Lud]). If we consider semi-direct products $G = K \ltimes N$ of compact connected Lie groups K acting on simply connected nilpotent Lie groups N, then again we have an orbit picture of the dual space of G (see [Lip]) and one can guess that the topology of $\widehat{G}$ is linked to the topology of the admissible co-adjoint orbits.

In this paper we consider the motion groups $M_n := SO(n) \ltimes \mathbb{R}^n$ and we show that in this case the topology of their unitary dual spaces $\hat{M}_n$ is determined by the topology of the space of admissible co-adjoint orbits. For every admissible linear functional ℓ of the Lie algebra $\mathfrak{m}_n$ of M_n, we can construct an irreducible unitary representation π_ℓ by holomorphic induction and every irreducible representation of M_n arises in this manner. We obtain in this fashion a map from the set $\mathfrak{m}_n^\ddagger$ of the admissible linear functionals onto the dual space $\hat{M}_n$ of M_n. Since π_ℓ is equivalent to $\pi_{\ell'}$ if and only if ℓ and ℓ' are in the same M_n-orbit, we obtain finally a homeomorphism between the space of admissible co-adjoint orbits $\mathfrak{m}_n^\ddagger/M_n$ and the dual space $\hat{M}_n$ of M_n in Theorem 2.4.6.

The dual topology of the semi-direct products $K \ltimes N$, where N is an abelian group and K is a compact group, is determined by Baggett in terms of the Fell topology (see Theorem 6.2-A of [Ba]). Other results have already been obtained on the topology of the dual space of M_n. For instance the cortex for general motion groups $K \ltimes \mathbb{R}^n$ has been determined in [Be-Ka] and it has been shown in [Kan-Ta] that for all compact subsets L of M_n, the mapping defined by

$$\psi_L(\pi) = \inf_{\xi \in \mathcal{H}_\pi^1} (\max_{x \in L} \|\pi(x)\xi - \xi\|)$$

is continuous on $\hat{M}_n \backslash \widehat{SO(n)}$, that is, on the set of infinite dimensional representations of M_n, where $\mathcal{H}_\pi^1$ is the unit sphere in $\mathcal{H}_\pi$, the Hilbert space of π.

Here is a brief section-by-section description of the contents of the paper. In paragraph 2, we describe the motion groups and we determine their dual spaces ; the representations attached to an admissible linear functional are obtained via Mackey's little-group method and the dual space of M_n is given by the parameter space $\mathcal{P}_n := \{(r, \rho), r > 0, \rho \in \widehat{SO(n-1)}\} \bigcup \widehat{SO(n)}$. In section 3, referring to the paper [Ba] of Baggett, we shall link the convergence of sequences of elements of $\hat{M}_n$ to the convergence in $\mathcal{P}_n$. In the last section, we use the convergence in the parameter space to show that the orbit space $\mathfrak{m}_n^\ddagger/M_n$ and $\hat{M}_n$ are homeomorphic.

Let us remark that similar results are true for other kinds of motion groups, for instance the groups $SU(n) \ltimes \mathbb{C}^n$. It suffices to adapt our proofs.

2.2 The Motion groups and their dual spaces.

We consider now the rotation group $SO(n)$ acting on the abelian group $\mathbb{R}^n$ by rotation. In this text, $\mathbb{R}^n$ is identified with the space of $n\times 1$ real matrices. Let M_n be the semi-direct product $SO(n)\ltimes\mathbb{R}^n$, equipped with the group law

$$(A,x)(B,y) := (AB, x+Ay). \tag{2.1}$$

We denote by $\mathfrak{m}_n = \mathfrak{so}(n)\oplus\mathbb{R}^n$ the Lie algebra of M_n, and $\mathfrak{m}_n^*$ the vector dual space of $\mathfrak{m}_n$. Then, for all $(A,a)\in M_n$ and all $(B,b)\in\mathfrak{m}_n$ we get

$$\begin{aligned}
Ad((A,a)^{-1})(B,b) &= \frac{d}{ds}\Big|_{s=0}(A,a)^{-1}(e^{sB},sb)(A,a)\\
&= \frac{d}{ds}\Big|_{s=0}(A^t,-A^ta)(e^{sB},sb)(A,a)\\
&= \frac{d}{ds}\Big|_{s=0}(A^te^{sB}A, A^te^{sB}a+sA^tb-A^ta)\\
&= (A^tBA, A^tBa+A^tb).
\end{aligned}$$

From this identity we deduce the Lie bracket

$$[(A,x),(B,y)] = (AB-BA, Ay-Bx)\ (A,B\in\mathfrak{so}(n), x,y\in\mathbb{R}^n).$$

On the Lie algebra $\mathfrak{m}_n$, we have the natural scalar product :

$$\langle(A,x),(B,y)\rangle := \frac{1}{2}\mathrm{tr}(AB^t)+x^ty\ (A,B\in\mathfrak{so}(n), x,y\in\mathbb{R}^n).$$

This scalar product can now be used to identify $\mathfrak{m}_n^*$ with $\mathfrak{m}_n$ and $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$. Every linear functional F on $\mathfrak{m}_n$ corresponds to a unique element $\xi_F\in\mathfrak{m}_n$, such that

$$F(\eta) = \langle\xi_F,\eta\rangle, \eta\in\mathfrak{m}_n.$$

It follows that for all $(A,a)\in M_n$, all $(B,b)\in\mathfrak{m}_\mathfrak{n}$ and all $(U,u)\in\mathfrak{m}_n^*$

$$\begin{aligned}
\langle Ad^*((A,a))(U,u),(B,b)\rangle &:= \langle(U,u), Ad((A,a)^{-1})(B,b)\rangle\\
&= \frac{1}{2}tr(UA^tB^tA)+u^t(A^tBa)+u^t(A^tb)\\
&= \frac{1}{2}tr((AUA^t)B^t)+(Au)^t(Ba)+(Au)^tb.
\end{aligned}$$

On the other hand, the fact that $B=(B_{ij})_{1\le i,j\le n}$ is a skew-symmetric matrix implies that

$$\frac{1}{2}tr((va^t-av^t)B^t) = \frac{1}{2}\sum_{1\le i,j\le n}(v_ia_j-a_iv_j)B_{ij} = v^tBa, \quad \text{for all } v\in\mathbb{R}^n.$$

Hence, we obtain

$$\langle Ad^*((A,a))(U,u),(B,b)\rangle = \langle (AUA^t + ((Au)a^t - a(Au)^t), Au), (B,b)\rangle, \quad (2.2)$$

i.e.,

$$Ad^*((A,a))(U,u) = (AUA^t + [(Au)a^t - a(Au)^t], Au). \quad (2.3)$$

Therefore, for $u \neq 0$, the co-adjoint orbit $\mathcal{O}_{U,u}$ is given by

$$\begin{aligned} \mathcal{O}_{U,u} &= \mathrm{Ad}^*(M_n)(U,u) = \{(AUA^t + [(Au)a^t - a(Au)^t], Au), A \in SO(n), a \in \mathbb{R}^n\} \quad (2.4) \\ &= \{(AUA^t, Au), A \in SO(n)\} + (AW_uA^t, 0), \end{aligned}$$

where $W_u = \{ua^t - au^t, a \in \mathbb{R}^n\}$ is a subspace of dimension $n-1$ of $\mathfrak{so}(n)$.

Remark 2.2.1. We deduce from this expression that the orbit $\mathcal{O}_{U,u}$ is closed and that the M_n-invariant measure $d\beta_{U,u}$ of the orbit $\mathcal{O}_{U,u}$ can be written as

$$\int_{\mathcal{O}_{U,u}} \varphi(q) d\beta_{U,u}(q) = \int_{SO(n)} \int_{W_u} \varphi((AUA^t, Au) + (ABA^t, 0)) dB dA, \varphi \in C_c(\mathcal{O}_{U,u}). \quad (2.5)$$

2.2.1 The dual space of $SO(n)$.

We need a precise description of the irreducible representations of $SO(n)$ (see [Knapp] for details).

A Cartan subalgebra of $\mathfrak{so}(n)$ can be taken to consist of the two-by-two diagonal blocks $\begin{pmatrix} 0 & \theta_j \\ -\theta_j & 0 \end{pmatrix}$, $j = 1, \cdots, [n/2]$ starting from the upper left (here $[m], m \in \mathbb{R}$, denotes the largest integer smaller than m). For an integer $j \in [1, [n/2]]$ denote by e_j the associated evaluation functional on the complexification of the Cartan subalgebra. When n is even, say $n = 2d$, the roots are the functionals $\pm e_i \pm e_j$ with $1 \leq i < j \leq d$. When n is odd, say $n = 2d+1$, the roots are the functionals $\pm e_i \pm e_j$ with $1 \leq i < j \leq d$ and also the $\pm e_j$ with $1 \leq j \leq d$. We take the positive roots to be the $e_i \pm e_j$ with $i < j$ and, when n is odd, the e_j.

The dominant integral forms λ for $SO(n)$ are given by expressions

$$\lambda_1 e_1 + ... + \lambda_d e_d \longleftrightarrow \lambda = (\lambda_1, ..., \lambda_d) \quad (2.6)$$

such that $\lambda_1 \geq ... \geq \lambda_{d-1} \geq |\lambda_d|$ when $n = 2d$ is even, and $\lambda_1 \geq ... \geq \lambda_d \geq 0$ when $n = 2d+1$ is odd, with all the λ_j's understood to be integers. Hence the dual space of $SO(n)$ is determined by the representations τ_λ, given by its highest weight λ.

Let now τ_λ be an irreducible representation of $SO(2d+1)$ with highest weight $(\lambda_1, ..., \lambda_d)$ and let ρ_μ be an irreducible representation of $SO(2d)$ with highest weight $\mu = (\mu_1, ..., \mu_d)$. By the branching theorem for $SO(2d+1)$ with respect to $SO(2d)$ and by the Frobenius reciprocity, the induced representation $\pi_\mu := \mathrm{ind}_{SO(2d)}^{SO(2d+1)} \rho_\mu$ contains τ_λ if and only if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \lambda_{d-1} \geq \mu_{d-1} \geq \lambda_d \geq |\mu_d|. \tag{2.7}$$

Similarly, if τ_λ is an irreducible representation of $SO(2d)$ with highest weight $(\lambda_1, ..., \lambda_d)$ and if ρ_μ is an irreducible representation of $SO(2d-1)$ with highest weight $\mu = (\mu_1, ..., \mu_{d-1})$, then by the branching theorem for $SO(2d)$ with respect to $SO(2d-1)$ and by the Frobenius reciprocity, the representation τ_λ appears in $\pi_\mu := \mathrm{ind}_{SO(2d-1)}^{SO(2d)} \rho_\mu$ if and only if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \lambda_{d-1} \geq \mu_{d-1} \geq |\lambda_d|. \tag{2.8}$$

Furthermore, in the two cases τ_λ is a subrepresentation of multiplicity one in π_μ.

2.2.2 Description of $\hat{M}_n$.

The dual space of M_n has been described by G. Mackey (for details, see [Mackey1] and [Mackey2]).
For each linear form ℓ on $\mathbb{R}^n$ and any irreducible unitary representation ρ of the stabilizer S_ℓ of ℓ in $SO(n)$, we have that

$$\sigma_{(\rho,\ell)} := \rho \otimes \chi_\ell \tag{2.9}$$

is an irreducible unitary representation of $H_\ell = S_\ell \ltimes \mathbb{R}^n$ whose restriction to $\mathbb{R}^n$ is a multiple of the character χ_ℓ of $\mathbb{R}^n$ given by $\chi_\ell(x) = e^{-i\langle \ell, x\rangle}$ ($x \in \mathbb{R}^n$), and the induced representation $\pi_{(\rho,\ell)} := \mathrm{ind}_{H_\ell}^{M_n} \sigma_{(\rho,\ell)}$ is an irreducible representation of M_n. If ℓ and ℓ' are in the same sphere centered at 0, then $\ell' = A \cdot \ell$ for some $A \in SO(n)$ and $S_{\ell'} = AS_\ell A^t$. The representations $\pi_{(\rho,\ell)}$ and $\pi_{(\rho',\ell')}$ (where $\rho'(B) := \rho(A^t BA)$, $B \in S_{\ell'}$) are equivalent (cf. [Mackey1] paragraph 3.9). If $r > 0$ is the radius of the sphere, we denote by χ_r the character associated with the linear form ℓ_r which is identified with the vector $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ r \end{pmatrix}$. The stabilizer S_{ℓ_r} of ℓ_r is the subgroup $SO(n-1)$. Let us write ρ_μ

instead of ρ for the representation of $SO(n-1)$ with highest weight μ and $\pi_{(\mu,r)}$ instead of $\pi_{(\rho_\mu,\ell_r)}$. The representation $\pi_{(\mu,r)}$ is realized on $L^2(SO(n))$ as follows ; for all $(A,x) \in M_n$ and all $B \in SO(n)$

$$\pi_{(\mu,r)}(A,x)F(B) = e^{-i\langle B\ell_r, x\rangle}F(A^{-1}B), \quad (F \in L^2(SO(n))). \tag{2.10}$$

In this way we obtain all the irreducible representations of M_n, which are not trivial on its normal subgroup $\mathbb{R}^n$.
On the other hand, every irreducible unitary representation τ_λ of $SO(n)$ extends to an irreducible representation (also denoted by τ_λ) of the entire group M_n, defined by

$$\tau_\lambda(A,x) := \tau_\lambda(A), A \in SO(n), x \in \mathbb{R}^n.$$

Now Mackey's theory tells us that

Proposition 2.2.2. $\widehat{SO(n) \ltimes} \mathbb{R}^n$ *is in bijection with the set of parameters* $\mathcal{P}_n := \widehat{SO(n-}1) \times \mathbb{R}_+^* \bigcup \widehat{SO(n)}$.

2.2.3 Co-adjoint orbits attached to irreducible representations.

Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We associate to the representation $\pi_{(\mu,r)}$ the linear functional (J_μ, ℓ_r) in $\mathfrak{m}_n^*$ where

$$J_\mu = \begin{pmatrix} \mu_1 J & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \mu_d J & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix},$$

if $n = 2d+1$ is odd and if $n = 2d$ is even, then

$$J_\mu = \begin{pmatrix} \mu_1 J & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \mu_{d-1} J & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

We see that the stabilizer $M_n(\ell)$ of $\ell = (J_\mu, \ell_r)$ in M_n is equal to $M_n(\ell) = SO(n)(\ell) \ltimes \mathbb{R}^n(\ell)$. Indeed, by (2.3), we have that

$$\begin{aligned} M_n(\ell) &= \{(A,a) \in M_n; (AJ_\mu A^t + (A\ell_r a^t - a(A\ell_r)^t), A\ell_r) = (J_\mu, \ell_r)\} \\ &= \{(A,a) \in M_n; A \in SO(n-1), AJ_\mu A^t + (\ell_r a^t - a(\ell_r)^t) = J_\mu\} \\ &= \{(A,a) \in M_n; a \in \mathbb{R}\ell_r, A \in SO(n-1), AJ_\mu A^t = J_\mu\}, \end{aligned}$$

since $AJ_\mu A^t \in \mathfrak{so}(n-1)$ and

$$\ell_r a^t - a(\ell_r)^t = \begin{pmatrix} 0 & \dots & 0 & -ra_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -ra_{n-1} \\ ra_1 & \dots & ra_{n-1} & 0 \end{pmatrix}.$$

Therefore $a \in \mathbb{R}\ell_r = \mathbb{R}^n(\ell)$ and $A \in SO(n)(\ell)$. Hence, ℓ is aligned (see [Lip] Lemma 4.2). A linear functional $\ell \in \mathfrak{m}_n^*$ is called admissible, if there exists a unitary character χ of the connected component of $M_n(\ell)$, such that $d\chi = i\ell_{|\mathfrak{m}_n(\ell)}$. It is clear now that the linear functionals (J_μ, ℓ_r) are all admissible and so, according to [Lip], the representation of M_n obtained by holomorphic induction from the linear functional (J_μ, ℓ_r) is equivalent to the representation $\pi_{(\mu,r)}$ (see [Lip]).

For τ_λ we take the linear functional $(J_\lambda, 0)$ of $\mathfrak{m}_n^*$ defined in the following way :
We identify the linear form λ with the element J_λ in $\mathfrak{so}(n)$ where

$$J_\lambda = \begin{pmatrix} \lambda_1 J & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_d J \end{pmatrix},$$

if $n = 2d$ is even. If $n = 2d + 1$ is odd, then we put

$$J_\lambda = \begin{pmatrix} \lambda_1 J & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda_d J & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

Hence, the representation of M_n obtained by holomorphic induction from $(J_\lambda, 0)$ is equivalent to τ_λ.
We denote by $\mathcal{O}_\lambda$ the co-adjoint orbit of $(J_\lambda, 0)$ and by $\mathcal{O}_{(\mu,r)}$ the co-adjoint orbit of (J_μ, ℓ_r).
Let $\mathfrak{m}_n^\ddagger \subset \mathfrak{m}_n^*$ be the union of all the $\mathcal{O}_{(\mu,r)}$ and of all the $\mathcal{O}_\lambda$ and denote by $\mathfrak{m}_n^\ddagger/M_n$ the corresponding set in the orbit space. It follows now from [Lip], that $\mathfrak{m}_n^\ddagger$ is just the set of all admissible linear functionals of $\mathfrak{m}_n$.

2.3 The topology of the dual space of the motion group M_n.

In this paragraph, we shall describe the topology of the dual space of the semi-direct product $M_n = SO(n) \ltimes \mathbb{R}^n$ in terms of the data $(r > 0, \rho_\mu \in \widehat{SO(n-1)}, \tau_\lambda \in \widehat{SO(n)})$. Let us first recall the description of the dual topology of the semi-direct products of abelian groups with compact groups. This description has been given by L. Baggett in [Ba].

Let G be an abelian group and let K be a compact subgroup of $Aut(G)$, the group of automorphisms of G. One can form the semi-direct product $K \ltimes G$, with group law

$$(k_1, x_1)(k_2, x_2) = (k_1 k_2, x_1 k_1.x_2). \tag{2.11}$$

Let χ be in $\hat{G}$, i.e. a character of G, and K_χ be the stabilizer of χ under the action of K on $\hat{G}$, i.e. the set of all elements $k \in K$ verifying $k.\chi = \chi$. If ρ is an element of the dual space $\widehat{K_\chi}$ of K_χ, the triple $(\chi, (K_\chi, \rho))$ is called cataloguing triple. We denote by $\pi(\chi, K_\chi, \rho)$ the induced representation $\mathrm{ind}_{K_\chi \ltimes G}^{K \ltimes G} \rho \otimes \chi$ which is realized on $L^2(K)$ as follows : for all $x \in G$, and all $k, k_1 \in K$

$$\left(\mathrm{ind}_{K_\chi \ltimes G}^{K \ltimes G} \rho \otimes \chi\right)(k, x) F(k_1) = \chi(k_1^{-1}.x) F(k^{-1} k_1), \ \ (F \in L^2(K)). \tag{2.12}$$

Baggett, in [Ba] (paragraph 2.4-D), has shown that

Proposition 2.3.1. *The mapping* $(\chi, (K_\chi, \rho)) \longrightarrow \pi(\chi, K_\chi, \rho)$ *is onto* $\widehat{K \ltimes G}$.

Denote by $\mathcal{A}(K)$ the set of all pairs (K', ρ') where K' is a closed subgroup of K and ρ' is an irreducible unitary representation of K'. We equip $\mathcal{A}(K)$ with the Fell topology (see [Fe]). We catalogue thus the elements of $\widehat{K \ltimes G}$ by elements of the topological space $\hat{G} \times \mathcal{A}(K)$. Hence, we characterize the topology of $\widehat{K \ltimes G}$ in terms of these parameters, as given in the following theorem (Theorem 6.2-A of [Ba]).

Theorem 2.3.2. *Let* Y *be a subset of* $\widehat{K \ltimes G}$ *and* π *an element of* $\widehat{K \ltimes G}$. π *is weakly contained in* Y *if and only if there exist : a cataloguing triple* $(\chi, (K_\chi, \rho))$ *for* π, *an element* (K', ρ') *of* $\mathcal{A}(K)$, *and a net* $\{(\chi_n, (K_{\chi_n}, \rho_n))\}$ *of cataloguing triples, such that :*

(i) For each n, *the irreducible unitary representation* $\pi(\chi_n, (K_{\chi_n}, \rho_n))$ *of* $K \ltimes G$ *is an element of* Y.

(ii) The net $\{(\chi_n, (K_{\chi_n}, \rho_n))\}$ *converges to* $(\chi, (K', \rho'))$ *in* $\hat{G} \times \mathcal{A}(K)$.

(iii) K_χ *contains* K', *and* $\mathrm{ind}_{K'}^{K_\chi} \rho'$ *contains* ρ.

We come now to describe the dual topology of our motion groups. By $(\chi_r, (SO(n-1), \rho_\mu))$ and $(0, (SO(n), \tau_\lambda))$ we mean respectively the cataloguing triples of the induced representation $\pi_{(\mu,r)}$ and the trivial extension of τ_λ on M_n. Hence, by Theorem 2.3.2 it follows that

Theorem 2.3.3. *Let $r > 0$ and $\rho_\mu \in \widehat{SO(n-1)}$. Then a sequence $(\pi_{(\mu^k,r_k)})_k$ of irreducible representations of M_n converges in $\hat{M}_n$ to $\pi_{(\mu,r)}$ if and only if $(r_k)_k$ tends to r as $k \longrightarrow +\infty$ and $\mu^k = \mu$ for k large enough.*

and that

Theorem 2.3.4. *Let $(\pi_{(\mu^k,r_k)})_k$ be a sequence of irreducible representations of M_n. Then $(\pi_{(\mu^k,r_k)})_k$ converges to τ_λ in $\hat{M}_n$ if and only if $\lim\limits_{k\to\infty} r_k = 0$ and $\tau_\lambda \in \pi_{\mu^k}$ for k large enough.*

Remark 2.3.5. It follows from the preceding theorems that a sequence $(\pi_{(\mu^k,r_k)})_k$ can only have a limit point if the sequences $(\mu^k)_k$ and $(r_k)_k$ are bounded. Furthermore we see that the subset $\widehat{SO(n-1)} \times \mathbb{R}_+^*$ of $\hat{M}_n$ has a Hausdorff topology, but that sequences in $\widehat{SO(n-1)} \times \mathbb{R}_+^*$ which converge to elements in $\widehat{SO(n)}$ have infinitely many different limit points. Of course the subset $\widehat{SO(n)}$ has the discrete topology.

2.4 Convergence of co-adjoint orbits.

We have previously seen that the dual space of our motion group $M_n = SO(n) \ltimes \mathbb{R}^n$ consists of all induced representations $\pi_{(\mu,r)} := \mathrm{ind}_{\mathrm{SO(n-1)\ltimes\mathbb{R}^n}}^{\mathrm{SO(n)\ltimes\mathbb{R}^n}} \rho_\mu \otimes \chi_{\mathrm{r}}$ where r runs over $]0, +\infty[$ and $\rho_\mu \in \widehat{SO(n-1)}$, and all extensions of irreducible unitary representations τ_λ of $SO(n)$ on M_n. The subspace W_{ℓ_r} of Formula (2.4) is generated by the vectors $(E_{n,j} - E_{j,n})$ $1 \le j \le n-1$, where $\{E_{i,j}\}_{1\le i,j\le n}$ is the canonical basis of the space of $n \times n$ real matrices. Then, by definition, the space $\mathfrak{m}_n^\ddagger/M_n$ is the set of all orbits

$$\mathcal{O}_{(\mu,r)} = \{(A(J_\mu + W_{\ell_r})A^t, A\ell_r)/A \in SO(n)\} \tag{2.13}$$

and all orbits

$$\mathcal{O}_\lambda = \{(AJ_\lambda A^t, 0)/A \in SO(n)\}, \tag{2.14}$$

where J_μ and J_λ are as defined in the subsection 2.2.3. In this way we have

$$\mathfrak{m}_n^\ddagger/M_n \cong \mathbb{N}^d \cup \mathbb{N}^{d-1} \times \mathbb{Z} \times]0, +\infty[$$

if $n = 2d + 1$ is odd. If $n = 2d$ is even we have

$$\mathfrak{m}_n^{\ddagger}/M_n \cong \mathbb{N}^{d-1} \times \mathbb{Z} \cup \mathbb{N}^{d-1} \times]0, +\infty[.$$

Lemma 2.4.1. *Let G be a unimodular Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the vector dual space of $\mathfrak{g}$. We denote by $\mathfrak{g}^*/G$ the space of co-adjoint orbits and by $p_G : \mathfrak{g}^* \to \mathfrak{g}^*/G$ the canonical projection. We equip this space with the quotient topology, i.e, a subset U in $\mathfrak{g}^*/G$ is open if and only $p_G^{-1}(U)$ is open in $\mathfrak{g}^*$. Therefore, a sequence $(\mathcal{O}_k)_k$ of elements in $\mathfrak{g}^*/G$ converges to the orbit $\mathcal{O}$ in $\mathfrak{g}^*/G$ if and only if for any $\ell \in \mathcal{O}$, there exist $\ell_k \in \mathcal{O}_k, k \in \mathbb{N}$, such that $\ell = \lim_{k+\infty} \ell_k$.*

A proof of this Lemma can be found in [Lep-Lud].

Theorem 2.4.2. *Let $(\mathcal{O}_{(\mu^k, r_k)})_{k\in\mathbb{N}}$ be a sequence of orbits in $\mathfrak{m}_n^{\ddagger}/M_n$. Then $(\mathcal{O}_{(\mu^k, r_k)})_k$ converges to $\mathcal{O}_{(\mu,r)}$ in $\mathfrak{m}_n^{\ddagger}/M_n$ if and only if $\lim_{k\to\infty} r_k = r$ and $\mu^k = \mu$ for large k.*

Démonstration. If r_k tends to r and $J_{\mu^k} = J_\mu$ for k large enough, then of course $\lim_{k\to\infty} (J_{\mu^k}, \ell_{r_k}) = (J_\mu, \ell_r)$ and so $\lim_{k\to\infty} \mathcal{O}_{(\mu^k, r_k)} = \mathcal{O}_{(\mu,r)}$.
Suppose now that $(\mathcal{O}_{(\mu^k, r_k)})_k$ converges to $\mathcal{O}_{(\mu,r)}$. If $n = 2d + 1$ is odd, there are then two sequences

$$B_k = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -b_1(k) \\ 0 & 0 & \dots & 0 & 0 & -b_2(k) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & -b_{2d-1}(k) \\ 0 & 0 & \dots & 0 & 0 & -b_{2d}(k) \\ b_1(k) & b_2(k) & \dots & b_{2d-1}(k) & b_{2d}(k) & 0 \end{pmatrix} \tag{2.15}$$

in W_{ℓ_r} and $(A_k)_k \subset SO(n)$, such that $\lim_{k\to\infty} A_k(J_{\mu^k}+B_k)A_k^t = J_\mu$ and $\lim_{k\to\infty} A_k\ell_{r_k} = \ell_r$. Therefore, there exists a subsequence $(A_{k_j})_{j\in I}$ which converges to an element A_∞, which is necessarily contained in the stabilizer $SO(n-1)$ of the linear form ℓ_r. Then we obtain that $\lim_{j\to\infty} (J_{\mu^{k_j}} + B_{k_j}) = A_\infty^t J_\mu A_\infty$. In addition, we have for $\ell_r = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r \end{pmatrix}$ that $(J_{\mu^{k_j}} + B_{k_j})\ell_r = r \begin{pmatrix} -b_1(k_j) \\ -b_2(k_j) \\ \vdots \\ -b_{2d}(k_j) \\ 0 \end{pmatrix}$ and

$(A_\infty^t J_\mu A_\infty)\ell_r = 0$ since $A_\infty^t J_\mu A_\infty = \begin{pmatrix} * & \dots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \dots & * & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}$.

Hence it follows that $(B_{k_j})_j$ converges to zero and $\lim_{j\to\infty} J_{\mu^{k_j}} = A_\infty^t J_\mu A_\infty$. Since the matrices J_{μ^k} are diagonal, so is the matrix $A_\infty^t J_\mu A_\infty$ and the fact that $A_\infty \in SO(n-1)$ implies that $A_\infty^t J_\mu A_\infty = J_\mu$. By considering all possible converging subsequences $(A_{k_j})_j$, we have $\mu^k = \mu$ for k large enough. The argument for $n = 2d$ is similar. □

Theorem 2.4.3. *Let $(\mathcal{O}_{(\mu^k,r_k)})_{k\in\mathbb{N}}$ be a sequence of orbits in $\mathfrak{m}_n^\ddagger/M_n$. Then $(\mathcal{O}_{(\mu^k,r_k)})_k$ converges to $\mathcal{O}_\lambda$ in $\mathfrak{m}_n^\ddagger/M_n$ if and only if $\lim_{j+\infty} r_k = 0$ and $\lambda_1 \geq \mu_1^k \geq \lambda_2 \geq \mu_2^k \geq ... \geq \lambda_d \geq |\mu_d^k|$ for k large enough (if $n = 2d+1$ is odd) resp. $\lim_{j+\infty} r_k = 0$ and $\lambda_1 \geq \mu_1^k \geq \lambda_2 \geq \mu_2^k \geq ... \geq \mu_{d-1}^k \geq |\lambda_d|$ for k large enough (if $n = 2d$ is even).*

Before beginning the proof of this theorem, we need to show some technical lemmas.

Lemma 2.4.4. *For any integer $n \geq 2$ and any scalars $X_1, ..., X_{n-1}, Y_1, ..., Y_n$ with $Y_i \neq Y_j$ for every $i \neq j$, we have*

$$\sum_{j=1}^{n} \frac{\prod_{i=1}^{n-1}(X_i - Y_j)}{\prod_{i=1,i\neq j}^{n}(Y_i - Y_j)} = 1. \tag{2.16}$$

Démonstration. According to the Lagrange's interpolation theorem, if P is a polynomial of degree $\leq n-1$, then

$$P(X) = \sum_{j=1}^{n} P(Y_j) \prod_{i=1,i\neq j}^{n} \frac{(X - Y_i)}{(Y_j - Y_i)}. \tag{2.17}$$

In particular, for $P(X) = \prod_{i=1}^{n-1}(X - X_i)$ we have

$$\prod_{i=1}^{n-1}(X - X_i) = \sum_{j=1}^{n} \prod_{i=1}^{n-1}(Y_j - X_i) \prod_{i=1,i\neq j}^{n} \frac{(X - Y_i)}{(Y_j - Y_i)}. \tag{2.18}$$

By differentiating $(n-1)$ times the polynomial P, we obtain

$$(n-1)! = \sum_{j=1}^{n} \prod_{i=1}^{n-1}(Y_j - X_i) \frac{(n-1)!}{\prod_{i=1,i\neq j}^{n}(Y_j - Y_i)}.$$

□

Lemma 2.4.5. *Let $\mu_1 \geq ... \geq \mu_{d-1} \geq |\mu_d|$ and $\lambda_1 \geq ... \geq \lambda_d \geq 0$, where the λ's and μ's are integers. Then, we have $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq ... \geq \mu_{d-1} \geq \lambda_d \geq |\mu_d|$ if and only if there exists a skew-symmetric matrix*

$$B = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -b_1 \\ 0 & 0 & \dots & 0 & 0 & -b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & -b_{2d-1} \\ 0 & 0 & \dots & 0 & 0 & -b_{2d} \\ b_1 & b_2 & \dots & b_{2d-1} & b_{2d} & 0 \end{pmatrix} \tag{2.19}$$

such that spectrum$(J_\mu + B) = \{0, \pm i\lambda_1, \pm i\lambda_2, ..., \pm i\lambda_d\}$.

Démonstration. It is easy to prove that, for all $x \in \mathbb{R}$, $\det(J_\mu + B - ix\mathbb{I}) = i(-1)^{d+1} x P(x)$ where

$$P(x) = \prod_{i=1}^{d}(x^2 - \mu_i^2) - \sum_{j=1}^{d}\Big((b_{2j-1}^2 + b_{2j}^2)\prod_{i=1, i\neq j}^{d}(x^2 - \mu_i^2)\Big). \tag{2.20}$$

Hence we remark so that zero is always an element of the spectrum and that

$$\begin{aligned} \lim_{x\to+\infty} P(x) &= +\infty, \\ P(\mu_1) &= -(b_1^2 + b_2^2)\prod_{i=2}^{d}(\mu_1^2 - \mu_i^2) \leq 0, \\ P(\mu_2) &= -(b_3^2 + b_4^2)\prod_{i=1, i\neq 2}^{d}(\mu_2^2 - \mu_i^2) \geq 0, \\ P(\mu_3) &= -(b_5^2 + b_6^2)\prod_{i=1, i\neq 3}^{d}(\mu_3^2 - \mu_i^2) \leq 0, \\ P(\mu_4) &= -(b_7^2 + b_8^2)\prod_{i=1, i\neq 4}^{d}(\mu_4^2 - \mu_i^2) \geq 0, \end{aligned}$$

and so on, i.e $P(\mu_i) \leq 0$ if i is odd and $P(\mu_i) \geq 0$, if i is even. We deduce that if $\pm i\lambda_1, \pm i\lambda_2, ..., \pm i\lambda_d$ are the elements of the spectrum of $J_\mu + B$, (i.e. $\pm\lambda_1, \pm\lambda_2, ..., \pm\lambda_d$ are all possible roots of the polynomial P), then we necessarily have

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq ... \geq \mu_{d-1} \geq \lambda_d \geq |\mu_d|. \tag{2.21}$$

Conversely, assume first that all μ_j are pairwise distinct. We can choose the skew-symmetric matrix B such that

$$b_{2j-1}^2 + b_{2j}^2 = \frac{\prod_{i=1}^{i=j}(\lambda_i^2 - \mu_j^2)\prod_{i=j+1}^{i=d}(\mu_j^2 - \lambda_i^2)}{\prod_{i=1}^{i=j-1}(\mu_i^2 - \mu_j^2)\prod_{i=j+1}^{i=d}(\mu_j^2 - \mu_i^2)} = \frac{\prod_{i=1}^{i=d}(\lambda_i^2 - \mu_j^2)}{\prod_{i=1, i\neq j}^{i=d}(\mu_i^2 - \mu_j^2)} \tag{2.22}$$

for all $j = 1, ..., d$. It follows, by the preceding lemma, that for all $1 \leq k \leq d$

$$\begin{aligned} P(\pm\lambda_k) &= \prod_{i=1}^{d}(\lambda_k^2 - \mu_i^2) - \sum_{j=1}^{d}\left(\frac{\prod_{i=1}^{i=d}(\lambda_i^2 - \mu_j^2)}{\prod_{i=1, i\neq j}^{i=d}(\mu_i^2 - \mu_j^2)} \prod_{i=1, i\neq j}^{d}(\lambda_k^2 - \mu_i^2)\right) \\ &= \prod_{i=1}^{d}(\lambda_k^2 - \mu_i^2)\left[1 - \sum_{j=1}^{d}\frac{\prod_{i=1, i\neq k}^{i=d}(\lambda_i^2 - \mu_j^2)}{\prod_{i=1, i\neq j}^{i=d}(\mu_i^2 - \mu_j^2)}\right] = 0. \end{aligned}$$

Then the spectrum of the matrix $J_\mu + B$ is equal to the set $\{0, \pm i\lambda_1, \pm i\lambda_2, ..., \pm i\lambda_d\}$.

Assume now that there exist two families of integers $\{p_l\}_{1\leq l\leq s}$ and $\{q_l\}_{1\leq l\leq s}$ such that $1 \leq p_1 < q_1 < p_2 < q_2 < ... < p_s < q_s \leq d$, and for all $1 \leq l \leq s$ $\mu_{p_l} = \mu_{p_l+1} = ... = \mu_{q_l-1} = \mu_{q_l}$, $\mu_{q_l} \neq \mu_{q_l+1}$ and $\mu_{p_l-1} \neq \mu_{p_l}$. Hence, if we set

$$Q(x) = \prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{d}(x^2 - \mu_i^2) \quad , \quad \tilde{Q}_l(x) = \prod_{\substack{i=1\\ i\neq p_l}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq p_l}}^{p_2}\cdots\prod_{\substack{i=q_{s-1}+1\\ i\neq p_l}}^{p_s}\prod_{i=q_s+1}^{d}(x^2 - \mu_i^2)$$

$$\text{and } Q_j(x) = \prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{d}(x^2 - \mu_i^2),$$

then $\det(J_\mu + B - ix\mathbb{I}) = i(-1)^{d+1}x\prod_{l=1}^{s}(x^2 - \mu_{p_l}^2)^{q_l - p_l}P(x)$ where

$$\begin{aligned} P(x) &= Q(x) - \Big(\sum_{l=1}^{s}\Big(\sum_{j=p_l}^{q_l} b_{2j-1}^2 + b_{2j}^2\Big)\tilde{Q}_l(x)\Big) \\ &- \sum_{j=1}^{p_1-1}\sum_{j=q_1+1}^{p_2-1}\cdots\sum_{j=q_s+1}^{d}\Big((b_{2j-1}^2 + b_{2j}^2)Q_j(x)\Big). \end{aligned}$$

We can choose the skew-symmetric matrix B such that

$$b_{2j-1}^2 + b_{2j}^2 = \frac{\prod_{i=1}^{i=d}(\lambda_i^2 - \mu_j^2)}{\prod_{i=1, i\neq j}^{i=d}(\mu_i^2 - \mu_j^2)} = \frac{\prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{d}(\lambda_i^2 - \mu_j^2)}{\prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{d}(\mu_i^2 - \mu_j^2)}$$

for all $j = 1, ..., p_1 - 1, q_1 + 1, ..., p_s - 1, q_s + 1, ..., d$ and

$$b_{2p_l-1}^2 + ... + b_{2q_l-1}^2 + b_{2q_l}^2 = \frac{\prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{d}(\lambda_i^2 - \mu_{p_l}^2)}{\prod_{\substack{i=1\\ i\neq p_l}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq p_l}}^{p_2}\cdots\prod_{\substack{i=q_{s-1}+1\\ i\neq p_l}}^{p_s}\prod_{i=q_s+1}^{d}(\mu_i^2 - \mu_{p_l}^2)}$$

for all $l = 1, ..., s$. It is easy to see that if $\lambda_k = \mu_{p_l}$ then $P(\pm\lambda_k) = Q(\pm\lambda_k) = 0$. On the other hand for all $\lambda_k \neq \mu_{p_l}$

$$\begin{aligned}
P(\pm\lambda_k) &= Q(\pm\lambda_k) - \Big(\sum_{l=1}^{s} \frac{\prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{d}(\lambda_i^2-\mu_{p_l}^2)}{\prod_{\substack{i=1\\ i\neq p_l}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq p_l}}^{p_2}\cdots\prod_{\substack{i=q_{s-1}+1\\ i\neq p_l}}^{p_s}\prod_{i=q_s+1}^{d}(\mu_i^2-\mu_{p_l}^2)}\tilde{Q}_l(\pm\lambda_k)\Big)\\
&- \sum_{j=1}^{p_1-1}\sum_{j=q_1+1}^{p_2-1}\cdots\sum_{j=q_s+1}^{d}\Big(\frac{\prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{d}(\lambda_i^2-\mu_j^2)}{\prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{d}(\mu_i^2-\mu_j^2)}Q_j(\pm\lambda_k)\Big)\\
&= Q(\pm\lambda_k)\Big[1-\Big(\sum_{l=1}^{s}\frac{\prod_{\substack{i=1\\ i\neq k}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq k}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq k}}^{d}(\lambda_i^2-\mu_{p_l}^2)}{\prod_{\substack{i=1\\ i\neq p_l}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq p_l}}^{p_2}\cdots\prod_{\substack{i=q_{s-1}+1\\ i\neq p_l}}^{p_s}\prod_{i=q_s+1}^{d}(\mu_i^2-\mu_{p_l}^2)}\Big)\\
&- \sum_{j=1}^{p_1-1}\sum_{j=q_1+1}^{p_2-1}\cdots\sum_{j=q_s+1}^{d}\Big(\frac{\prod_{\substack{i=1\\ i\neq k}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq k}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq k}}^{d}(\lambda_i^2-\mu_j^2)}{\prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{d}(\mu_i^2-\mu_j^2)}\Big)\Big]\\
&= Q(\pm\lambda_k)\Big[1-\sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}\cdots\sum_{j=q_s+1}^{d}\Big(\frac{\prod_{\substack{i=1\\ i\neq k}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq k}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq k}}^{d}(\lambda_i^2-\mu_j^2)}{\prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{d}(\mu_i^2-\mu_j^2)}\Big)\Big]\\
&= 0
\end{aligned}$$

by using the preceding Lemma. Thus, $\det(J_\mu + B \pm i\lambda_k\mathbb{I}) = 0$ for all $k = 1, ..., d$. □

Démonstration. (of the theorem 2.4.3) Let $n = 2d+1$ be odd. Suppose that $\lambda_1 \geq \mu_1^k \geq \lambda_2 \geq \mu_2^k \geq ... \geq \lambda_d \geq |\mu_d^k|$ for k large enough. So there is at least one subsequence $(\mu^{k_j})_{j\in I}$ such that $\mu^{k_j} = \mu$ for all j in I where μ depends on I and $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \lambda_d \geq |\mu_d|$. We have proved in the preceding Lemma that there exists a skew-symmetric matrix B defined in (2.19) such that the spectrum of the matrix $J_\mu + B$ is given by zero and the complex numbers $\pm i\lambda_1$, $\pm i\lambda_2, ...,$ $\pm i\lambda_d$. On the other hand, there is an orthogonal matrix A such that $A(J_\mu + B)A^t = J_\lambda$ (c.f. for instance [BJLR], Proposition 7.3 for a similar statement in the complex case). If $A \in SO(2d+1)$, then we can take $A_{k_j} = A$ (if not, we take $A_{k_j} = -A$) and $B_{k_j} = B$ for all j in I.
Conversely, it is clear that $\lim\limits_{k\to\infty} r_k = \lim\limits_{k\to\infty} \|A_k\ell_{r_k}\| = 0$, and for all $j = 1, 2, ..., d$ one has $\lim\limits_{k\to\infty}\det(J_{\mu^k} + B_k \pm i\lambda_j\mathbb{I}) = \lim\limits_{k\to\infty}\det(A_k(J_{\mu^k} + B_k)A_k^t \pm i\lambda_j\mathbb{I}) = 0$. Then, by the preceding Lemma, $\lambda_1 \geq \mu_1^k \geq \lambda_2 \geq \mu_2^k \geq ... \geq \lambda_d \geq |\mu_d^k|$ for k large enough large.

If n is even i.e. $n = 2d$, then the same proof applies. The only difference is the choice of the matrix A in $O(2d)$ satisfying $A(J_\mu + B)A^t = J_\lambda$, if $\det(A) = -1$. In this situation, we multiply the last line of the matrix A by -1. Then we obtain $\det(A) = 1$ and $A(J_\mu + B)A^t = J_{\tilde{\lambda}}$ such that $\tilde{\lambda} = (\lambda_1, ..., \lambda_{d-1}, -\lambda_d)$. □

We have finished the proof of

Theorem 2.4.6. *The dual space of the group $M_n = SO(n) \ltimes \mathbb{R}^n$ is homeomorphic with its space of admissible co-adjoint orbits $\mathfrak{m}_n^\ddagger / M_n$.*

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Chapitre 3

On the dual topology of the groups $\mathbf{U(n)} \ltimes \mathbb{H}_\mathbf{n}$

Résumé : Soit $\mathbb{H}_n$, $n \geq 1$, le groupe de Heisenberg de dimention $2n+1$ et soit $U(n)$ le groupe des matrices unitaires agissant sur $\mathbb{H}_n$ par automorphisme. Dans ce chapitre, on décrit l'espace quotient des orbites coadjointes admissibles du produit semi-direct $G_n = U(n) \ltimes \mathbb{H}_n$, et on détermine la topologie de cet espace. On montre que la bijection entre le dual unitaire de G_n et l'espace des orbites coadjointes admissibles est continue sur $\hat{G}_n$, et dans le cas où $n = 1$, cette identification est un homéomorphisme.

Abstract : Let $\mathbb{H}_n$, $n \geq 1$, be the $(2n+1)$-dimensional Heisenberg Lie group and let $U(n)$ be the unitary group acting on $\mathbb{H}_n$ by automorphisms. In this paper, we describe the space of admissible coadjoint orbits of the semi-direct product $G_n = U(n) \ltimes \mathbb{H}_n$ and we determine the topology of this space. We show that the bijection between the dual space $\hat{G}_n$ of G_n and its admissible coadjoint orbit space is continuous onto $\hat{G}_n$, and that for the group G_1, this identification is a homeomorphism.

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Keywords : Unitary group, semi-direct product, dual topology, admissible coadjoint orbit space.

3.1 Introduction.

Let G be a locally compact group and $\widehat{G}$ the unitary dual of G, i.e., the set of equivalence classes of irreducible unitary representations of G, endowed with the pullback of the hull-kernel topology on the primitive ideal space of $C^*(G)$, the C^*-algebra of G. Besides the fondamental problem of determining $\widehat{G}$ as a set, there is a genuine interest in a precise and neat description of the topology on $\widehat{G}$. For several classes of Lie groups, such as simply connected nilpotent Lie groups or, more generally, exponential solvable Lie groups, the Euclidean motion groups and also the extension groups $U(n) \ltimes \mathbb{H}_n$ considered in this paper, there is a nice geometric object parametrizing $\widehat{G}$, namely the space of admissible coadjoint orbits in the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G.

In such a situation, the natural and important question arises of whether the bijection between the orbit space, equipped with the quotient topology, and $\widehat{G}$ is a homeomorphism. In [Lep-Lud], H. Leptin and J. Ludwig have proved that for an exponential solvable Lie group $G = \exp \mathfrak{g}$, the dual space $\widehat{G}$ is homeomorphic to the space of coadjoint orbits $\mathfrak{g}^*/G$ through the Kirillov mapping. On the other hand, we have recently shown in [El-Lu] that the dual topology of the classical motion groups $SO(n) \ltimes \mathbb{R}^n$, $n \geq 2$, can be linked to the topology of the quotient space of admissible coadjoint orbits.

In this paper we consider the semi-direct product $G_n = U(n) \ltimes \mathbb{H}_n$, $n \geq 1$, and we identify its dual space $\hat{G}_n$ with the lattice of admissible coadjoint orbits. Lipsman showed in [Lip] that each irreducible unitary representation of G_n can be constructed by holomorphic induction from an admissible linear functional ℓ of the Lie algebra $\mathfrak{g}_n$ of G_n. Furthermore, two irreducible representations in $\hat{G}_n$ are equivalent if and only if their respective linear functionals are in the same G_n-orbit. We guess then that this identification is a homeomorphism and we prove this conjecture for $G_1 = U(1) \ltimes \mathbb{H}_1$.

This paper is structured in the following way. Section 2 contains preliminary material and summarizes results from previous work concerning the dual space of G_n which is identified with its admissible coadjoint orbit space. The representations attached to an admissible linear functional are obtained via Mackey's little-group method and the dual space $\hat{G}_n$ is given by the parameter space $\mathcal{P}_n = \{\alpha \in \mathbb{R}^*, r > 0, \rho_\mu \in \widehat{U(n-1)}, \tau_\lambda \in \widehat{U(n)}\}$. In section 3, we shall link the convergence of sequences of admissible coadjoint orbits to the convergence in $\mathcal{P}_n$. Section 4 describes the dual topology of a second countable locally compact group. In the last paragraph, we discuss the topology of the dual space of our groups G_n.

3.2 Preliminaries.

Given the n-dimensional complex vector space $\mathbb{C}^n$ with the standard scalar product $\langle .,. \rangle$, we denote by $(.,.)$ and $\omega(.,.)$ the real and imaginary parts of $\langle .,. \rangle$ so that

$$\langle .,. \rangle = (.,.) + i\omega(.,.).$$

The bilinear forms $(.,.)$ and $\omega(.,.)$ define respectively a positive definite inner product and a symplectic structure on the underlying real vector space $\mathbb{R}^{2n}$ of $\mathbb{C}^n$. The associated Heisenberg group $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ of dimension $2n+1$ over $\mathbb{R}$ is given by the group multiplication

$$(z,t)(z',t') := (z+z', t+t' - \frac{1}{2}\omega(z,z')).$$

We consider the unitary group $U(n)$ of automorphisms of $\mathbb{H}_n$ preserving $\langle .,. \rangle$ on $\mathbb{C}^n$ which embeds into $Aut(\mathbb{H}_n)$ via

$$A.(z,t) = (Az, t).$$

Furthermore, $U(n)$ yields a maximal compact connected subgroup of $Aut(\mathbb{H}_n)$ (cf. [Ho]). The symbol $G_n = U(n) \ltimes \mathbb{H}_n$ denotes the semi-direct product of $U(n)$ with the Heisenberg group $\mathbb{H}_n$. Our convention for the semi-direct product group law is

$$(A,z,t)(B,z',t') = (AB, z+Az', t+t' - \frac{1}{2}\omega(z,Az')).$$

We identify the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$ with $\mathbb{H}_n$ via the exponential map. The Lie bracket of $\mathfrak{h}_n$ is given by

$$[(z,t),(w,s)] = (0, -\omega(z,w))$$

and the derived action of the Lie algebra $\mathfrak{u}(n)$ of $U(n)$ on $\mathfrak{h}_n$ is

$$A.(z,t) = (Az, 0).$$

By $\mathfrak{g}_n = \mathfrak{u}(n) \ltimes \mathfrak{h}_n$ we mean the Lie algebra of G_n. Then, for all $(A,z,t) \in G_n$ and all $(B,w,s) \in \mathfrak{g}_n$ we get

$$\begin{aligned} Ad(A,z,t)(B,w,s) &= \frac{d}{dy}\Big|_{y=0} Ad(A,z,t)(e^{yB}, yw, ys) \\ &= (ABA^*, -ABA^*z + Aw, s - \omega(z,Aw) + \tfrac{1}{2}\omega(A^*z, BA^*z)). \end{aligned} \tag{3.1}$$

In particular

$$Ad(A)(B,w,s) = (ABA^*, Aw, s). \tag{3.2}$$

From the identity (3.1) we deduce the Lie bracket

$$\begin{aligned}[(A,z,t),(B,w,s)] &= \frac{d}{dy}\Big|_{y=0} Ad((e^{yA},yz,yt))(B,w,s) \\ &= (AB-BA, Aw-Bz, -\omega(z,w)),\end{aligned}$$

for all $(A,z,t),(B,w,s) \in \mathfrak{g}_n$.

3.2.1 Coadjoint orbits in G_n.

In this subsection, we describe the coadjoint orbit space of G_n according to [BJLR].

We identify $\mathfrak{u}(n)$ with its vector dual space $\mathfrak{u}^*(n)$ through the $U(n)$-invariant inner product

$$\langle A,B\rangle = \mathrm{t}r(AB)$$

and for $z \in \mathbb{C}^n$ we define the linear form z^* in $(\mathbb{C}^n)^*$ by

$$z^*(w) := \omega(z,w).$$

One defines a map $\times : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathfrak{u}^*(n)$, $(z,w) \mapsto z \times w$ by

$$\langle z\times w, B\rangle = z\times w(B) := w^*(Bz) = \omega(w,Bz), \ \ B \in \mathfrak{u}(n).$$

It is easy to verify that for $A \in U(n)$, $B \in \mathfrak{u}(n)$ and $z,w \in \mathbb{C}^n$ one has

$$\begin{aligned} Az^* &:= z^*\circ A^{-1} = (Az)^* \\ z^*\circ B &= -(Bz)^* \\ z\times w &= w\times z \\ A(z\times w)A^* &= (Az)\times(Aw). \end{aligned} \tag{3.3}$$

Hence we will identify the dual $\mathfrak{g}_n^* = (\mathfrak{u}(n)\ltimes\mathfrak{h}_n)^*$ with $\mathfrak{u}(n)\oplus\mathfrak{h}_n$, i.e., each element $\ell \in \mathfrak{g}_n^*$ can be identified with an element $(U,u,x) \in \mathfrak{u}(n)\times\mathbb{C}^n\times\mathbb{R}$ such that

$$\langle (U,u,x),(B,w,s)\rangle = \langle U,B\rangle + u^*(w) + xs, \quad (B,w,s) \in \mathfrak{g}_n.$$

From (3.2) and (3.3), we obtain

$$Ad^*(A)(U,u,x) = (AUA^*, Au, x) \tag{3.4}$$

and

$$Ad^*(A,z,t)(U,u,x) = (AUA^* + z \times (Au) + \frac{x}{2} z \times z, Au + xz, x), \quad (3.5)$$

where $z \times w(B) = w^*(Bz) = \omega(w, Bz)$.
Letting A and z vary over $U(n)$ and $\mathbb{C}^n$ respectively, the coadjoint orbit $\mathcal{O}_{(U,u,x)}$ through the linear form (U,u,x) can be written

$$\mathcal{O}_{(U,u,x)} = \{(AUA^* + z \times (Au) + \frac{x}{2} z \times z, Au + xz, x) |\, A \in U(n), z \in \mathbb{C}^n\} \quad (3.6)$$

or equivalently, by replacing z by Az and using the identity (3.4),

$$\mathcal{O}_{(U,u,x)} = \{Ad^*(A)(U + z \times u + \frac{x}{2} z \times z, u + xz, x) |\, A \in U(n), z \in \mathbb{C}^n\}. \quad (3.7)$$

Remark 3.2.1. Here we regard z as a column vector $z = (z_1, \ldots, z_n)^T$ and $z^* := \overline{z}^t$. Then $z \times u \in \mathfrak{u}^*(n) \cong \mathfrak{u}(n)$ is the n by n skew Hermitian matrix $\frac{i}{2}(uz^* + zu^*)$. Indeed, for all $B \in \mathfrak{u}(n)$ we compute

$$\langle uz^* + zu^*, B\rangle = \mathrm{tr}((uz^* + zu^*)B) = \sum_{1 \le i,j \le n} B_{ji} z_i \overline{u}_j - \sum_{1 \le i,j \le n} u_i \overline{B}_{ij} \overline{z}_j = -2iz \times u(B).$$

In particular, $z \times z$ is the skew Hermitian matrix izz^* whose entries are determined by $(izz^*)_{lj} = iz_l \overline{z}_j$.

3.2.2 The dual space of $U(n)$.

Let

$$\mathbb{T}_n = \{T = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \cdots, \mathrm{e}^{\mathrm{i}\theta_n}), \theta_j \in \mathbb{R}, \text{ for } j = 1, \cdots, n\}$$

be a maximal torus of the unitary group $U(n)$ and let $\mathfrak{t}_n$ be its Lie algebra. By complexification of $\mathfrak{u}(n)$ and $\mathfrak{t}_n$, we get respectively the complex Lie algebras $\mathfrak{u}^{\mathbb{C}}(n) = \mathfrak{gl}(n, \mathbb{C}) = M(n, \mathbb{C})$ and

$$\mathfrak{t}_n^{\mathbb{C}} = \{H = \mathrm{diag}(h_1, \cdots, h_n), h_j \in \mathbb{C}, \text{ for } j = 1, \cdots, n\},$$

which is a Cartan subalgebra of $\mathfrak{u}^{\mathbb{C}}(n)$. For $j = 1, \cdots, n$, we define a linear functional

$$e_j \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix} = h_j.$$

Let P_n be the set of all dominant integral forms λ for $U(n)$ which may be written in the form $\sum_{j=1}^n i\lambda_j e_j$, or simply in the more traditional form $\lambda = (\lambda_1, \cdots, \lambda_n)$ with all the λ_j's understood to be integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. P_n is a lattice in the vector dual space $\mathfrak{t}_n^*$ of $\mathfrak{t}_n$, $P_n \cong \mathbb{Z}^n$. Each irreducible unitary representation τ_λ of $U(n)$ is determined by its highest weight $\lambda \in P_n$. Therefore, the dual space $\widehat{U(n)}$ of $U(n)$ is in bijection with the set P_n.

For each λ in P_n, the highest weight vector ϕ^λ in the space $\mathcal{H}_\lambda$ of τ_λ verifies that $\tau_\lambda(T)\phi^\lambda = \chi_\lambda(T)\phi^\lambda$, where χ_λ is the character of $\mathbb{T}_n$ associated to the linear functional λ and defined by

$$\chi_\lambda(T = \mathrm{diag}(e^{i\theta_1}, \cdots, e^{i\theta_n})) = e^{-i\lambda_1\theta_1} \times \cdots \times e^{-i\lambda_n\theta_n}.$$

For two irreducible unitary representations $(\tau_\lambda, \mathcal{H}_\lambda)$ and $(\tau_{\lambda'}, \mathcal{H}_{\lambda'})$, the Schur orthogonality relation says that for all $\xi, \eta \in \mathcal{H}_\lambda$, $\xi', \eta' \in \mathcal{H}_{\lambda'}$,

$$\int_{U(n)} \langle \tau_\lambda(g)\xi, \eta\rangle \overline{\langle \tau_{\lambda'}(g)\xi', \eta'\rangle} dg = \begin{cases} 0 & \text{if } \lambda \neq \lambda', \\ \frac{\langle \xi, \xi'\rangle \langle \eta', \eta\rangle}{d_\lambda} & \text{if } \lambda = \lambda', \end{cases} \tag{3.8}$$

where d_λ denotes the dimension of the representation τ_λ.

According to Frobenius reciprocity and Weyl's theorem (cf. [We]), if ρ_μ is an irreducible representation of $U(n-1)$ with highest weight $\mu = (\mu_1, ..., \mu_{n-1})$, the induced representation $\pi_\mu := \mathrm{ind}_{U(n-1)}^{U(n)} \rho_\mu$ of $U(n)$ decomposes with multiplicity one, and the representations of $U(n)$ that appear are exactly those with highest weights $\lambda = (\lambda_1, ..., \lambda_n)$ such that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

3.2.3 Irreducible representations and admissible coadjoint orbits of G_n.

The description of the dual space of G_n is based on the Mackey "machine" (cf. [Ma]). We recall first the representation theory of the Heisenberg group $\mathbb{H}_n$. The infinite dimensional irreducible representations of $\mathbb{H}_n$ are parametrized by $\mathbb{R}^*$. For each $\alpha \in \mathbb{R}^*$, the Kirillov orbit $\mathcal{O}_\alpha$ of the irreducible representation σ_α is the hyperplane $\mathcal{O}_\alpha = \{(z, \lambda), z \in \mathbb{C}^n\}$. It is clear that for every α the coadjoint orbit $\mathcal{O}_\alpha$ is invariant under the action of the unitary group $U(n)$. Therefore $U(n)$ preserves the equivalence class of σ_α.

The representation σ_α can be realized in the Fock space

$$\mathcal{F}_\alpha(n) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \text{ entire} \mid \int_{\mathbb{C}^n} |f(w)|^2 e^{-\frac{|\alpha|}{2}|w|^2} dw < \infty\}$$

as

$$\sigma_\alpha(z,t)f(w) = e^{i\alpha t - \frac{\alpha}{4}|z|^2 - \frac{\alpha}{2}\langle w,z\rangle} f(w+z)$$

for $\alpha > 0$ and

$$\sigma_\alpha(z,t)f(\overline{w}) = e^{i\alpha t + \frac{\alpha}{4}|z|^2 + \frac{\alpha}{2}\langle \overline{w},\overline{z}\rangle} f(\overline{w}+\overline{z})$$

for $\alpha < 0$. We refer the reader to [Ho] or [Fo] for a discussion of the Fock space.

For each $A \in U(n)$, the operator $W_\alpha(A) : \mathcal{F}_\alpha(n) \to \mathcal{F}_\alpha(n)$ defined by

$$W_\alpha(A)f(z) = f(A^{-1}z)$$

intertwines σ_α and $(\sigma_\alpha)_A$ given by $(\sigma_\alpha)_A(z,t) := \sigma_\alpha(Az,t)$. It is easy to see that $U(n)$ stabilizes σ_α. W_α is said to be the projective intertwining representation of $U(n)$ on the Fock space. Then by Mackey , for each nonzero $\alpha \in \mathbb{R}$ and each element τ_λ in $\widehat{U(n)}$

$$\pi_{(\lambda,\alpha)}(A,z,t) = \tau_\lambda(A) \otimes \sigma_\alpha(z,t) \circ W_\alpha(A), \;\; (A,z,t) \in G_n,$$

is an irreducible unitary representation of G_n realized on $\mathcal{H}_\lambda \otimes \mathcal{F}_\alpha(n)$, where $\mathcal{H}_\lambda$ is the Hilbert space of τ_λ.

We associate to $\pi_{(\lambda,\alpha)}$ the linear functional $\ell_{\lambda,\alpha} = (J_\lambda, 0, \alpha)$ in $\mathfrak{g}_n^*$ where

$$J_\lambda = \begin{pmatrix} i\lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & i\lambda_n \end{pmatrix}.$$

Denote by $G_n[\ell_{\lambda,\alpha}]$, $U(n)[\ell_{\lambda,\alpha}]$ and $\mathbb{H}_n[\ell_{\lambda,\alpha}]$ the stabilizers of $\ell_{\lambda,\alpha}$ respectively in G_n, $U(n)$ and $\mathbb{H}_n$. By formula (3.5)

$$\begin{aligned} G_n[\ell_{\lambda,\alpha}] &= \{(A,z,t) \in G_n; (AJ_\lambda A^* + \frac{i}{2}\alpha z z^*, \alpha z, \alpha) = (J_\lambda, 0, \alpha)\} \\ &= \{(A,0,t) \in G_n; AJ_\lambda A^* = J_\lambda\}, \end{aligned}$$

$$\begin{aligned} U(n)[\ell_{\lambda,\alpha}] &= \{A \in U(n); (AJ_\lambda A^*, 0, \alpha) = (J_\lambda, 0, \alpha)\} \\ &= \{A \in U(n); AJ_\lambda A^* = J_\lambda\}, \end{aligned}$$

and

$$\mathbb{H}_n[\ell_{\lambda,\alpha}] = \{(z,t) \in \mathbb{H}(n); (J_\lambda + \frac{i}{2}\alpha z z^*, \alpha z, \alpha) = (J_\lambda, 0, \alpha)\} = \{0\} \times \mathbb{R}.$$

It follows that $G_n[\ell_{\lambda,\alpha}] = U(n)[\ell_{\lambda,\alpha}] \ltimes \mathbb{H}_n[\ell_{\lambda,\alpha}]$. Hence, $\ell_{\lambda,\alpha}$ is aligned in the sense of Lipsman (see Lemma 4.2 in [Lip]).

The finite dimensional irreducible representations of $\mathbb{H}_n$ are the characters χ_v, $v \in \mathbb{C}^n$, defined by

$$\chi_v(z,t) = e^{-i(v,z)}.$$

We denote by $U(n)_v$ the stabilizer of the character χ_v, equivalently of the vector v, under the action of $U(n)$. For any irreducible unitary representation ρ of $U(n)_v$, $\rho \otimes \chi_v$ is an irreducible representation of $U(n)_v \ltimes \mathbb{H}_n$ whose restriction to $\mathbb{H}_n$ is a multiple of χ_v, and the induced representation $\pi_{(\rho,v)} = \mathrm{ind}_{U(n)_v \ltimes \mathbb{H}_n}^{U(n)\ltimes\mathbb{H}_n} \rho \otimes \chi_v$ is an irreducible representation of G_n. The restriction of $\pi_{(\rho,v)}$ on $U(n)$ is equivalent to the induced representation $\mathrm{ind}_{U(n)_v}^{U(n)} \rho$. We remark that for any $v' = Av$, $A \in U(n)$, i.e. v and v' belong to the same sphere centered at zero and of radius $r = \|v\|$, we have $U(n)_{v'} = AU(n)_v A^*$ and the representations $\pi_{(\rho',v')}$ and $\pi_{(\rho,v)}$ are equivalent, where ρ' is an element of $\widehat{U(n)_{v'}}$ so that $\rho'(B) = \rho(A^*BA)$ for each $B \in U(n)_{v'}$. Hence, let χ_r denotes the character associated to the linear form v_r which is identified with the vector $(0,\ldots,0,r)^T$ in $\mathbb{C}^n$. Throughout this text, we denote ρ_μ the representation of the subgroup $U(n-1) = U(n)_{v_r}$ with highest weight μ and $\pi_{(\mu,r)}$ the representation $\pi_{(\rho_\mu,v_r)}$.

We link the representation $\pi_{(\mu,r)}$ to the linear functional $\ell_{\mu,r} = (J_\mu, v_r, 0)$ in $\mathfrak{g}_n^*$ where

$$J_\mu = \begin{pmatrix} i\mu_1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & i\mu_{n-1} & 0 \\ 0 & \ldots & 0 & 0 \end{pmatrix}.$$

By the expression in (3.5), we check that

$$\begin{aligned} G_n[\ell_{\mu,r}] &= \{(A,z,t) \in G_n; (AJ_\mu A^* + z \times (Av_r), Av_r, 0) = (J_\mu, v_r, 0)\} \\ &= \{(A,z,t) \in G_n; A \in U(n-1), AJ_\mu A^* + \frac{i}{2}(v_r z^* + z(v_r)^*) = J_\mu\} \\ &= \{(A,z,t) \in G_n; z \in i\mathbb{R}v_r, A \in U(n-1), AJ_\mu A^* = J_\mu\}, \end{aligned}$$

since $AJ_\mu A^* \in \mathfrak{u}(n-1)$ and

$$v_r z^* + z v_r^* = \begin{pmatrix} 0 & \ldots & 0 & rz_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & rz_{n-1} \\ r\overline{z}_1 & \ldots & r\overline{z}_{n-1} & 2r\,\mathrm{Re}\,(z_n) \end{pmatrix}. \tag{3.9}$$

In addition, we evidently have $U(n)[\ell_{\mu,r}] = \{A \in U(n-1) | AJ_\mu A^* = J_\mu\}$ and $\mathbb{H}_n[\ell_{\mu,r}] = i\mathbb{R}v_r \times \mathbb{R}$. Hence, similarly to the first case, $\ell_{\mu,r}$ is aligned.

We obtain in this way all the finite dimensional irreducible unitary representations of G_n which are not trivial on $\mathbb{H}_n$. On the other hand, the trivial extension of each element τ_λ of $\widehat{U(n)}$ to the entire group G_n is an irreducible representation which will be also denoted by τ_λ. The corresponding linear functional is $\ell_\lambda = (J_\lambda, 0, 0)$. Therfore, by Mackey's theory the dual space $\hat{G}_n$ is in bijection with the set

$$(P_n \times \mathbb{R}^*) \bigcup (P_{n-1} \times \mathbb{R}_+^*) \bigcup P_n.$$

By definition, a linear functional ℓ in $\mathfrak{g}_n^*$ is said to be admissible if there exists a unitary character χ of the connected component of $G_n[\ell]$ such that $d\chi = i\ell_{|\mathfrak{g}_n[\ell]}$. It is obvious that all the linear functionals $\ell_{\lambda,\alpha}$, $\ell_{\mu,r}$ and ℓ_λ are admissible. Then, according to [Lip], the representations $\pi_{(\lambda,\alpha)}$, $\pi_{(\mu,r)}$ and τ_λ described above are equivalent to the representations of G_n obtained by holomorphic induction from their respective linear functionals $\ell_{\lambda,\alpha}$, $\ell_{\mu,r}$ and ℓ_λ.

We denote respectively by $\mathcal{O}_{(\lambda,\alpha)}$, $\mathcal{O}_{(\mu,r)}$ and $\mathcal{O}_\lambda$ the co-adjoint orbits associated to the linear forms $\ell_{\lambda,\alpha}$, $\ell_{\mu,r}$ and ℓ_λ. Let $\mathfrak{g}_n^\ddagger \subset \mathfrak{g}_n^*$ be the union of all the $\mathcal{O}_{(\lambda,\alpha)}$, all the $\mathcal{O}_{(\mu,r)}$, and all the $\mathcal{O}_\lambda$ and denote by $\mathfrak{g}_n^\ddagger/G_n$ the corresponding set in the orbit space. It follows now from [Lip], that $\mathfrak{g}_n^\ddagger$ is just the set of all admissible linear functionals of $\mathfrak{g}_n$.

3.3 Convergence in the quotient space $\mathfrak{g}_n^\ddagger/G_n$.

In the last paragraph, we have seen that the dual space of G_n is parametrized by the dominant integral forms λ for $U(n)$ and μ for $U(n-1)$, the non zero $\alpha \in \mathbb{R}$ attached to the generic orbit $\mathcal{O}_\alpha$ in $\mathfrak{h}_n^*$ and the positive real r derived from the natural action of the unitary group $U(n)$ on the characters of the Heisenberg $\mathbb{H}_n$. Moreover, we have seen that the quotient space $\mathfrak{g}_n^\ddagger/G_n$ of admissible coadjoint orbits is in bijection with $\widehat{G_n}$.

Let $\mathcal{W}$ be the subspace of $\mathfrak{u}(n)$ generated by the matrices $z \times v_r = \frac{i}{2}(v_r z^* + z v_r^*)$, $z \in \mathbb{C}^n$, then the space $\mathfrak{g}_n^\ddagger/G_n$ is the set of all orbits

$$\mathcal{O}_{(\lambda,\alpha)} = \{(AJ_\lambda A^* + \frac{i\alpha}{2} zz^*, \alpha z, \alpha) | z \in \mathbb{C}^n, A \in U(n)\},$$

all orbits

$$\mathcal{O}_{(\mu,r)} = \{(A(J_\mu + \mathcal{W})A^*, Av_r, 0) | A \in U(n)\},$$

and all orbits

$$\mathcal{O}_\lambda = \{(AJ_\lambda A^*, 0, 0) | A \in U(n)\}.$$

Before beginning our discussion on the convergence of the admissible co-adjoint orbits, we need to state the following basic lemma.

Lemma 3.3.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. We denote by $\mathfrak{g}^*/G$ the space of co-adjoint orbits and by $p_G : \mathfrak{g}^* \to \mathfrak{g}^*/G$ the canonical projection. We equip this space with the quotient topology, i.e, a subset U in $\mathfrak{g}^*/G$ is open if and only $p_G^{-1}(U)$ is open in $\mathfrak{g}^*$. Then, a sequence $(\mathcal{O}_k)_k$ of elements in $\mathfrak{g}^*/G$ converges to the orbit $\mathcal{O}$ in $\mathfrak{g}^*/G$ if and only if for any $\ell \in \mathcal{O}$, there exist $\ell_k \in \mathcal{O}_k, k \in \mathbb{N}$, such that $\ell = \lim\limits_{k+\infty} \ell_k$.*

For the proof, see [Lep-Lud].

Lemma 3.3.2. *For $n \geq 2$ and for any scalars $X_1, ..., X_n, Y_1, ..., Y_{n-1}$ such that $Y_i \neq Y_j$ for $i \neq j$, we have*

$$\sum_{j=1}^{n-1} \frac{\prod_{\substack{i=1\\ i\neq k}}^{n}(X_i - Y_j)}{\prod_{\substack{i=1\\ i\neq j}}^{n-1}(Y_i - Y_j)} = \sum_{\substack{j=1\\ j\neq k}}^{n} X_j - \sum_{j=1}^{n-1} Y_j$$

for each $k = 1, \cdots, n$.

Démonstration. For $n = 1$ the formula is trivial. Suppose that it is true for n. For $k = n + 1$, a simple calculation gives the result. If $k \neq n + 1$ we have

$$\sum_{j=1}^{n} \frac{\prod_{\substack{i=1\\ i\neq k}}^{n+1}(X_i - Y_j)}{\prod_{\substack{i=1\\ i\neq j}}^{n}(Y_i - Y_j)} = \frac{\prod_{\substack{i=1\\ i\neq k}}^{n+1}(X_i - Y_n)}{\prod_{i=1}^{n-1}(Y_i - Y_n)} + \sum_{j=1}^{n-1} \frac{\prod_{\substack{i=1\\ i\neq k}}^{n+1}(X_i - Y_j)}{\prod_{\substack{i=1\\ i\neq j}}^{n}(Y_i - Y_j)}$$

$$= (X_{n+1} - Y_n)\frac{\prod_{\substack{i=1\\ i\neq k}}^{n}(X_i - Y_n)}{\prod_{i=1}^{n-1}(Y_i - Y_n)} + \sum_{j=1}^{n-1} \frac{\prod_{\substack{i=1\\ i\neq k}}^{n}(X_i - Y_j)}{\prod_{\substack{i=1\\ i\neq j}}^{n-1}(Y_i - Y_j)} \frac{(X_{n+1} - Y_j)}{Y_n - Y_j}$$

$$= (X_{n+1} - Y_n)\frac{\prod_{\substack{i=1\\ i\neq k}}^{n}(X_i - Y_n)}{\prod_{i=1}^{n-1}(Y_i - Y_n)} + \sum_{j=1}^{n-1} \frac{\prod_{\substack{i=1\\ i\neq k}}^{n}(X_i - Y_j)}{\prod_{\substack{i=1\\ i\neq j}}^{n-1}(Y_i - Y_j)} \frac{(X_{n+1} - Y_n)}{Y_n - Y_j} + \underbrace{\sum_{j=1}^{n-1} \frac{\prod_{\substack{i=1\\ i\neq k}}^{n}(X_i - Y_j)}{\prod_{\substack{i=1\\ i\neq j}}^{n-1}(Y_i - Y_j)}}_{=\sum_{\substack{j=1\\ j\neq k}}^{n} X_j - \sum_{j=1}^{n-1} Y_j}$$

$$= (X_{n+1} - Y_n) \underbrace{\sum_{j=1}^{n} \frac{\prod_{\substack{i=1\\ i\neq k}}^{n}(X_i - Y_j)}{\prod_{\substack{i=1\\ i\neq j}}^{n}(Y_i - Y_j)}}_{=1 \text{ by Lemma 4.4 of [El-Lu]}} + \sum_{\substack{j=1\\ j\neq k}}^{n} X_j - \sum_{j=1}^{n-1} Y_j = \sum_{\substack{j=1\\ j\neq k}}^{n+1} X_j - \sum_{j=1}^{n} Y_j.$$

□

Lemma 3.3.3. *Given $\mu \in P_{n-1}$ and $\lambda \in P_n$, then $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq ... \geq \mu_{n-1} \geq \lambda_n$ if and only if there is a skew-hermitian matrix*

$$B = \begin{pmatrix} 0 & 0 & \dots & 0 & -z_1 \\ 0 & 0 & \dots & 0 & -z_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -z_{n-1} \\ \overline{z}_1 & \overline{z}_2 & \dots & \overline{z}_{n-1} & ix \end{pmatrix} \tag{3.10}$$

in $\mathcal{W}$ such that $A(J_\mu + B)A^* = J_\lambda$ for some $A \in U(n)$.

Démonstration. For $y \in \mathbb{R}$, we get $\det(J_\mu + B - iy\mathbb{I}) = (-i)^n P(y)$ where

$$P(y) = (y - x)\prod_{i=1}^{n-1}(y - \mu_i) - \sum_{j=1}^{n-1}\Big(|z_j|^2 \prod_{\substack{i=1 \\ i\neq j}}^{n-1}(y - \mu_i)\Big).$$

It is easy to see that $\lim_{y\to+\infty} P(y) = +\infty$, $P(\mu_j) \leq 0$ if j is odd and $P(\mu_j) \geq 0$ if j is even. Now if $A(J_\mu + B)A^* = J_\lambda$ for some $A \in U(n)$ then $i\lambda_1, i\lambda_2, \cdots, i\lambda_n$ are all the elements of the spectrum of the matrix $J_\mu + B$ with $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq ... \geq \mu_{n-1} \geq \lambda_n$.

Conversely, we suppose first that all μ_j are pairwise distinct. In this case, we can take the skew-hermitian matrix B with entries $z_1, \cdots, z_{n-1}, x$ satisfying

$$|z_j|^2 = -\frac{\prod_{i=1}^n(\lambda_i - \mu_j)}{\prod_{\substack{i=1 \\ i\neq j}}^{n-1}(\mu_i - \mu_j)}$$

for every $1 \leq j \leq n-1$, and

$$x = \sum_{j=1}^n \lambda_j - \sum_{j=1}^{n-1}\mu_j.$$

From Lemma 3.3.2,

$$\begin{aligned} P(\lambda_k) &= \Big(\sum_{j=1}^{n-1}\mu_j - \sum_{\substack{j=1 \\ j\neq k}}^{n}\lambda_j\Big)\prod_{i=1}^{n-1}(\lambda_k - \mu_i) + \sum_{j=1}^{n-1}\left(\frac{\prod_{\substack{i=1 \\ i\neq k}}^n(\lambda_i - \mu_j)}{\prod_{\substack{i=1 \\ i\neq j}}^{n-1}(\mu_i - \mu_j)}\prod_{i=1}^{n-1}(\lambda_k - \mu_i)\right) \\ &= \prod_{i=1}^{n-1}(\lambda_k - \mu_i)\left[\sum_{j=1}^{n-1}\mu_j - \sum_{\substack{j=1 \\ j\neq k}}^{n}\lambda_j + \sum_{j=1}^{n-1}\frac{\prod_{\substack{i=1 \\ i\neq k}}^n(\lambda_i - \mu_j)}{\prod_{\substack{i=1 \\ i\neq j}}^{n-1}(\mu_i - \mu_j)}\right] = 0. \end{aligned}$$

Hence the spectrum of the matrix $J_\mu + B$ is the set $\{i\lambda_1, i\lambda_2, \cdots, i\lambda_n\}$.
Now, if the μ_j are not pairwise distinct, there exist two families of integers $\{p_l\}_{1\le l\le s}$ and $\{q_l\}_{1\le l\le s}$ such that $1 \le p_1 < q_1 < p_2 < q_2 < \cdots < p_s < q_s \le n-1$, and for all $1 \le l \le s$ $\mu_{p_l} = \mu_{p_l+1} = \cdots = \mu_{q_l-1} = \mu_{q_l}$, $\mu_{q_l} \neq \mu_{q_l+1}$ and $\mu_{p_l-1} \neq \mu_{p_l}$. Put

$$Q(y) = \prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{n-1}(y-\mu_i),\ \ \tilde{Q}_l(y) = \prod_{\substack{i=1\\ i\neq p_l}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq p_l}}^{p_2}\cdots\prod_{\substack{i=q_{s-1}+1\\ i\neq p_l}}^{p_s}\prod_{i=q_s+1}^{n-1}(y-\mu_i)$$

$$\text{and}\quad Q_j(y) = \prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{n-1}(y-\mu_i).$$

Hence $\det(J_\mu + B - iy\mathbb{I}) = (-i)^n \prod_{l=1}^{s}(y-\mu_{p_l})^{q_l-p_l}P(y)$ where

$$P(y) \;=\; (y-x)Q(y) - \sum_{l=1}^{s}\Big(\sum_{j=p_l}^{q_l}|z_j|^2\Big)\tilde{Q}_l(y) - \sum_{j=1}^{p_1-1}\sum_{j=q_1+1}^{p_2-1}\cdots\sum_{j=q_s+1}^{n-1}\Big(|z_j|^2 Q_j(y)\Big).$$

The skew-hermitian matrix B can be taken as follows :

$$|z_j|^2 = -\frac{\prod_{i=1}^{i=n}(\lambda_i-\mu_j)}{\prod_{i=1,i\neq j}^{i=n-1}(\mu_i-\mu_j)} = -\frac{\prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{n}(\lambda_i-\mu_j)}{\prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)}$$

for each $j = 1, \cdots, p_1-1, q_1+1, \cdots, p_s-1, q_s+1, \cdots, n-1$,

$$|z_{p_l}|^2 + \cdots + |z_{q_l-1}|^2 + |z_{q_l}|^2 = -\frac{\prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{n}(\lambda_i-\mu_{p_l})}{\prod_{\substack{i=1\\ i\neq p_l}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq p_l}}^{p_2}\cdots\prod_{\substack{i=q_{s-1}+1\\ i\neq p_l}}^{p_s}\prod_{i=q_s+1}^{n-1}(\mu_i-\mu_{p_l})}$$

for each $l = 1, \cdots, s$, and

$$x = \sum_{j=1}^{n}\lambda_j - \sum_{j=1}^{n-1}\mu_j = \sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}\cdots\sum_{j=q_s+1}^{n}\lambda_j - \sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}\cdots\sum_{j=q_s+1}^{n-1}\mu_j.$$

We evidently have $P(\lambda_k) = Q(\lambda_k) = 0$ if $\lambda_k = \mu_{p_l}$, and for all $\lambda_k \neq \mu_{p_l}$

$$
\begin{aligned}
P(\lambda_k) &= \Big(\lambda_k - \sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}\cdots\sum_{j=q_s+1}^{n}\lambda_j + \sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}\cdots\sum_{j=q_s+1}^{n-1}\mu_j\Big)Q(\lambda_k)\\
&+ \sum_{j=1}^{p_1-1}\sum_{j=q_1+1}^{p_2-1}\cdots\sum_{j=q_s+1}^{n-1}\Big(\frac{\prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{n}(\lambda_i-\mu_j)}{\prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)}Q_j(\lambda_k)\Big)\\
&+ \sum_{l=1}^{s}\Big(\frac{\prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{n}(\lambda_i-\mu_{p_l})}{\prod_{\substack{i=1\\ i\neq p_l}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq p_l}}^{p_2}\cdots\prod_{\substack{i=q_{s-1}+1\\ i\neq p_l}}^{p_s}\prod_{i=q_s+1}^{n-1}(\mu_i-\mu_{p_l})}\tilde{Q}_l(\lambda_k)\Big)\\
&= Q(\lambda_k)\Big(\sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}\cdots\sum_{j=q_s+1}^{n-1}\mu_j - \sum_{\substack{j=1\\ j\neq k}}^{p_1}\sum_{\substack{j=q_1+1\\ j\neq k}}^{p_2}\cdots\sum_{\substack{j=q_s+1\\ j\neq k}}^{n}\lambda_j\\
&+ \sum_{j=1}^{p_1-1}\sum_{j=q_1+1}^{p_2-1}\cdots\sum_{j=q_s+1}^{n-1}\frac{\prod_{\substack{i=1\\ i\neq k}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq k}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq k}}^{n}(\lambda_i-\mu_j)}{\prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)}\\
&+ \sum_{l=1}^{s}\frac{\prod_{\substack{i=1\\ i\neq k}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq k}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq k}}^{n}(\lambda_i-\mu_{p_l})}{\prod_{\substack{i=1\\ i\neq p_l}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq p_l}}^{p_2}\cdots\prod_{\substack{i=q_{s-1}+1\\ i\neq p_l}}^{p_s}\prod_{i=q_s+1}^{n-1}(\mu_i-\mu_{p_l})}\Big)\\
&= Q(\lambda_k)\Big(\sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}\cdots\sum_{j=q_s+1}^{n-1}\mu_j - \sum_{\substack{j=1\\ j\neq k}}^{p_1}\sum_{\substack{j=q_1+1\\ j\neq k}}^{p_2}\cdots\sum_{\substack{j=q_s+1\\ j\neq k}}^{n}\lambda_j\\
&+ \sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}\cdots\sum_{j=q_s+1}^{n-1}\frac{\prod_{\substack{i=1\\ i\neq k}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq k}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq k}}^{n}(\lambda_i-\mu_j)}{\prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)}\Big) = 0.
\end{aligned}
$$

Hence the spectrum of the matrix $J_\mu + B$ equals the set $\{i\lambda_1, i\lambda_2, \cdots, i\lambda_n\}$. The spectral theorem implies that $A(J_\mu + B)A^* = J_\lambda$ for some $A \in U(n)$. This completes the proof. □

Lemma 3.3.4. *Given $\lambda \in P_n^+$, $\alpha \in \mathbb{R}^*$ and $z \in \mathbb{C}^n$, then the matrix $J_\lambda + \frac{i}{\alpha}zz^*$ admits n eigenvalues $i\beta_1, i\beta_2, \ldots, i\beta_n$ such that $\beta_1 \geq \lambda_1 \geq \beta_2 \geq \lambda_2 \geq \cdots \geq \beta_n \geq \lambda_n$ if $\alpha > 0$ and $\lambda_1 \geq \beta_1 \geq \lambda_2 \geq \beta_2 \geq \cdots \geq \lambda_n \geq \beta_n$ if $\alpha < 0$.*

Démonstration. We can prove by induction that the characteristic polynomial of the matrix $J_\lambda + \frac{i}{\alpha}zz^*$ is equal to $(-i)^n Q_n^{\lambda,z,\alpha}(x)$ where

$$Q_n^{\lambda,z,\alpha}(x) = \prod_{i=1}^{n}(x-\lambda_i) - \sum_{j=1}^{n}\prod_{\substack{i=1\\ i\neq j}}^{n}(x-\lambda_i)\frac{|z_j|^2}{\alpha}.$$

Assume that α is negative. We remark that $\lim_{x\to+\infty} Q_n^{\lambda,z,\alpha}(x) = +\infty$, $Q_n^{\lambda,z,\alpha}(\lambda_j) \geq 0$ if j is odd and $Q_n^{\lambda,z,\alpha}(\lambda_j) \leq 0$ if j is even. Using that $\lim_{x\to-\infty} Q_n^{\lambda,z,\alpha}(x) = -\infty$ if n odd and $\lim_{x\to-\infty} Q_n^{\lambda,z,\alpha}(x) = +\infty$ if n is even, we deduce that $J_\lambda - \frac{i}{\alpha} zz^*$ admits n eigenvalues $i\beta_1, i\beta_2, \ldots, i\beta_n$ verifying $\lambda_1 \geq \beta_1 \geq \lambda_2 \geq \beta_2 \geq \cdots \geq \lambda_n \geq \beta_n$. The same reasoning applies when α is positive. □

Theorem 3.3.5. *Given $\alpha \in \mathbb{R}^*$, $r > 0$, $\mu \in P_{n-1}$ and $\lambda \in P_n$, then*

1) A sequence of coadjoint orbits $(\mathcal{O}_{(\mu^k,r_k)})_k$ converges to $\mathcal{O}_{(\mu,r)}$ in $\mathfrak{g}_n^\ddagger/G_n$ if and only if $\lim_{k\to\infty} r_k = r$ and $\mu^k = \mu$ for large k.

2) A sequence of coadjoint orbits $(\mathcal{O}_{(\mu^k,r_k)})_k$ converges to $\mathcal{O}_\lambda$ in $\mathfrak{g}_n^\ddagger/G_n$ if and only if $(r_k)_k$ tends to zero and $\lambda_1 \geq \mu_1^k \geq \lambda_2 \geq \mu_2^k \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1}^k \geq \lambda_n$ for k large enough.

3) A sequence of coadjoint orbits $(\mathcal{O}_{(\lambda^k,\alpha_k)})_k$ converges to the orbit $\mathcal{O}_{(\lambda,\alpha)}$ in $\mathfrak{g}_n^\ddagger/G_n$ if and only if $\lim_{k\to\infty} \alpha_k = \alpha$ and $\lambda^k = \lambda$ for large k.

4) A sequence of coadjoint orbits $(\mathcal{O}_{(\lambda^k,\alpha_k)})_{k\in\mathbb{N}}$ converges to the orbit $\mathcal{O}_{(\mu,r)}$ in $\mathfrak{g}_n^\ddagger/G_n$ if and only if $\lim_{k\to\infty} \alpha_k = 0$ and the sequence $(\mathcal{O}_{(\lambda^k,\alpha_k)})_{k\in\mathbb{N}}$ satisfies one of the following conditions
i) for k large enough, $\alpha_k > 0$, $\lambda_j^k = \mu_j$ for all $1 \leq j \leq n-1$ and $\lim_{k\to\infty} \alpha_k \lambda_n^k = -\frac{r^2}{2}$,
ii) for k large enough, $\alpha_k < 0$, $\lambda_j^k = \mu_{j-1}$ for all $2 \leq j \leq n$ and $\lim_{k\to\infty} \alpha_k \lambda_1^k = -\frac{r^2}{2}$.

5) A sequence of coadjoint orbits $(\mathcal{O}_{(\lambda^k,\alpha_k)})_{k\in\mathbb{N}}$ converges to the orbit $\mathcal{O}_\lambda$ in $\mathfrak{g}_n^\ddagger/G_n$ if and only if $\lim_{k\to\infty} \alpha_k = 0$ and the sequence $(\mathcal{O}_{(\lambda^k,\alpha_k)})_{k\in\mathbb{N}}$ satisfies one of the following conditions
i) $\lim_{k\to\infty} \alpha_k \lambda_n^k = 0$, $\alpha_k > 0$ and $\lambda_1 \geq \lambda_1^k \geq \cdots \geq \lambda_{n-1} \geq \lambda_{n-1}^k \geq \lambda_n \geq \lambda_n^k$ (for k large enough),
ii) $\lim_{k\to\infty} \alpha_k \lambda_1^k = 0$, $\alpha_k < 0$ and $\lambda_1^k \geq \lambda_1 \geq \lambda_2^k \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n^k \geq \lambda_n$ (for k large enough).

6) A sequence of coadjoint orbits $(\mathcal{O}_{\lambda^k,})_k$ converges to the orbit $\mathcal{O}_\lambda$ in $\mathfrak{g}_n^\ddagger/G_n$ if and only if $\lambda^k = \lambda$ for k large k.

Démonstration. 3) and 6) are trivial. The proof of 1) is similar to that of Theorem 4.2 in [El-Lu] and the assertion 2) follows immediately from Lemma 3.3.3.

4) Assume that $(\mathcal{O}_{(\lambda^k,\alpha_k)})_{k\in\mathbb{N}}$ converges to the orbit $\mathcal{O}_{(\mu,r)}$. Then there exist a sequence $(A_k)_{k\in\mathbb{N}}$ in $U(n)$ and a sequence of vectors $(z(k))_{k\in\mathbb{N}}$ in $\mathbb{C}^n$

so that

$$\lim_{k\to\infty}(A_k(J_{\lambda^k}+\frac{i}{\alpha_k}z(k)z(k)^*)A_k^*, \sqrt{2}A_kz(k), \alpha_k) = (J_\mu, v_r, 0).$$

Let $A = (a_{ij})_{1\leq j\leq n}$ be the limit of a subsequence $(A_s)_{s\in I}$ $(I \subset \mathbb{N})$. So we can say that $\lim_{s\to\infty} J_{\lambda^s} + \frac{i}{\alpha_s}z(s)z(s)^* = A^*J_\mu A$ and $\lim_{s\to\infty} z_j(s) = \frac{r}{\sqrt{2}}\overline{a}_{nj}$ for $j = 1, \cdots, n$. On the other hand, we have $(A^*J_\mu A)_{ij} = i\sum_{l=1}^{n-1}\mu_l\overline{a}_{li}a_{lj}$ and

$$J_{\lambda^s} + \frac{i}{\alpha_s}z(s)z(s)^* = \begin{pmatrix} i\lambda_1^s + i\frac{|z_1(s)|^2}{\alpha_s} & i\frac{z_1(s)\overline{z}_2(s)}{\alpha_s} & \dots & i\frac{z_1(s)\overline{z}_n(s)}{\alpha_s} \\ i\frac{z_2(s)\overline{z}_1(s)}{\alpha_s} & i\lambda_2^s + i\frac{|z_2(s)|^2}{\alpha_s} & \dots & i\frac{z_2(s)\overline{z}_n(s)}{\alpha_s} \\ \vdots & \vdots & \ddots & \vdots \\ i\frac{z_n(s)\overline{z}_1(s)}{\alpha_s} & i\frac{z_n(s)\overline{z}_2(s)}{\alpha_s} & \dots & i\lambda_n^s + i\frac{|z_n(s)|^2}{\alpha_s} \end{pmatrix}.$$

Hence, for $i \neq j$, $\lim_{s\to\infty}|\frac{z_i(s)\overline{z}_j(s)}{\alpha_s}| = |\sum_{l=1}^{n-1}\mu_l\overline{a}_{li}a_{lj}| < \infty$, and since $\lim_{s\to\infty}\|z(s)\| = \frac{r}{\sqrt{2}} \neq 0$, there is a unique $1 \leq i_0 \leq n$ such that $\lim_{s\to\infty} z_{i_0}(s) = \frac{r}{\sqrt{2}}e^{i\theta}$ $(\theta \in \mathbb{R})$ and $\lim_{s\to\infty} z_j(s) = 0$ for $j \neq i_0$. We obtain $a_{ni_0} = e^{-i\theta}$ and $a_{nj} = 0$ for $j \neq i_0$, i.e., the matrices A and $A^*J_\mu A$ can be written in the following way

$$A = \begin{pmatrix} * & \cdots & * & 0 & * & \cdots & * \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * \\ 0 & \cdots & 0 & e^{-i\theta} & 0 & \cdots & 0 \end{pmatrix}$$
$$\underbrace{}_{i_0^{\text{th}}\text{ position}}$$

and

$$A^*J_\mu A = \begin{pmatrix} * & \cdots & * & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * \end{pmatrix} \}i_0^{\text{th}}\text{ position}$$
$$\underbrace{}_{i_0^{\text{th}}\text{ position}}$$

since $\overline{(A^*J_\mu A)_{i_0j}} = -(A^*J_\mu A)_{ji_0} = -i\sum_{l=1}^{n-1}\mu_l\overline{a}_{lj}a_{li_0} = 0$ for $j = 1, \cdots, n$. It follows that $\lim\limits_{s\to\infty}\lambda_{i_0}^s + \frac{|z_{i_0}(s)|^2}{\alpha_s} = 0$ which implies that $\lim\limits_{s\to\infty}|\lambda_{i_0}^s| = \infty$ and that for each $j \neq i_0$

$$\lim_{s\to\infty}\lambda_j^s = \sum_{l=1}^{n-1}\mu_l|a_{lj}|^2, \ \lim_{s\to\infty}\frac{z_j(s)\overline{z}_{i_0}(s)}{\alpha_s} = 0, \ \lim_{s\to\infty}\frac{z_j(s)}{\alpha_s} = 0, \ \text{and} \ \lim_{s\to\infty}\frac{|z_j(s)|^2}{\alpha_s} = 0. \tag{3.11}$$

This proves that i_0 can only take the value 1 if $\alpha_s < 0$ and n if $\alpha_s > 0$. Otherwise, since $\lambda_{i_0-1}^s \geq \lambda_{i_0}^s \geq \lambda_{i_0+1}^s$ we get $\lim\limits_{s\to\infty}\lambda_{i_0-1}^s = +\infty$ if $\alpha_s < 0$ and $\lim\limits_{s\to\infty}\lambda_{i_0+1}^s = -\infty$ if $\alpha_s > 0$ which contradicts the fact that $\lim\limits_{s\to\infty}\lambda_j^s$ is finite for all $j \neq i_0$.

<u>Case $i_0 = n$</u> : In this case, it is clear that $\lim\limits_{s\to\infty}\alpha_s\lambda_n^s = -\frac{r^2}{2}$ and $\lim\limits_{s\to\infty}\lambda_j^s = \sum_{l=1}^{n-1}\mu_l|a_{lj}|^2$ $j = 1, \cdots, n-1$. Furthermore, the matrices A and $A^*J_\mu A$ have the form

$$A = \begin{pmatrix} & & & 0 \\ & \tilde{A} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & e^{-i\theta} \end{pmatrix} \text{ and } A^*J_\mu A = \begin{pmatrix} * & \dots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \dots & * & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

where $\tilde{A} \in U(n-1)$. However, the limit matrix of the subsequence $(J_{\lambda^s} + \frac{i}{\alpha_s}z(s)z(s)^*)_{s\in I}$ must be diagonal since $\lim\limits_{s\to\infty}\frac{z_i(s)\overline{z}_j(s)}{\alpha_s} = 0$ for all $i \neq j$. This implies that $A^*J_\mu A = \mathrm{diag}(i\mu_1, \ldots, i\mu_{n-1}, 0)$, and consequently, for $j = 1, \ldots, n-1$, $\lambda_j^s = \mu_j$ for large s.

<u>Case $i_0 = 1$</u> : In this case, it is easy to check that $\lim\limits_{s\to\infty}\alpha_s\lambda_1^s = -\frac{r^2}{2}$ and $\lim\limits_{s\to\infty}\lambda_j^s = \sum_{l=1}^{n-1}\mu_l|a_{lj}|^2$ $j = 2, \cdots, n$. Moreover, there is $\tilde{A} \in U(n-1)$ so that the matrix A is given by

$$A = \begin{pmatrix} 0 & & & \\ \vdots & & \tilde{A} & \\ 0 & & & \\ e^{-i\theta} & 0 & \cdots & 0 \end{pmatrix} \text{ and hence } A^*J_\mu A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix}.$$

Using the same arguments as above, we have $\lambda_{j+1}^s = \mu_j$ for s large enough and for every $j = 1, \ldots, n-1$.

Conversely, suppose that $\lim_{k\to\infty}\alpha_k = 0$. If our sequence satisfies the first condition, then we take $z(k) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{-\alpha_k\lambda_n^k} \end{pmatrix}$ and $A_k = \mathbb{I}$ for $k \geq N$ (N large enough in $\mathbb{N}$). In the other case, we just set

$$z(k) = \begin{pmatrix} \sqrt{-\alpha_k\lambda_1^k} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \text{ and } A_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ for } k \geq N.$$

Then $\lim_{k\to\infty}(A_k(J_{\lambda^k} + \frac{i}{\alpha_k}z(k)z(k)^*)A_k^*, \sqrt{2}A_k z(k), \alpha_k) = (J_\mu, v_r, 0)$.

5) Suppose that $(\mathcal{O}_{(\lambda^k,\alpha_k)})_{k\in\mathbb{N}}$ converges to the orbit $\mathcal{O}_\lambda$. Then, there exist a sequence $(A_k)_{k\in\mathbb{N}}$ in $U(n)$ and a sequence $(z(k))_{k\in\mathbb{N}}$ in $\mathbb{C}^n$ such that

$$\lim_{k\to\infty}(A_k(J_{\lambda^k} + \frac{i}{\alpha_k}z(k)z(k)^*)A_k^*, \sqrt{2}A_k z(k), \alpha_k) = (J_\lambda, 0, 0).$$

It follows that $\lim_{k\to\infty}\alpha_k = 0$ and that $(z(k))_k$ tends to zero in $\mathbb{C}^n$. Denote by $A = (a_{ij})_{1\leq i,j\leq n}$ the limit matrix of a subsequence $(A_s)_{s\in I}$ ($I \subset \mathbb{N}$). So, we have $\lim_{s\to\infty} J_{\lambda^s} + \frac{i}{\alpha_s}z(s)z(s)^* = A^*J_\lambda A$ with $(A^*J_\lambda A)_{ij} = i\sum_{l=1}^n \lambda_l \overline{a}_{li}a_{lj}$.
Let $\sqrt{\alpha_s}$ be the square root of α_s. The fact that $\lim_{s\to\infty}\frac{z_i(s)\overline{z}_j(s)}{\alpha_s}$ is finite implies that there exists at most one integer $1 \leq i_0 \leq n$ such that $\lim_{s\to\infty}\frac{z_{i_0}(s)}{\sqrt{\alpha_s}} = \infty$.
Therefore, we get

$$\lim_{s\to\infty}\frac{|z_j(s)|^2}{\alpha_s} = \lim_{s\to\infty}\frac{z_j(s)}{\sqrt{\alpha_s}} = \lim_{s\to\infty}\frac{z_i(s)\overline{z}_j(s)}{\alpha_s} = 0$$

for all i and j distinct from i_0. Hence, for the same reasons as in the proof of 4), necessarily $i_0 \in \{1, n\}$.

<u>**Case** $i_0 = n$</u> : In this case, it is easy to see that $\lim_{s\to\infty}\alpha_s\lambda_n^s = 0$ (since $\lim_{s\to\infty}|\lambda_n^s + \frac{|z_n(s)|^2}{\alpha_s}| < \infty$), $\lim_{s\to\infty}\lambda_n^s = -\infty$ and $\lim_{s\to\infty}\lambda_j^s = \sum_{l=1}^n \lambda_l|a_{lj}|^2$ for $j = 1, \cdots, n-1$. Also, α_s must be positive for s large.
Choose

$$x = \lim_{s\to\infty}\lambda_n^s + \frac{|z_n(s)|^2}{\alpha_s}, \;\; \lambda_j' = \lim_{s\to\infty}\lambda_j^s \text{ and } w_j = -i\lim_{s\to\infty}\frac{z_j(s)\overline{z}_n(s)}{\alpha_s}$$

for $j = 1, 2, \ldots, n-1$. Then the limit matrix $A^* J_\lambda A$ of $J_{\lambda^s} + \frac{i}{\alpha_s} z(s) z(s)$ has the form

$$\begin{pmatrix} i\lambda'_1 & 0 & \ldots & 0 & -w_1 \\ 0 & i\lambda'_2 & \ldots & 0 & -w_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & i\lambda'_{n-1} & -w_{n-1} \\ \overline{w}_1 & \overline{w}_2 & \ldots & \overline{w}_{n-1} & ix \end{pmatrix}.$$

By Lemma 3.3.3 we obtain $\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda'_{n-1} \geq \lambda_n$, i.e., $\lambda_1 \geq \lambda_1^s \geq \lambda_2 \geq \lambda_2^s \geq \cdots \geq \lambda_{n-1} \geq \lambda_{n-1}^s \geq \lambda_n \geq \lambda_n^s$ for large s.

__Case $i_0 = 1$__ : Here, it is clear that $\lim\limits_{s\to\infty} \alpha_s \lambda_1^s = 0$ (since $\lim\limits_{s\to\infty} |\lambda_1^s + \frac{|z_1(s)|^2}{\alpha_s}| < \infty$), $\lim\limits_{s\to\infty} \lambda_1^s = +\infty$ and $\lim\limits_{s\to\infty} \lambda_j^s = \sum_{l=1}^n \lambda_l |a_{lj}|^2$ for $j = 2, \cdots, n$. Hence $\alpha_s < 0$ for s large enough. If we set

$$x = \lim_{s\to\infty} \lambda_1^s + \frac{|z_1(s)|^2}{\alpha_s}, \;\; \lambda'_j = \lim_{s\to\infty} \lambda_{j+1}^s \text{ and } w_j = -i \lim_{s\to\infty} \frac{\overline{z}_1(s) z_{j+1}(s)}{\alpha_s}$$

for $j = 1, 2, \ldots, n-1$, the limit matrix $A^* J_\lambda A$ of $J_{\lambda^s} + \frac{i}{\alpha_s} z(s) z(s)$ can be written as follows :

$$\begin{pmatrix} ix & \overline{w}_1 & \overline{w}_2 & \ldots & \overline{w}_{n-1} \\ -w_1 & i\lambda'_1 & 0 & \ldots & 0 \\ -w_2 & 0 & i\lambda'_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_{n-1} & 0 & 0 & \ldots & i\lambda'_{n-1} \end{pmatrix} = \tilde{A}^* \begin{pmatrix} i\lambda'_1 & 0 & \ldots & 0 & -w_1 \\ 0 & i\lambda'_2 & \ldots & 0 & -w_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & i\lambda'_{n-1} & -w_{n-1} \\ \overline{w}_1 & \overline{w}_2 & \ldots & \overline{w}_{n-1} & ix \end{pmatrix} \tilde{A} \tag{3.12}$$

where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.13}$$

This proves that $\lambda_1^s \geq \lambda_1 \geq \lambda_2^s \geq \lambda_2 \geq \cdots \geq \lambda_{n-1}^s \geq \lambda_{n-1} \geq \lambda_n^s \geq \lambda_n$ for large s.

Take now the special case where all $\lim\limits_{s\to\infty} \lambda_j^s = \lambda'_j$ and all $\lim\limits_{s\to\infty} \frac{z_j(s)}{\sqrt{\alpha_s}} = \frac{z_j}{\sqrt{\alpha}}$ are finite for $j = 1, \ldots, n$ with $\alpha > 0$ when $\alpha_s > 0$ and $\alpha < 0$ when $\alpha_s < 0$ for large s. We evidently have $A^* J_\lambda A = J_{\lambda'} + \frac{i}{\alpha} z z^*$. It follows by Lemma 3.3.4

that for k large enough

$$\begin{cases} \lambda_1 \geq \lambda_1^s \geq \lambda_2 \geq \lambda_2^s \geq \cdots \geq \lambda_n \geq \lambda_n^s & \text{if } \alpha_s > 0, \\ \lambda_1^s \geq \lambda_1 \geq \lambda_2^s \geq \lambda_2 \geq \cdots \geq \lambda_n^s \geq \lambda_n & \text{if } \alpha_s < 0. \end{cases}$$

Conversely, suppose that the sequence $(\mathcal{O}_{(\lambda^k,\alpha_k)})_{k\in\mathbb{N}}$ satisfies the first condition. In this case there is a subsequence $(\lambda^s)_{s\in I}$ $(I \subset \mathbb{N})$ with $\lambda_j^s = \lambda'_j$ for all $1 \leq j \leq n-1$ and all $s \in I$. From the lemma 3.3.3, there exist $w_1, w_2, \ldots, w_{n-1}$ in $\mathbb{C}$, $x \in \mathbb{R}$ and $A \in U(n)$ such that

$$A^* J_\lambda A = \begin{pmatrix} i\lambda'_1 & 0 & \ldots & 0 & -w_1 \\ 0 & i\lambda'_2 & \ldots & 0 & -w_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & i\lambda'_{n-1} & -w_{n-1} \\ \overline{w}_1 & \overline{w}_2 & \ldots & \overline{w}_{n-1} & ix \end{pmatrix}.$$

In the sequel, we assume that $\lambda^k \neq \lambda$ (if $\lambda^k = \lambda$ for k large enough, we can take $z(k) = 0$ and $A_k = \mathbb{I}$). We choose $x = \sum_{j=1}^n \lambda_j - \sum_{j=1}^{n-1} \lambda'_j$ (see the proof of Lemma 3.3.3). It follows that

$$\alpha_s(x - \lambda_n^s) = \sum_{j=1}^n \alpha_s(\lambda_j - \lambda_j^s) > 0.$$

Let $(z(s))_{s\in I}$ be a sequence in $\mathbb{C}^n$ with $z_n(s) = \sqrt{\alpha_s(x - \lambda_n^s)}$ and $z_j(s) = i\frac{\alpha_s w_j}{\sqrt{\alpha_s(x-\lambda_n^s)}}$ for $j = 1, 2, \ldots, n-1$. We can easly see that

$$\lim_{s\to\infty} z(s) = 0$$
$$\lambda_n^s + \tfrac{|z_n(s)|^2}{\alpha_s} = x$$
$$\lim_{s\to\infty} \tfrac{|z_j(s)|^2}{\alpha_s} = \lim_{s\to\infty} \tfrac{\alpha_s |w_j|^2}{x-\lambda_n^s} = 0 \quad (j = 1, \ldots, n-1)$$
$$\lim_{s\to\infty} \tfrac{z_i(s)\overline{z_j(s)}}{\alpha_s} = \lim_{s\to\infty} \tfrac{\alpha_s w_i \overline{w}_j}{x-\lambda_n^s} = 0 \quad (1 \leq i \neq j \leq n-1)$$
$$\lim_{s\to\infty} \tfrac{z_j(s)\overline{z_n(s)}}{\alpha_s} = iw_j \quad (j = 1, \ldots, n-1).$$

Hence, $(A(J_{\lambda^s} + \frac{i}{\alpha_s} z(s)z(s)^*)A^*)_{s\in I}$ converges to J_λ.

Suppose now that, for k large enough $\alpha_k < 0$, $\lambda_1^k \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}^k \geq \lambda_{n-1} \geq \lambda_n^k \geq \lambda_n$ and $\lim_{k\to\infty} \alpha_k \lambda_1^k = 0$. In this case, there is a subsequence $(\lambda^s)_{s\in I}$ $(I \subset \mathbb{N})$ such that $\lambda_j^s = \lambda'_{j-1}$ for all $2 \leq j \leq n$ and all $s \in I$. By the identity (3.12) and the Lemma 3.3.3 , there exist $w_1, w_2, \ldots, w_{n-1}$ in $\mathbb{C}$, $x \in \mathbb{R}$ and $A \in U(n)$ such that

$$A^* J_\lambda A = \begin{pmatrix} ix & \overline{w}_1 & \overline{w}_2 & \ldots & \overline{w}_{n-1} \\ -w_1 & i\lambda'_1 & 0 & \ldots & 0 \\ -w_2 & 0 & i\lambda'_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_{n-1} & 0 & 0 & \ldots & i\lambda'_{n-1} \end{pmatrix}.$$

Similarly to the last case, we take $x = \sum_{j=1}^n \lambda_j - \sum_{j=1}^{n-1} \lambda'_j$. We have then

$$\alpha_s(x - \lambda_1^s) = \sum_{j=1}^n \alpha_s(\lambda_j - \lambda_j^s) > 0.$$

This allows us to define the sequence $(z(s))_{s\in I}$ in $\mathbb{C}^n$ by $z_1(s) = \sqrt{\alpha_s(x - \lambda_1^s)}$ and $z_j(s) = -i\frac{\alpha_s w_{j-1}}{\sqrt{\alpha_s(x-\lambda_1^s)}}$ for $j = 2, \ldots, n$. It is clear that

$$\begin{aligned} \lim_{s\to\infty} z(s) &= 0 \\ \lambda_1^s + \tfrac{|z_1(s)|^2}{\alpha_s} &= x \\ \lim_{s\to\infty} \tfrac{|z_j(s)|^2}{\alpha_s} = \lim_{s\to\infty} \tfrac{\alpha_s |w_{j-1}|^2}{x-\lambda_1^s} &= 0 \quad (j = 2, \ldots, n) \\ \lim_{s\to\infty} \tfrac{z_i(s)\overline{z_j(s)}}{\alpha_s} = \lim_{s\to\infty} \tfrac{\alpha_s w_{i-1}\overline{w}_{j-1}}{x-\lambda_1^s} &= 0 \quad (2 \le i \ne j \le n) \\ \lim_{s\to\infty} \tfrac{z_j(s)\overline{z_1(s)}}{\alpha_s} &= i w_{j-1} \quad (j = 2, \ldots, n). \end{aligned}$$

We conclude that $((A(J_{\lambda^s} + \frac{i}{\alpha_s} z(s)z(s)^*)A^*, \sqrt{2}Az(s), \alpha_s))_{s\in I}$ converges to $(J_\lambda, 0, 0)$. $\square$

3.4 Some theorems on the dual topology.

Let G be a second countable locally compact group, and let $\widehat{G}$ be the space of the equivalence classes of irreducible unitary representations of G.

Definition 3.4.1. *A continuous function $\varphi : G \longrightarrow \mathbb{C}$ is said to be of positive type if the kernel function defined on $G \times G$ by $(g_1, g_2) \mapsto \varphi(g_j^{-1} g_i)$ is of positive type, i.e. for all $g_1, g_2, \ldots, g_n \in G$ and all $c_1, c_2, \ldots, c_n \in \mathbb{C}$,*

$$\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} \varphi(g_j^{-1} g_i) \ge 0.$$

Let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation on the Hilbert space $\mathcal{H}_\pi$.

Proposition 3.4.2. *Let ξ be a vector in $\mathcal{H}_\pi$. Then the function $C_\xi^\pi : G \longrightarrow \mathbb{C}$, $g \longmapsto \langle \pi(g)\xi, \xi\rangle$ is of positive type.*

Theorem 3.4.3. *([Dix]) Let $(\pi_k, \mathcal{H}_{\pi_k})_{k\in\mathbb{N}}$ be a family of irreducible unitary representations of G. Then $(\pi_k)_k$ converges to π in $\widehat{G}$ if and only if for some non-zero (resp. for every) vector ξ in $\mathcal{H}_\pi$, there exist $\xi_k \in \mathcal{H}_{\pi_k}, k \in \mathbb{N}$, such that the sequence $(C_{\xi_k}^{\pi_k})_k$ of functions converges uniformly on compacta to C_ξ^π.*

The topology of $\widehat{G}$ can also be expressed by the weak convergence of the coefficient functions.

Theorem 3.4.4. *([Dix]) Let $(\pi_k, \mathcal{H}_{\pi_k})_{k\in\mathbb{N}}$ be a sequence of irreducible unitary representations of G. Then $(\pi_k)_k$ converges to π in $\widehat{G}$ if and only if for some non-zero (resp. for every) vector ξ in $\mathcal{H}_\pi$, there are $\xi_k \in \mathcal{H}_{\pi_k}$ such that the sequence of linear functionals $(C_{\xi_k}^{\pi_k})_k \subset C^*(G)'$ converges weakly on some dense subspace of the C^*-algebra $C^*(G)$ of G to the linear functional C_ξ^π.*

If G is a Lie group, then we denote respectively by $\mathfrak{g}$ the Lie algebra of G and by $\mathcal{U}(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. For a unitary representation $(\pi, \mathcal{H}_\pi)$ of G, let $\mathcal{H}_\pi^\infty$ be the subspace of $\mathcal{H}_\pi$ consisting of the C^∞-vectors for π.

Corollary 3.4.5. *Let $(\pi_k, \mathcal{H}_{\pi_k})_{k\in\mathbb{N}}$ be a sequence of irreducible unitary representations of the Lie group G. If $(\pi_k)_k$ converges to π in $\widehat{G}$ then for every unit vector ξ in $\mathcal{H}_\pi^\infty$, there exist $\xi_k \in \mathcal{H}_{\pi_k}^\infty, k \in \mathbb{N}$, such that the numerical sequence $(\langle d\pi_k(D)\xi_k, \xi_k\rangle)_k$ converges to $\langle d\pi(D)\xi, \xi\rangle$, for each $D \in \mathcal{U}(\mathfrak{g})$.*

Démonstration. Let $\xi \in \mathcal{H}_\pi^\infty$ be a unit vector . It follows from [Dix-Mal], that there exist $f_1, \cdots, f_s \in C_c^\infty(G)$ and linearly independent vectors $\xi_1, \cdots, \xi_s \in \mathcal{H}_\pi$, such that $\xi = \pi(f_1)\xi_1 + \cdots + \pi(f_s)\xi_s$. Since π is irreducible, we can find for any non-zero vector $\eta \in \mathcal{H}_\pi$, elements q_j in the C^*-algebra of G, such that $\xi_j = \pi(q_j)\eta, j = 1\cdots, s$. Hence $\xi = \sum_{j=1}^s \pi(f_j)\pi(q_j)\eta$. Choose now for $k \in \mathbb{N}$ vectors $\eta_k \in \mathcal{H}_{\pi_k}$, such that the coefficients $C_{\eta_k}^{\pi_k}$ converge weakly to the coefficient C_η^π. Let $\xi_k := \sum_{j=1}^s \pi_k(f_j)\pi_k(q_j)\eta_k, k \in \mathbb{N}$. Then, for $D \in \mathcal{U}(\mathfrak{g})$

it follows that

$$\begin{aligned}\lim_{k+\infty}\langle d\pi_k(D)\xi_k,\xi_k\rangle &= \lim_{k+\infty}\langle\sum_{j=1}^{s}\pi_k(D*f_j)\pi_k(q_j)\eta_k,\sum_{i=1}^{s}\pi_k(f_i)\pi_k(q_i)\eta_k\rangle\\ &= \sum_{i,j=1}^{s}\lim_{k+\infty}\langle\pi_k(q_i^* * f_i^* * D * f_j * q_j)\eta_k,\eta_k\rangle\\ &= \sum_{i,j=1}^{s}\langle\pi(q_i^* * f_i^* * D * f_j * q_j)\eta,\eta\rangle\\ &= \langle d\pi(D)\xi,\xi\rangle.\end{aligned}$$

□

The question is whether the topology of the dual space of $U(n) \ltimes \mathbb{H}_n$ is determined by the topology of its admissible quotient space.

3.5 The topology of the dual space of G_n.

In this section we give some results on convergence in the dual space of the semi-direct product $G_n = U(n) \ltimes \mathbb{H}_n$ in terms of the Mackey data.

Let us first write down explicitly the representation $\pi_{(\mu,r)} = \mathrm{ind}_{U(n-1)\ltimes\mathbb{H}_n}^{G_n}\rho_\mu\otimes\chi_r$. Its Hilbert space $\mathcal{H}_{(\mu,r)}$ can be identified with the space

$$L^2(G_n/U(n-1)\ltimes\mathbb{H}_n,\rho_\mu\otimes\chi_r)\simeq L^2(U(n)/U(n-1),\rho_\mu).$$

Let ξ be a unit vector in $\mathcal{H}_{(\mu,r)}$. For all $(z,t)\in\mathbb{H}_n$, and all $A,B\in U(n)$ we have

$$(\pi_{(\mu,r)}(A,z,t)\xi)(B) = e^{-i(Bv_r,z)}\xi(A^{-1}B). \tag{3.14}$$

Therefore

$$\begin{aligned}C_\xi^{\pi_{(\mu,r)}}(A,z,t) &= \langle\pi_{(\mu,r)}(A,z,t)\xi,\xi\rangle_{L^2(U(n)/U(n-1),\rho_\mu)}\\ &= \int_{U(n)} e^{-i(Bv_r,z)}\langle\xi(A^{-1}B),\xi(B)\rangle_{\mathcal{H}_{\rho_\mu}}dB.\end{aligned} \tag{3.15}$$

Let us use the notations of the subsection 3.2.2. By the theorems of Weyl and Frobenius (see subsection 3.2.2), we have

$$\pi_\mu := \pi_{(\mu,r)|U(n)} \simeq \mathrm{ind}_{U(n-1)}^{U(n)}\rho_\mu = \sum_{\substack{\tau_\lambda\in\widehat{U(n)}\\ \lambda_1\geq\mu_1\geq\lambda_2\geq\mu_2\geq\ldots\geq\lambda_{n-1}\geq\mu_{n-1}\geq\lambda_n}}\tau_\lambda. \tag{3.16}$$

Since ρ_μ is a subrepresentation of $\mathrm{ind}_{\mathbb{T}_{n-1}}^{U(n-1)}\chi_\mu$, we can identify the Hilbert space $\mathcal{H}_{(\mu,r)}$ of the representation $\pi_{(\mu,r)}$ with a closed subspace L^2_μ of the space $L^2(U(n)/\mathbb{T}_{n-1},\chi_\mu)$. Here $\mathbb{T}_{n-1}\subset\mathbb{T}_n$ denotes the maximal torus of $U(n-1)$. Now every irreducible representation τ_λ of $U(n)$ can be realized as a subrepresentation of $L^2(U(n))$ via the intertwining operator

$$U_\lambda:\mathcal{H}_\lambda\to L^2(U(n)); U_\lambda(\xi)(A):=\langle\xi,\tau_\lambda(A)\xi_\lambda\rangle, A\in U(n).$$

For $\tau_\lambda\in\widehat{U(n)}$ we take an orthonormal basis $\mathcal{B}^\lambda=\{\phi_j^\lambda; j=1,\cdots,d_\lambda\}$ of $\mathcal{H}_\lambda$ consisting of eigenvectors for $\mathbb{T}_n$ of $\mathcal{H}_\lambda$, and for every eigenvalue χ_ν of $\mathbb{T}_{n-1}$ appearing in τ_λ we denote by $I(\lambda,\nu)$ the set of indices i for which $\tau_\lambda(A)\phi_i^\lambda=\chi_\nu(A)\phi_i^\lambda, A\in\mathbb{T}_{n-1}$. It follows then from the theorem of Peter-Weyl, that

$$L^2_\mu\subset\sum_{\substack{\tau_\lambda\in\widehat{U(n)}\\ \tau_\lambda\in\pi_\mu}}\sum_{1\le j\le d_\lambda}\sum_{i\in I(\lambda,\mu)}\mathbb{C}\overline{C^\lambda_{i,j}}, \tag{3.17}$$

where for simplicity of notations, we have written $C^\lambda_{i,j}:=C^{\tau_\lambda}_{\phi_i^\lambda,\phi_j^\lambda}, 1\le i,j\le d_\lambda$.

We take as basis of the Lie algebra $\mathfrak{h}_n$ of the Heisenberg group the left invariant vector fields $\{Z_1,Z_2,\ldots,Z_n,\overline{Z}_1,\overline{Z}_2,\ldots,\overline{Z}_n,T\}$ where

$$Z_j=2\frac{\partial}{\partial\overline{z}_j}+i\frac{z_j}{2}\frac{\partial}{\partial t},\ \ \overline{Z}_j=2\frac{\partial}{\partial z_j}-i\frac{\overline{z}_j}{2}\frac{\partial}{\partial t}, \tag{3.18}$$

and

$$T:=\frac{\partial}{\partial t}. \tag{3.19}$$

With these conventions one has $[Z_j,\overline{Z}_j]=-2iT$. One differential operator will play a key role in this paper. This is the Heisenberg sub-Laplacian defined by

$$\mathcal{L}=\frac{1}{2}\sum_{j=1}^n(Z_j\overline{Z}_j+\overline{Z}_jZ_j). \tag{3.20}$$

The operator $\mathcal{L}$ is $U(n)$-invariant.

Lemma 3.5.1. *For every irreducible representation $\pi_{(\mu,r)}(r>0,\rho_\mu\in\widehat{U(n-1)})$ of G_n, we have that*

$$d\pi_{(\mu,r)}(\mathcal{L})=-r^2\mathbb{I}.$$

Démonstration. Since the representation $\pi_{(\mu,r)}$ is trivial on the center of $\mathfrak{h}_n$, we have

$$d\pi_{(\mu,r)}(\mathcal{L})\xi(B)=2\sum_{1\le j\le n}\left(\frac{\partial^2}{\partial z_j\partial\overline{z}_j}+\frac{\partial^2}{\partial\overline{z}_j\partial z_j}\right)\left[e^{-i(Bv_r,z)}\right]\xi(B).$$

Let $\mathbb{D} = \{e_1, \ldots, e_n\}$ be an $\langle .,. \rangle$-orthonormal basis for $\mathbb{C}^n$. By writing $(Bv_r, z) = \frac{1}{2}(\langle Bv_r, z\rangle + \overline{\langle Bv_r, z\rangle})$, we get

$$d\pi_{(\mu,r)}(\mathcal{L})\xi(B) = -\sum_{1\leq j\leq n} |\langle Bv_r, e_j\rangle|^2\xi(B) = -r^2\xi(B).$$

□

The two following theorems 3.5.2 and 3.5.3 can be read off from Theorem 6.2.A of [Ba], but we give here a direct proof which might be useful for later studies of the dual topology of more complicated groups.

Theorem 3.5.2. *Let $r > 0$ and $\rho_\mu \in \widehat{U(n-1)}$. Then a sequence $(\pi_{(\mu^k,r_k)})_k$ of irreducible representations of G_n converges to $\pi_{(\mu,r)}$ in $\hat{G}_n$ if and only if $(r_k)_k$ tends to r as $k \longrightarrow +\infty$ and $\mu^k = \mu$ for k large enough.*

Démonstration. Suppose at first that $\lim_{k\to\infty} r_k = r$ and $\mu^k = \mu$ for k large enough. We choose $\xi_k = \xi$ for all $k \in \mathbb{N}$. Thus for $f \in C_c^\infty(G_n)$ and for every $k \in \mathbb{N}$ we have

$$\langle C_{\xi_k}^{\pi_{(\mu^k,r_k)}}, f\rangle = \int_{\mathbb{H}_n}\int_{U(n)}\int_{U(n)} e^{-i(Bv_{r_k},z)} f(A,z,t)\xi(A^{-1}B)\overline{\xi(B)}dBdAdzdt.$$

Then, by Lebesgue's theorem $(\langle C_{\xi_k}^{\pi_{(\mu^k,r_k)}}, f\rangle)_k$ converges to $\langle C_\xi^{\pi_{(\mu,r)}}, f\rangle$.
Conversely, suppose that $(\pi_{(\mu^k,r_k)})_k$ converges to $\pi_{(\mu,r)}$. It follows from Corollary 3.4.5 that for a unit vector $\xi \in \mathcal{H}^\infty_{(\mu,r)}$, there exist $\xi_k \in \mathcal{H}^\infty_{(\mu^k,r_k)}$ such that $\|\xi_k\|_{\mathcal{H}_{(\mu^k,r_k)}} = 1$ and $(\langle d\pi_{(\mu^k,r_k)}(\mathcal{L})\xi_k, \xi_k\rangle)_k$ converges to $\langle d\pi_{(\mu,r)}(\mathcal{L})\xi,\xi\rangle$. By Lemma 3.5.1 we have

$$-r_k^2 = \langle d\pi_{(\mu^k,r_k)}(\mathcal{L})\xi_k,\xi_k\rangle \to \langle d\pi_{(\mu,r)}(\mathcal{L})\xi,\xi\rangle = -r^2.$$

Thus, r_k tends to r as $k \longrightarrow +\infty$. It remains for us to show that $\mu^k = \mu$ for k large enough.

Let ξ be any unit vector in $\mathcal{H}_{(\mu,r)}$. So by Theorem 3.4.3 there are $\xi_k \in \mathcal{H}_{(\mu^k,r_k)}$ such that $\|\xi_k\|_{\mathcal{H}_{(\mu^k,r_k)}} = 1$ and $(C_{\xi_k}^{\pi_{(\mu^k,r_k)}})_k$ converges uniformly on compacta to $C_\xi^{\pi_{(\mu,r)}}$. In particular, we have

$$\begin{aligned}\lim_{k\to\infty} C_{\xi_k}^{\pi_{(\mu^k,r_k)}}(A,0,0) &= \lim_{k\to\infty}\langle \pi_{(\mu^k,r_k)}(A,0,0)\xi_k,\xi_k\rangle \\ &= \lim_{k\to\infty}\int_{U(n)} \xi_k(A^{-1}B)\overline{\xi_k(B)}dB \\ &= \int_{U(n)} \xi(A^{-1}B)\overline{\xi(B)}dB \\ &= C_\xi^{\pi_{(\mu,r)}}(A,0,0)\end{aligned} \tag{3.21}$$

uniformly in $A \in U(n)$. However, by (3.17) we can write

$$\xi_k = \sum_{\substack{\tau_\lambda \in \widehat{U(n)} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1 \leq j \leq d_\lambda} \sum_{i \in I(\lambda, \mu^k)} a_{i,j}^{(\lambda,k)} \overline{C_{i,j}^\lambda},$$

and

$$\sum_{\substack{\tau_\lambda \in \widehat{U(n)} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1 \leq j \leq d_\lambda} \sum_{i \in I(\lambda, \mu^k)} \frac{|a_{i,j}^{(\lambda,k)}|^2}{d_\lambda} = \|\xi_k\|^2_{\mathcal{H}_{(\mu^k, r_k)}} = 1. \tag{3.22}$$

In addition, for all $A, B \in U(n)$

$$C_{i,j}^\lambda(A^{-1}B) = \langle \tau_\lambda(A^{-1}B)\phi_i^\lambda, \phi_j^\lambda \rangle = C^{\tau_\lambda}_{\phi_i^\lambda, \tau_\lambda(A)\phi_j^\lambda}(B).$$

Consequently, by using the orthogonality relation (3.8), we have

$$\begin{aligned} C_{\xi_k}^{\pi(\mu^k, r_k)}(A,0,0) &= \sum_{\substack{\tau_\lambda \in \widehat{U(n)} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1 \leq j,j' \leq d_\lambda} \sum_{i,i' \in I(\lambda, \mu^k)} a_{i,j}^{(\lambda,k)} \overline{a_{i',j'}^{(\lambda,k)}} \overline{\langle C^{\tau_\lambda}_{\phi_i^\lambda, \tau_\lambda(A)\phi_j^\lambda}, C^{\tau_\lambda}_{\phi_{i'}^\lambda, \phi_{j'}^\lambda} \rangle} \\ &= \sum_{\substack{\tau_\lambda \in \widehat{U(n)} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1 \leq j,j' \leq d_\lambda} \sum_{i,i' \in I(\lambda, \mu^k)} \frac{a_{i,j}^{(\lambda,k)} \overline{a_{i',j'}^{(\lambda,k)}}}{d_\lambda} \overline{\langle \phi_i^\lambda, \phi_{i'}^\lambda \rangle \langle \phi_{j'}^\lambda, \tau_\lambda(A)\phi_j^\lambda \rangle} \\ &= \sum_{\substack{\tau_\lambda \in \widehat{U(n)} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1 \leq j,j' \leq d_\lambda} \sum_{i \in I(\lambda, \mu^k)} \frac{a_{i,j}^{(\lambda,k)} \overline{a_{i,j'}^{(\lambda,k)}}}{d_\lambda} C_{j,j'}^\lambda(A). \end{aligned} \tag{3.23}$$

Let $\tilde{\mu} = (\mu_1, ..., \mu_{n-1}, \mu_{n-1})$. Then $I(\tilde{\mu}, \mu)$ consists of one point, since $\tilde{\mu}$ is dominant integral and we can take $I(\tilde{\mu}, \mu) = \{1\}$. We choose now $\overline{\xi_\mu} := \overline{\xi} := \sqrt{d_{\tilde{\mu}}} C^{\tau_{\tilde{\mu}}}_{\phi_1^{\tilde{\mu}}, \phi_1^{\tilde{\mu}}} \in L^2_\mu$ and we obtain

$$\begin{aligned} C_\xi^{\pi(\mu,r)}(A,0,0) &= \int_{U(n)} \xi(A^{-1}B)\overline{\xi(B)}dB \\ &= d_{\tilde{\mu}} \overline{\langle C^{\tau_{\tilde{\mu}}}_{\phi_1^{\tilde{\mu}}, \tau_{\tilde{\mu}}(A)\phi_1^{\tilde{\mu}}}, C^{\tau_{\tilde{\mu}}}_{\phi_1^{\tilde{\mu}}, \phi_1^{\tilde{\mu}}} \rangle} \\ &= \overline{\langle \phi_1^{\tilde{\mu}}, \tau_{\tilde{\mu}}(A)\phi_1^{\tilde{\mu}} \rangle} = C^{\tau_{\tilde{\mu}}}_{\phi_1^{\tilde{\mu}}, \phi_1^{\tilde{\mu}}}(A), A \in U(n). \end{aligned}$$

It follows from (3.21) and (3.23), that the numerical series defined by

$$\begin{aligned} S_k &:= \langle C_{\xi^k}^{\pi(\mu^k, r_k)}(.,0,0), C_\xi^{\pi(\mu,r)}(.,0,0) \rangle \\ &= \sum_{\substack{\tau_\lambda \in \widehat{U(n)} \\ \tau_\lambda \in \pi_{\mu^k}}} \sum_{1 \leq j,j' \leq d_\lambda} \sum_{i \in I(\lambda, \mu^k)} \frac{a_{i,j}^{(\lambda,k)} \overline{a_{i,j'}^{(\lambda,k)}}}{d_\lambda} \langle C_{j,j'}^\lambda, C_{1,1}^{\tilde{\mu}} \rangle \end{aligned}$$

converges to the number $\langle C_\xi^{\pi(\mu,r)}(.,0,0), C_\xi^{\pi(\mu,r)}(.,0,0)\rangle = \frac{1}{d_{\tilde{\mu}}} \neq 0$. Hence by the orthogonality relation (3.8), we must have that $\tau_{\tilde{\mu}} \in \pi_{\mu^k}$ for k large enough, since otherwise the right hand side of S_k is zero for infinitely many k. Therefore we deduce from (3.16) that

$$\mu_1 \geq \mu_1^k \geq \mu_2 \geq \mu_2^k \geq \cdots \geq \mu_{n-2} \geq \mu_{n-2}^k \geq \mu_{n-1} = \mu_{n-1}^k$$

and also that

$$\lim_{k\to\infty} \sum_{i\in I(\tilde{\mu},\mu^k)} \frac{|a_{i,1}^{(\tilde{\mu},k)}|^2}{d_{\tilde{\mu}}} = 1. \tag{3.24}$$

Whence, by (3.22)

$$\lim_{k\to\infty} \left[\sum_{\substack{\tau_\lambda\neq\tau_{\tilde{\mu}} \\ \tau_\lambda\in\pi_{\mu^k}}} \sum_{1\leq j\leq d_\lambda} \sum_{i\in I(\lambda,\mu^k)} \frac{|a_{i,j}^{(\lambda,k)}|^2}{d_\lambda} + \sum_{2\leq j\leq d_{\tilde{\mu}}} \sum_{i\in I(\tilde{\mu},\mu^k)} \frac{|a_{i,j}^{(\tilde{\mu},k)}|^2}{d_{\tilde{\mu}}} \right] = 0.$$

Thus, we have $\xi_k = \sum_{i\in I(\tilde{\mu},\mu^k)} a_{1,i}^{(\tilde{\mu},k)} \overline{C_{1,i}^{\tilde{\mu}}} + E_k$ where $E_k \in L^2_{\mu^k}$ with $\lim_{k\to\infty} \|E_k\|_2 = 0$ $(k \in \mathbb{N})$. Let $\eta_k := \sum_{i\in I(\tilde{\mu},\mu^k)} a_{1,i}^{(\tilde{\mu},k)} \overline{C_{1,i}^{\tilde{\mu}}}$, $k \in \mathbb{N}$. Since the sequence $(\mu^k)_k$ is seen to be bounded, we can decompose it (apart from a finite number of indices) in a finite union of constant subsequences. Let us show that all these constant subsequences are equal to μ. Take such a constant subsequence $(\mu^s)_{s\in I}$ $(I \subset \mathbb{N})$, i.e, we have that $\mu^s = \mu'$, $s \in I$, with

$$\mu_1 \geq \mu'_1 \geq \mu_2 \geq \mu'_2 \geq \cdots \geq \mu_{n-2} \geq \mu'_{n-2} \geq \mu_{n-1} = \mu'_{n-1}.$$

Then, we obtain for $z \in \mathbb{C}^n$ that

$$\begin{aligned} & C_{\xi_s}^{\pi(\mu^s,r_s)}(\mathbb{I}, z, 0) = C_{\xi_s}^{\pi(\mu',r_s)}(\mathbb{I}, z, 0) \\ = & \int_{U(n)} e^{-i(Bv_{r_s},z)} |\xi_s(B)|^2 dB \\ = & \int_{U(n)} e^{-i(Bv_r,z)} |\eta_s(B)|^2 dB + \varepsilon_s(z), \end{aligned}$$

where $(\varepsilon_s)_s$ tends uniformly to zero as k tends to infinity. Since by (3.24) (for another subsequence) $\eta_s = \sum_{i\in I(\tilde{\mu},\mu')} a_{1,i}^{(\tilde{\mu},s)} \overline{C_{1,i}^{\tilde{\mu}}}$ tends to an element $\xi_{\mu'} = \sum_{i\in I(\tilde{\mu},\mu')} a_{1,i}^{(\tilde{\mu})} \overline{C_{1,i}^{\tilde{\mu}}} \in L^2_{\mu'}$, we have

$$\lim_{j\to\infty} C_{\xi_s}^{\pi(\mu^s,r_s)}(\mathbb{I}, z, 0) = \int_{U(n)} e^{-i(Bv_r,z)} |\xi_{\mu'}(B)|^2 dB.$$

Consequently, we find

$$\int_{U(n)} e^{-i(Bv_r,z)}|\xi_{\mu'}(B)|^2 dB = \int_{U(n)} e^{-i(Bv_r,z)}|\xi_{\mu}(B)|^2 dB. \tag{3.25}$$

We define two measures δ_μ and $\delta_{\mu'}$ on $\mathbb{C}^n$ by

$$\delta_\mu(f) = \int_{U(n)} f(Bv_r)|\xi_\mu(B)|^2 dB$$

and

$$\delta_{\mu'}(f) = \int_{U(n)} f(Bv_r)|\xi_{\mu'}(B)|^2 dB,$$

for all $f \in C_c^\infty(\mathbb{C}^n)$. From (3.25), it follows that $\widehat{\delta_\mu} = \widehat{\delta_{\mu'}}$, i.e., $\delta_\mu = \delta_{\mu'}$ and $|\xi_\mu| = |\xi_{\mu'}|$. Hence

$$0 \neq \langle \phi_1^{\tilde{\mu}}, \phi_1^{\tilde{\mu}} \rangle = |\xi_\mu(\mathbb{I}_n)| = |\xi_{\mu'}(\mathbb{I}_n)| = |\sum_{i \in I(\tilde{\mu},\mu')} a_{1,i}^{(\tilde{\mu})} \langle \phi_i^{\tilde{\mu}}, \phi_1^{\tilde{\mu}} \rangle|.$$

This implies that $\langle \phi_i^{\tilde{\mu}}, \phi_1^{\tilde{\mu}} \rangle \neq 0$ for at least one $i \in I(\tilde{\mu}, \mu')$. This proves that $\mu = \mu'$. □

Theorem 3.5.3. *Let $(\pi_{(\mu^k,r_k)})_k$ be sequence of irreducible representations of G_n. Then $(\pi_{(\mu^k,r_k)})_k$ converges to τ_λ in $\hat{G}_n$ if and only if $\lim\limits_{k\to\infty} r_k = 0$ and $\tau_\lambda \in \pi_{\mu^k}$ for k large enough.*

Démonstration. Suppose that $\tau_\lambda \in \pi_{\mu^k}$, i.e. $\lambda_1 \geq \mu_1^k \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1}^k \geq \lambda_n$, for large k and that $\lim\limits_{k\to\infty} r_k = 0$. Hence the sequence $(\mu^k)_k$ is bounded and we can again write $(\mu^k)_k$ as a finite union of eventually constant sequences. Take such an infinite subset $I \subset \mathbb{N}$, such that $\mu^s = \mu = \mu(I)$ for all $s \in I$. We choose a unit vector $\xi \in \mathcal{H}_\lambda \subset \mathcal{H}_{(\mu^k,r_k)}$. Hence we have that

$$\begin{aligned}\langle \tau_\lambda(A)\xi, \xi \rangle_{\mathcal{H}_\lambda} &= \langle (\mathrm{ind}_{U(n-1)}^{U(n)} \rho_\mu)(A)\xi, \xi \rangle_{L^2} \\ &= \int_{U(n)} \langle \xi(A^{-1}B), \xi(B) \rangle_{\mathcal{H}_{\rho_\mu}} dB,\end{aligned}$$

for all $A \in U(n)$. Thus, we can choose $\xi_s = \xi$ for all $s \in I$. We obtain, for all f in $C_c^\infty(G_n)$

$$\begin{aligned}&\langle C_{\xi_s}^{\pi_{(\mu^s,r_s)}}, f \rangle = \langle C_{\xi}^{\pi_{(\mu,r_s)}}, f \rangle \\ = &\int_{\mathbb{H}_n} \int_{U(n)} \int_{U(n)} \chi_{r_s}(B^{-1}z) f(A,z,t) \langle \xi(A^{-1}B), \xi(B) \rangle_{\mathcal{H}_{\rho_\mu}} dBdAdzdt.\end{aligned}$$

This integral converges to

$$\begin{aligned} & \int_{U(n)} \int_{\mathbb{H}_n} f(A,z,t) \int_{U(n)} \langle \xi(A^{-1}B), \xi(B) \rangle_{\mathcal{H}_{\rho_\mu}} dB dA dz dt \\ = & \int_{U(n)} \int_{\mathbb{H}_n} f(A,z,t) \langle \tau_\lambda(A,z,t)\xi, \xi \rangle_{\mathcal{H}_\lambda} dA dz dt \\ = & \langle C_\xi^{\tau_\lambda}, f \rangle. \end{aligned}$$

By considering all possible subsets I of this kind, we see that $\pi_{(\mu^k, r_k)}$ has as limit point the representation τ_λ.

Conversely, it is clear from Lemma 3.5.1 and Corollary 3.4.5 that $\lim_{k\to\infty} r_k = 0$, since τ_λ is trivial on $\mathbb{H}_n$. It remains for us to show that $\lambda_1 \geq \mu_1^k \geq ... \geq \lambda_{n-1} \geq \mu_{n-1}^k \geq \lambda_n$ for k large enough. We use the notations and procedure of the proof of Theorem 3.5.2.
Let $\xi = \phi_1^\lambda \in \mathcal{H}_\lambda$ be a unit vector associated to the highest weight λ. Then there exist $\xi_k \in \mathcal{H}_{(\mu^k, r_k)}$ of length 1 such that for all $A \in U(n)$ we have $\lim_{k\to\infty} C_{\xi_k}^{\pi_{(\mu^k,r_k)}}(A,0,0) = C_\xi^{\tau_\lambda}(A)$. Then by (3.17) we can write

$$\xi_k = \sum_{\substack{\tau_{\lambda'} \in \widehat{U(n)} \\ \tau_{\lambda'} \in \pi_{\mu^k}}} \sum_{1 \leq j \leq d_\lambda} \sum_{i \in I(\lambda', \mu^k)} a_{i,j}^{(\lambda',k)} \overline{C_{i,j}^{\lambda'}},$$

and

$$\sum_{\substack{\tau_{\lambda'} \in \widehat{U(n)} \\ \tau_{\lambda'} \in \pi_{\mu^k}}} \sum_{1 \leq j \leq d_{\lambda'}} \sum_{i \in I(\lambda', \mu^k)} \frac{|a_{i,j}^{(\lambda',k)}|^2}{d_{\lambda'}} = \|\xi_k\|^2_{\mathcal{H}_{(\mu^k, r_k)}} = 1.$$

The numerical series S_k defined by

$$\begin{aligned} S_k & := \langle C_{\xi^k}^{\pi_{(\mu^k,r_k)}}(.,0,0), C_\xi^{\tau_\lambda} \rangle \\ & = \sum_{\substack{\tau_{\lambda'} \in \widehat{U(n)} \\ \tau_{\lambda'} \in \pi_{\mu^k}}} \sum_{1 \leq j,j' \leq d_{\lambda'}} \sum_{i \in I(\lambda', \mu^k)} \frac{a_{i,j}^{(\lambda',k)} \overline{a_{i,j'}^{(\lambda',k)}}}{d_{\lambda'}} \langle C_{j,j'}^{\lambda'}, C_{1,1}^\lambda \rangle \end{aligned}$$

converges to $\langle C_\xi^{\tau_\lambda}, C_\xi^{\tau_\lambda} \rangle = \frac{1}{d_\lambda} \neq 0$. By the orthogonality relation (3.8), it follows that $\tau_\lambda \in \pi_{\mu^k}$ for k large enough. $\square$

Let us now look at the representations $\pi_{(\lambda,\alpha)}$. Let a unit vector $\xi = \sum_{1\leq j\leq d_\lambda} \phi_j^\lambda \otimes f_j$ be in the Hilbert space $\mathcal{H}_{(\lambda,\alpha)} := \mathcal{H}_\lambda \otimes F_\alpha(n)$ of $\pi_{(\lambda,\alpha)}$, where $f_1, \ldots, f_{d_\lambda}$ belong to the Fock space $F_\alpha(n)$. For all $A \in U(n)$ and $(z,t) \in \mathbb{H}_n$

$$\pi_{(\lambda,\alpha)}(A,z,t)\xi(w) = \sum_{1\leq j\leq d_\lambda} \tau_\lambda(A)\phi_j^\lambda \otimes e^{i\alpha t - \frac{\alpha}{4}|z|^2 - \frac{\alpha}{2}\langle w,z\rangle} f_j(A^{-1}w + A^{-1}z) \quad \text{if } \alpha > 0 \tag{3.26}$$

and

$$\pi_{(\lambda,\alpha)}(A,z,t)\xi(\overline{w}) = \sum_{1\leq j\leq d_\lambda} \tau_\lambda(A)\phi_j^\lambda \otimes e^{i\alpha t + \frac{\alpha}{4}|z|^2 + \frac{\alpha}{2}\langle \overline{w},\overline{z}\rangle} f_j(\overline{A^{-1}w} + \overline{A^{-1}z}) \quad \text{if } \alpha < 0 \tag{3.27}$$

It follows that

$$C_\xi^{\pi_{(\lambda,\alpha)}}(A,z,t) = \langle \pi_{(\lambda,\alpha)}(A,z,t)\xi, \xi\rangle_{\mathcal{H}_{(\lambda,\alpha)}} \tag{3.28}$$

$$= \begin{cases} \sum_{1\leq j,j'\leq d_\lambda} \langle \tau_\lambda(A)\phi_j^\lambda, \phi_{j'}^\lambda\rangle \int_{\mathbb{C}^n} e^{i\alpha t - \frac{\alpha}{4}|z|^2 - \frac{\alpha}{2}\langle w,z\rangle} f_j(A^{-1}w + A^{-1}z)\overline{f_{j'}(w)} e^{-\frac{\alpha}{2}|w|} dw & \text{if } \alpha > 0, \\ \sum_{1\leq j,j'\leq d_\lambda} \langle \tau_\lambda(A)\phi_j^\lambda, \phi_{j'}^\lambda\rangle \int_{\mathbb{C}^n} e^{i\alpha t + \frac{\alpha}{4}|z|^2 + \frac{\alpha}{2}\langle \overline{w},\overline{z}\rangle} f_j(\overline{A^{-1}w} + \overline{A^{-1}z})\overline{f_{j'}(\overline{w})} e^{\frac{\alpha}{2}|w|} dw & \text{if } \alpha < 0. \end{cases}$$

Lemma 3.5.4. *For each irreducible representation $\pi_{(\lambda,\alpha)}$ ($\alpha \in \mathbb{R}^*$, $\tau_\lambda \in \widehat{U(n)}$) of G_n, we have*

$$d\pi_{(\lambda,\alpha)}(T) = i\alpha\mathbb{I}.$$

Démonstration. Let $\xi = \sum_{1\leq j\leq d_\lambda} \phi_j^\lambda \otimes f_j$ be a unit vector in $\mathcal{H}_{(\lambda,\alpha)}$. Then

$$d\pi_{(\lambda,\alpha)}(T)\xi, \xi\rangle = \frac{d}{dt}\Big|_{t=0} \langle \pi_{(\lambda,\alpha)}(\mathbb{I},0,t)\xi,\xi\rangle = \frac{d}{dt}\Big|_{t=0} e^{i\alpha t} \sum_{1\leq j\leq d_\lambda} \|f_j\|^2 = i\alpha.$$

□

Given $\mathcal{R}_\alpha = \{h_{m,\alpha};\ m = (m_1,\ldots,m_n) \in \mathbb{N}^n\}$ be the orthonormal basis of the Fock space $F_\alpha(n)$ defined by the Hermite functions

$$h_{m,\alpha}(z) = \Big(\frac{|\alpha|}{2\pi}\Big)^{\frac{n}{2}} \sqrt{\frac{|\alpha|^{|m|}}{2^{|m|}m!}}\, z^m$$

with $|m| = m_1 + \cdots + m_n$, $m! = m_1! \cdots m_n!$ and $z^m = z_1^{m_1} \cdots z_n^{m_n}$(cf. [Fo]).

Theorem 3.5.5. *Let $\alpha \in \mathbb{R}^*$ and $\tau_\lambda \in \widehat{U(n)}$. Then a sequence $(\pi_{(\lambda^k,\alpha_k)})_k$ of elements in $\widehat{G_n}$ converges to $\pi_{(\lambda,\alpha)}$ in $\hat{G}_n$ if and only if $\lim_{k\to\infty} \alpha_k = \alpha$ and $\lambda^k = \lambda$ for large k.*

Démonstration. We consider first the case when α is positive. Assume that α_k tends to the real α and that $\lambda^k = \lambda$ for k large enough. Let $f \in C_c^\infty(G_n)$ and ξ be a unit vector in $\mathcal{H}_\lambda$. Then

$$\begin{aligned}\langle C^{\pi(\lambda_k,\alpha_k)}_{\xi\otimes h_{0,\alpha_k}}, f\rangle &= \int_{U(n)}\int_{\mathbb{H}_n} f(A,z,t)\langle\tau_\lambda(A)\xi,\xi\rangle e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \times \\ & \int_{\mathbb{C}^n}\left(\frac{1}{2\pi}\right)^n e^{-\frac{1}{2}\sqrt{\alpha_k}\langle w,z\rangle - \frac{1}{2}|w|^2} dwdAdzdt\end{aligned}$$

tends to $\langle C^{\pi(\lambda,\alpha)}_{\xi\otimes h_{0,\alpha}}, f\rangle$. Hence $(\pi_{(\lambda_k,\alpha_k)})_k$ converges to $\pi_{(\lambda,\alpha)}$. The same reasoning applies when α is negative.

Conversely, the fact that the sequence $(\pi_{(\lambda^k,\alpha_k)})_k$ converges to $\pi_{(\lambda,\alpha)}$ implies by Corollary 3.4.5 that for a unit vector $\xi \in \mathcal{H}^\infty_{(\lambda,\alpha)}$, there is $\xi_k \in \mathcal{H}^\infty_{(\lambda^k,\alpha_k)}$ of length 1 such that $(\langle d\pi_{(\lambda^k,\alpha_k)}(T)\xi_k,\xi_k\rangle)_k$ converges to $\langle d\pi_{(\lambda,\alpha)}(T)\xi,\xi\rangle$. Thus, by Lemma 3.5.4 α_k tends to α. It remains for us to show that $\lambda^k = \lambda$ for k large enough.

Let ξ a unit vector in $\mathcal{H}_\lambda$. Hence, by theorem 3.4.3, there exist $\xi_k = \sum_{m\in\mathbb{N}^n}\zeta^k_m \otimes h_{m,\alpha_k} \in \mathcal{H}_{(\lambda^k,\alpha_k)}$ of length 1 such that $(C^{\pi(\lambda^k,\alpha_k)}_{\xi_k})_k$ converges uniformly on compacta to $C^{\pi(\lambda,\alpha)}_{\xi\otimes h_{0,\alpha}}$.

Take now a positive real δ such that $0 \notin I_{\alpha,\delta} =]\alpha-\delta,\alpha+\delta[$, and given a Schwartz function φ on $\mathbb{R}$ with $\varphi_{|I_{\alpha,\delta}} \equiv 1$ and $\varphi \equiv 0$ at neighbourhood of zero. Then, it is easy to see that there is Schwartz function ψ on $\mathbb{H}_n$ verifying

$$\sigma_\beta(\psi) = \varphi(\beta)P_\beta \text{ for all } \beta \in \mathbb{R}^*,$$

where $P_\beta : \mathcal{F}_\beta(n) \longrightarrow \mathbb{C}$ is the orthogonal projection onto the one dimensional subspace $\mathbb{C}.h_{0,\beta}$ of all constant functions in $\mathcal{F}_\beta(n)$. On the other hand, there exists $k_\delta \in \mathbb{N}$ such that $\alpha_k \in I_{\alpha,\delta}$ for all $k \geq k_\delta$. We obtain $\sigma_\alpha(\psi)h_{0,\alpha} = h_{0,\alpha}$ and for all $k \geq k_\delta$ $\sigma_{\alpha_k}(\psi)h_{0,\alpha_k} = h_{0,\alpha_k}$. It follows that

$$\begin{aligned}\lim_{k\to\infty}\|\zeta^k_0\|^2 &= \lim_{k\to\infty}\sum_{m,m'\in\mathbb{N}^n}\langle\zeta^k_m,\zeta^k_{m'}\rangle\langle\sigma_{\alpha_k}(\psi)h_{m,\alpha_k},h_{m',\alpha_k}\rangle \\ &= \lim_{k\to\infty}\langle C^{\pi(\lambda^k,\alpha_k)}_{\sum_{m\in\mathbb{N}^n}\zeta^k_m\otimes h_{m,\alpha_k}}(\mathbb{I},.,.),\psi\rangle \\ &= \langle\sigma_\alpha(\psi)h_{0,\alpha},h_{0,\alpha}\rangle = 1.\end{aligned}$$

Then we get

$$\lim_{k\to\infty}\|\xi_k - \zeta^k_0\otimes h_{0,\alpha_k}\| = 0. \tag{3.29}$$

We deduce that

$$\lim_{k\to\infty}\langle\tau_{\lambda^k}(A)\zeta^k_0,\zeta^k_0\rangle = \langle\tau_\lambda(A)\xi,\xi\rangle \tag{3.30}$$

uniformly in $A \in U(n)$. Therefore, we just take

$$\phi_k = \frac{\zeta_0^k}{\|\zeta_0^k\|}$$

as a unit vector in $\mathcal{H}_{\lambda^k}$ $(k \in \mathbb{N})$ to obtain finally the uniform convergence on compacta of $C_{\phi_k}^{\tau_{\lambda^k}}$ to $C_{\xi}^{\tau_\lambda}$. Whence $\lambda^k = \lambda$ for large k. □

Lemma 3.5.6. *For each irreducible representation $\pi_{(\lambda,\alpha)}$ ($\alpha \in \mathbb{R}^*$, $\tau_\lambda \in \widehat{U(n)}$) of G_n, we have*

$$\langle d\pi_{(\lambda,\alpha)}(\mathcal{L})h_{m,\alpha}, h_{m,\alpha}\rangle = -|\alpha|(n+2|m|) \text{ for each } m \in \mathbb{N}^n.$$

The proof follows from Proposition 3.20 in [BJR] together with Lemma 3.4 in [BJRW].

Theorem 3.5.7. *Let $\lambda \in P_n$, $\mu \in P_{n-1}$ and $r > 0$.*

1) If a sequence $(\pi_{(\lambda^k,\alpha_k)})_{k\in\mathbb{N}}$ of elements of $\hat{G}_n$ converges to the representation $\pi_{(\mu,r)}$ in $\hat{G}_n$, then $\lim_{k\to\infty} \alpha_k = 0$ and the sequence $(\pi_{(\lambda^k,\alpha_k)})_{k\in\mathbb{N}}$ satisfies one of the following conditions

i) for k large enough, $\alpha_k > 0$, $\lambda_j^k = \mu_j$ for all $1 \le j \le n-1$ and $\lim_{k\to\infty} \alpha_k\lambda_n^k = -\frac{r^2}{2}$,

ii) for k large enough, $\alpha_k < 0$, $\lambda_j^k = \mu_{j-1}$ for all $2 \le j \le n$ and $\lim_{k\to\infty} \alpha_k\lambda_1^k = -\frac{r^2}{2}$.

2) If a sequence $(\pi_{(\lambda^k,\alpha_k)})_{k\in\mathbb{N}}$ of elements of $\hat{G}_n$ converges to the representation τ_λ in $\hat{G}_n$, then $\lim_{k\to\infty} \alpha_k = 0$ and the sequence $(\pi_{(\lambda^k,\alpha_k)})_{k\in\mathbb{N}}$ satisfies one of the following conditions

i) $\lim_{k\to\infty} \alpha_k\lambda_n^k = 0$, $\alpha_k > 0$ and $\lambda_1 \ge \lambda_1^k \ge \cdots \ge \lambda_{n-1} \ge \lambda_{n-1}^k \ge \lambda_n \ge \lambda_n^k$ (for k large enough),

ii) $\lim_{k\to\infty} \alpha_k\lambda_1^k = 0$, $\alpha_k < 0$ and $\lambda_1^k \ge \lambda_1 \ge \lambda_2^k \ge \lambda_2 \ge \cdots \ge \lambda_{n-1} \ge \lambda_n^k \ge \lambda_n$ (for k large enough).

Démonstration. Throughout this proof we take α_k positive for large k. The same reasoning applies when α is negative.

1) Let $\tilde{\mu}^s = (\mu_1, \ldots, \mu_s, \mu_s, \mu_{s+1}, \ldots, \mu_{n-1})$, $1 \le s \le n-1$. Then, for each s, the set $I(\tilde{\mu}^s, \mu)$ consists of one point, since $\tilde{\mu}^s$ is dominant integral and we can take $I(\tilde{\mu}^s, \mu) = \{1\}$. By hypothesis the sequence $\pi_{(\lambda^k,\alpha_k)}$ which converges to the representation $\pi_{(\mu,r)}$ in $\hat{G}_n$, hence by Corollary 3.4.5, for $\xi^s =$

$\sqrt{d_{\tilde{\mu}^s}}\overline{C^{\tilde{\mu}^s}_{\phi_1^{\tilde{\mu}^s},\phi_1^{\tilde{\mu}^s}}} \in \mathcal{H}^\infty_{(\mu,r)}$, there is a sequence of unit vectors $\xi_k = \sum_{m\in\mathbb{N}^n} \zeta^k_m \otimes h_{m,\alpha_k} \in \mathcal{H}^\infty_{(\lambda^k,\alpha_k)}$ such that

$$\langle d\pi_{(\lambda^k,\alpha_k)}(T)\xi_k, \xi_k\rangle \longrightarrow \langle d\pi_{(\mu,r)}(T)(\xi^s), \xi^s\rangle = 0,$$

$$\langle d\pi_{(\lambda^k,\alpha_k)}(\mathcal{L})\xi_k, \xi_k\rangle \longrightarrow \langle d\pi_{(\mu,r)}(\mathcal{L})(\xi^s), \xi^s\rangle = -r^2$$

and

$$\langle d\pi_{(\lambda^k,\alpha_k)}(\mathbb{T})\xi_k, \xi_k\rangle \longrightarrow \langle d\pi_{(\mu,r)}(\mathbb{T})(\xi^s), \xi^s\rangle,$$

for $\mathbb{T} \in \mathfrak{t}_n$. It follows that $\lim\limits_{k\to\infty} \alpha_k = 0$,

$$2\alpha_k \sum_{m\in\mathbb{N}^n} |m|\|\zeta^k_m\|^2 \longrightarrow r^2$$

and

$$\sum_{j=1}^n \lambda^k_j + \sum_{m\in\mathbb{N}^n} |m|\|\zeta^k_m\|^2 \longrightarrow \mu_s + \sum_{j=1}^{n-1} \mu_j.$$

This shows that

$$\lim_{k\to\infty} \sum_{j=1}^n \lambda^k_j = +\infty.$$

On the other hand we can say that $\langle \tau_{\lambda^k}\otimes W_{\alpha_k}(A)\xi_k, \xi_k\rangle$ converges to $C^{\pi(\mu,r)}_\xi(A,0,0) = C^{\tilde{\mu}^s}_{\phi_1^{\tilde{\mu}^s},\phi_1^{\tilde{\mu}^s}}(A)$ uniformly in each $A \in U(n)$. Hence

$$\lim_{k\to\infty} \int_{U(n)} \langle \tau_{\lambda^k} \otimes W_{\alpha_k}(A)\xi_k, \xi_k\rangle \overline{\langle \tau_{\tilde{\mu}^s}(A)\phi_1^{\tilde{\mu}^s}, \phi_1^{\tilde{\mu}^s}\rangle} = \frac{1}{d_{\tilde{\mu}^s}} \neq 0.$$

By the classical Pieri's rule (see [Fu-Ha]), the representation $\tau_{\lambda^k} \otimes W_{\alpha_k}$ is decomposed as follows

$$\tau_{\lambda^k} \otimes W_{\alpha_k} = \sum_{\substack{\lambda'\in P_n \\ \lambda'_1\geq\lambda^k_1\geq\cdots\geq\lambda'_n\geq\lambda^k_n}} \tau_{\lambda'} \tag{3.31}$$

Then we have $\mu_1 \geq \lambda^k_1 \geq ... \geq \mu_s = \lambda^k_s \geq ... \geq \mu_{n-1} \geq \lambda^k_n$ for k large enough. This is true for all $1 \leq s \leq n-1$. Thus $\lim\limits_{k\to\infty} \alpha_k\lambda^k_n = -\frac{r^2}{2}$ and $\lambda^k_j = \mu_j$ for

$j = 1, \cdots, n-1$.

2) The fact that the sequence $(\pi_{(\lambda^k,\alpha_k)})_k$ converges to τ_λ implies that there is $\xi_k = \sum_{m\in\mathbb{N}^n} \zeta_m^k \otimes h_{m,\alpha_k} \in \mathcal{H}^\infty_{(\lambda^k,\alpha_k)}$ of length 1 such that $(\langle d\pi_{(\lambda^k,\alpha_k)}(T)\xi_k, \xi_k\rangle)_k$ converges to $\langle d\tau_\lambda(T)\phi_1^\lambda, \phi_1^\lambda\rangle$. Thus, by Lemma 3.5.4 α_k tends to zero.
We remark now that

$$\langle d\pi_{(\lambda^k,\alpha_k)}(\mathcal{L})\xi_k, \xi_k\rangle \longrightarrow \langle d\tau_\lambda(\mathcal{L})\phi_1^\lambda, \phi_1^\lambda\rangle = 0$$

and

$$\langle d\pi_{(\lambda^k,\alpha_k)}(\mathbb{T})\xi_k, \xi_k\rangle \longrightarrow \langle d\tau_\lambda(\mathbb{T})\phi_1^\lambda, \phi_1^\lambda\rangle,$$

for $\mathbb{T} \in \mathfrak{t}_n$. It follows that

$$2\alpha_k \sum_{m\in\mathbb{N}^n} |m| \|\zeta_m^k\|^2 \longrightarrow 0$$

and

$$\sum_{j=1}^n \lambda_j^k + \sum_{m\in\mathbb{N}^n} |m| \|\zeta_m^k\|^2 \longrightarrow \sum_{j=1}^n \lambda_j.$$

Then

$$\lim_{k\to\infty} \alpha_k \sum_{j=1}^n \lambda_j^k = 0. \tag{3.32}$$

On the other hand, we have that $\langle \tau_{\lambda^k} \otimes W_{\alpha_k}(A)\xi_k, \xi_k\rangle$ converges to $C^\lambda_{\phi_1^\lambda,\phi_1^\lambda}(A)$ uniformly in each $A \in U(n)$. Hence

$$\lim_{k\to\infty} \int_{U(n)} \langle \tau_{\lambda^k} \otimes W_{\alpha_k}(A)\xi_k, \xi_k\rangle \overline{\langle \tau_\lambda(A)\phi_1^\lambda, \phi_1^\lambda\rangle} = \frac{1}{d_\lambda} \neq 0.$$

By formula 3.31, we get $\lambda_1 \geq \lambda_1^k \geq .... \geq \lambda_n \geq \lambda_n^k$ for large k and thus by equation (3.32) $\lim\limits_{k\to\infty} \alpha_k \lambda_n^k = 0$. □

The arguments above show that

Theorem 3.5.8. *The mapping*

$$\begin{array}{ccc} \hat{G}_n & \longrightarrow & \mathfrak{g}_n^\ddagger/G_n \\ \pi_\ell & \mapsto & \mathcal{O}_\ell \end{array}$$

is continuous.

Theorem 3.5.9. *The dual space of the semi-direct product $U(1) \ltimes \mathbb{H}_1$ is homeomorphic to its admissible co-adjoint orbit space.*

Démonstration. Assume that α_k tends to zero and that $\lim\limits_{k\to\infty} \lambda_k\alpha_k = -\frac{r^2}{2}$. If α_k is positive (resp. negative) for k large enough, we can take the sequence $(f_k)_{k\in\mathbb{N}}$ of elements in the Fock space $\mathcal{F}_{\alpha_k}(1)$ defined by $f_k(w) = c_{\alpha_k,\lambda_k} w^{-\lambda_k}$ (resp. $f_k(w) = c_{\alpha_k,\lambda_k} w^{\lambda_k}$) with $\|f_k\| = 1$. Then, for $f \in C_c^\infty(G_1)$ we have

$$\begin{aligned}
\langle C_{f_k}^{\pi(\lambda_k,\alpha_k)}, f\rangle &= \int_{G_1} f(\theta,z,t)\chi_{\lambda_k}(e^{i\theta})e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \int_{\mathbb{C}} |c_{\alpha_k,\lambda_k}|^2 e^{-\frac{\alpha_k}{2}\langle w,z\rangle} \times \\
&\qquad (e^{-i\theta}w + e^{-i\theta}z)^{-\lambda_k}\overline{w}^{-\lambda_k}e^{-\frac{\alpha_k}{2}|w|^2} dwd\theta dzdt \\
&= \int_{G_1} f(\theta,z,t)e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \int_{\mathbb{C}} |c_{\alpha_k,\lambda_k}|^2 e^{-\frac{\alpha_k}{2}\langle w,z\rangle}(w+z)^{-\lambda_k} \times \\
&\qquad \overline{w}^{-\lambda_k}e^{-\frac{\alpha_k}{2}|w|^2} dwd\theta dzdt \\
&= \int_{G_1} f(\theta,z,t)e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \sum_{j=0}^{\infty}\sum_{l=0}^{-\lambda_k} \Big(\int_{\mathbb{C}} |c_{\alpha_k,\lambda_k}|^2 \frac{C_{-\lambda_k}^l}{j!}(\frac{\alpha_k}{2})^j \times \\
&\qquad w^{j+l}\overline{w}^{-\lambda_k}(-\overline{z})^j z^{(-\lambda_k-l)}e^{-\frac{\alpha_k}{2}|w|^2}dw\Big) d\theta dzdt \\
&= \int_{G_1} f(\theta,z,t)e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2}\Big(\sum_{j=0}^{-\lambda_k}\frac{(-\lambda_k)!}{(-\lambda_k-j)!(j!)^2}(\frac{\alpha_k}{2})^j(-|z|^2)^j\Big)d\theta dzdt.
\end{aligned}$$

Since the sequence

$$(\frac{\alpha_k}{2})^j\frac{(-\lambda_k)!}{(-\lambda_k-j)!} = (\frac{-\lambda_k\alpha_k}{2})^j(1+\frac{1}{\lambda_k})(1+\frac{2}{\lambda_k})\cdots(1+\frac{j-1}{\lambda_k})$$

converges to $(\frac{r^2}{4})^j$, we have

$$\begin{aligned}
\lim_{k\to\infty}\langle C_{f_k}^{\pi(\lambda_k,\alpha_k)}, f\rangle &= \int_{G_1} f(\theta,z,t)\underbrace{\Big(\sum_{j=0}^{\infty}\frac{(\frac{-r^2|z|^2}{4})^j}{(j!)^2}\Big)}_{\text{Bessel function}} d\theta dzdt \\
&= \int_{G_1} f(\theta,z,t)\frac{1}{2\pi}\int_0^{2\pi} e^{-ir\,\mathrm{Re}\,(e^{i\beta}\overline{z})}d\beta d\theta dzdt \\
&= \int_{G_1} f(\theta,z,t)\langle(\mathrm{ind}_{\mathbb{H}_1}^{G_1}\chi_r)(\theta,z,t)(1),1\rangle.
\end{aligned}$$

Hence $(\pi_{(\lambda_k,\alpha_k)})_k$ converges to the irreducible unitary representation $\pi_r := \mathrm{ind}_{\mathbb{H}_1}^{G_1}\chi_r$.

Assume now that $\lim_{k\to\infty} \lambda_k \alpha_k = 0$. For $\lambda \geq \lambda_k$ (resp. for $\lambda \leq \lambda_k$), k is large enough, we define the sequence $(f_k)_k$ by $f_k(w) = c_{\alpha_k,\lambda_k,\lambda} w^{\lambda-\lambda_k}$ (resp. $f_k(w) = c_{\alpha_k,\lambda_k,\lambda} w^{\lambda_k-\lambda}$) with $\|f_k\| = 1$. Then by the same computation as above we get

$$\lim_{k\to\infty} \langle C_{f_k}^{\pi(\lambda_k,\alpha_k)}, f\rangle = \int_{G_1} f(\theta, z, t)\chi_\lambda(\theta) d\theta dz dt = \langle \chi_\lambda, f\rangle.$$

Hence $(\pi_{(\lambda_k,\alpha_k)})_k$ converges to the character χ_λ of $U(1)$. □

Conjecture. *The dual space of the group $G_n = U(n) \ltimes \mathbb{H}_n$ is homeomorphic with its space of admissible coadjoint orbits $\mathfrak{g}_n^\ddagger / G_n$.*

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Chapitre 4

Flat orbits and kernels of irreducible representations of the group algebra of a completely solvable Lie group

Résumé : Dans ce chapitre on prouve que le noyau d'une représentation unitaire irréductible π de l'algèbre involutive $L^1(G)$ d'un groupe complètement résoluble est déterminé par les fonctions dont la transformée de Fourier s'annule sur l'orbite coadjointe $\mathcal{O}_\pi$ associé à π si et seulement si $\mathcal{O}_\pi$ est plate.

Abstract : We show that the kernel of an irreducible unitary representation π of the group algebra $L^1(G)$ of a completely solvable Lie group G is given by the functions, whose abelian Fourier transform vanish on the Kirillov orbit $\mathcal{O}_\pi$ of π if and only if this orbit $\mathcal{O}_\pi$ is flat. This is a generalization of a result obtained before for nilpotent Lie groups.

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Keywords : completely solvable Lie groups, flat orbits, group algebras, kernels of induced representations.

4.1 Introduction

Let $G = \exp(\mathfrak{g})$ be a connected simply connected exponential Lie group with Lie algebra $\mathfrak{g}$. The unitary dual $\hat{G}$ of G, the set of equivalence classes of irreducible unitary representations of G, has a nice geometric parametrization via the Kirillov orbit method. It is known that there is a one to one correspondence $\pi \mapsto \mathcal{O}_\pi$ between the equivalence classes of irreducible representations π of G and the co-adjoint orbits $\mathcal{O}_\pi$ in $\mathfrak{g}^*$, the dual vector space of $\mathfrak{g}$. Furthermore, every unitary representation π of $C^*(G)$, the C^*-algebra of G, is uniquely determined by its kernel $ker(\pi)$. We can therefore expect a description of the kernel in $C^*(G)$ of an irreducible unitary representation in terms of the corresponding co-adjoint orbit.

If now $G = \exp(\mathfrak{g})$ is a connected simply connected nilpotent Lie group, then the mapping $L^1(G) \to L^1(\mathfrak{g}), f \mapsto f \circ \exp$ is an isometry and J. Ludwig in [Lud] has shown that the kernel $\ker(\pi)$ in the group algebra $L^1(G)$ is given by the subspace $\ker(\pi) := \{f \in L^1(G);\ \widehat{f \circ exp} = 0 \text{ on } \mathcal{O}_\pi\}$ if and only if the orbit $\mathcal{O}_\pi$ of π is flat, i.e. an affine linear subset of $\mathfrak{g}^*$. Nilpotent Lie groups are $*$-regular, i.e. the canonical mapping from the primitive ideal space $Prim(C^*(G))$ to the $*$-primitive ideal space $Prim_*(L^1(G)) : I \to I \cap L^1(G)$ is a homeomorphism. In particular the kernel $ker_{C^*(G)}(\pi)$ of π in the C^*-algebra of G is given by the closure $\overline{ker_{L^1(G)}(\pi)}$ in $C^*(G)$ of the kernel $ker_{L^1(G)}(\pi)$ in $L^1(G)$. This shows that in general a "nice" description of the kernel $ker(\pi)$ in $C^*(G)$ in terms of its Kirillov orbit is not available, if the orbit is not flat (see [Lud1]).

In the exponential case, the group G is no longer $*$-regular in general (see [Boi]), but it may still be that $ker_{C^*(G)}(\pi) = \overline{ker_{L^1(G)}(\pi)}$ for every $\pi \in \widehat{G}$ (see [Ung]). In this paper, we extend the result obtained for nilpotent Lie groups in [Lud] to the completely solvable ones and show the following. Let $\pi \in \hat{G}$ such that the co-adjoint orbit $\mathcal{O}_\pi$ of π is closed. Then $\mathcal{O}_\pi$ is flat if and only if $ker(\pi) = \{f \in L^1(G) : [(f \circ \exp).j_\mathfrak{g}\widehat{]}(\mathcal{O}_\pi) = \{0\}\}$, where exp is the exponential mapping of G and $j_\mathfrak{g}dx$ denotes the pull-back of the Haar measure of the group G to the Lie algebra $\mathfrak{g}$ via the exponential mapping.

The paper contains three sections. In the first, we give the necessary definitions and properties of completely solvable Lie groups and of induced representations. In the second section we present several characterizations of flat co-adjoint orbits of completely solvable Lie groups. In the last section, we determine the kernels in the group algebra of the irreducible representations associated to flat orbits.

4.2 Preliminaries

4.2.1 Some Notations and Basic Facts

A connected, simply connected solvable Lie group with Lie algebra $\mathfrak{g}$ is called *exponential* if the exponential mapping $\exp : \mathfrak{g} \to G$ is a C^∞ diffeomorphism. In this case we denote by log the inverse mapping of exp. It is well-known that G is exponential if and only if for every $X \in \mathfrak{g}$, the spectrum of the endomorphism $\mathrm{a}d(X) : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}, \mathrm{a}d(X)U := [X,U]$, does not contain any number of the form λi with $\lambda \in \mathbb{R}^*$. If in particular the spectrum of $\mathrm{a}d(X)$ is real for every $X \in \mathfrak{g}$, then we say that G is *completely solvable*. In this case, there exists a Jordan-Hölder sequence $\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_n \supset \mathfrak{g}_{n+1} = \{0\}$ of ideals in $\mathfrak{g}$, such that the dimension of $\mathfrak{g}_j/\mathfrak{g}_{j+1} = 1$ for every $j = 1, \cdots, n$. Choosing for every j an element $Z_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j+1}$, we obtain a Jordan-Hölder basis $\mathcal{Z} = \{Z_1, \cdots, Z_n\}$ of $\mathfrak{g}$ and for every $X \in \mathfrak{g}$, we have a real number $\rho_j(X)$, such that $[X, Z_j] = \rho_j(X)Z_j$ modulo $\mathfrak{g}_{j+1}, j = 1, \cdots, n$. The linear functionals $\rho_j : \mathfrak{g} \to \mathbb{R}$ are called the *roots* of $\mathfrak{g}$. If $\mathfrak{g}$ is nilpotent then of course all the roots are 0.
Since for exponential solvable groups G the exponential mapping is a diffeomorphism, we can transfer the multiplication in G to the vector space $\mathfrak{g}$ and we obtain the so-called Campbell-Baker-Hausdorff multiplication $\cdot_\mathfrak{g}$ on $\mathfrak{g}$:

$$X \cdot_\mathfrak{g} Y = \log(\exp X \cdot \exp Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \cdots, X, Y \in \mathfrak{g}.$$

Let dg denote a left Haar measure on G. The pull-back $\exp_*(dg)$ of the measure dg is the measure $j_\mathfrak{g}(X)dX$ on $\mathfrak{g}$, where $j_\mathfrak{g}(X)$ is the Jacobian of the left translation by X on $\mathfrak{g}$:

$$j_\mathfrak{g}(X) = \left|\det\Big(\frac{1 - e^{-\mathrm{a}d_\mathfrak{g}(X)}}{\mathrm{a}d_\mathfrak{g}(X)}\Big)\right|$$

(see [Wal]). The group G acts on $\mathfrak{g}$ by the adjoint representation $\mathrm{A}d_G$, i.e.,

$$\mathrm{A}d_G(g)(X) = \mathrm{A}d(g)(X) = e^{\mathrm{ad}(\log(g))}X,\ g \in G, X \in \mathfrak{g},$$

and on $\mathfrak{g}^*$ by the co-adjoint representation $\mathrm{A}d_G^*$, i.e.,

$$< \mathrm{A}d_G^*(g)l, X >:=< l, \mathrm{A}d_G(g^{-1})(X) >=:< g.l, X >,\ g \in G, l \in \mathfrak{g}^*, X \in \mathfrak{g}.$$

We denote by $\mathfrak{g}^*/G$ the space of the co-adjoint G-orbits $\mathcal{O}(l) = \{g.l : g \in G\}$, $l \in \mathfrak{g}^*$. Let $\mathfrak{g}(l) = \{X \in \mathfrak{g} :< l, [X, \mathfrak{g}] >= \{0\}\}$ be the stabilizer of $l \in \mathfrak{g}^*$ in $\mathfrak{g}$. It is also the Lie algebra of $G(l) = \{g \in G : g.l = l\}$. A co-adjoint orbit $\mathcal{O}(l)$ of $l \in \mathfrak{g}^*$ is said to be saturated with respect to an ideal $\mathfrak{g}_0$ of $\mathfrak{g}$, if $\mathcal{O}(l) = \mathcal{O}(l) + \mathfrak{g}_0^{\perp}$. In this case we have that $\mathfrak{g}(l) \subset \mathfrak{g}_0$. So we can say, in the case where $\mathfrak{g}_0$ is of codimension 1 in $\mathfrak{g}$, that

$$\dim(\mathcal{O}(l_0)) = \dim(\mathcal{O}(l)) - 2,\ l_0 = l_{|\mathfrak{g}_0},\ \mathcal{O}(l_0) = \exp(\mathfrak{g}_0) \cdot l_0.$$

Let dg be a left Haar measure on G and let Δ_G be the modular function of G, which is defined by the formula

$$\int_G \xi(gx^{-1})dg = \Delta_G(x) \int_G \xi(g)dg,$$

for all $x \in G$ and for every ξ belonging to the space $C_c(G)$ of continuous functions on G with compact support. We have thus that

$$\Delta_G(x) = |\det(\mathrm{A}d(x))|^{-1} = \exp(-\mathrm{t}r\ \mathrm{a}d(\log x))\ (x \in G).$$

Let H be a closed connected subgroup of G with corresponding Lie algebra $\mathfrak{h}$. We denote by $\Delta_{H,G}$ the real character of H defined by

$$\Delta_{H,G}(h) = \frac{\Delta_H(h)}{\Delta_G(h)}\ (h \in H).$$

Hence, we have

$$\Delta_{H,G}(h) = \exp(\mathrm{t}r\ \mathrm{a}d_{\mathfrak{g}/\mathfrak{h}}(\log h))\ (h \in H).$$

It is well known that if H is a normal subgroup of G, then $\Delta_{H,G}(h) = 1$ for all $h \in H$.

4.2.2 Induced Representation

We consider the space

$$\begin{aligned}\mathcal{E}(G,H) \;\; = \;\; &\{\xi : G \to \mathbb{C},\ \text{continuous with compact support modulo H},\\ &\quad \text{such that } \xi(gh) = \Delta_{H,G}(h)\xi(h),\ g \in G, h \in H\}.\end{aligned}$$

The group G acts on $\mathcal{E}(G,H)$ by left translation and there exists a unique (up to a positive multiple) positive G-invariant linear functional on this space, which is denoted by $\nu_{G/H}$. Therefore, we can write it in the form of an integral

$$\nu_{G/H}(\xi) = \oint_{G/H} \xi(g) d\nu_{G/H}(g).$$

We remark that if $\Delta_{H,G} = 1$, then $\nu_{G/H}$ is simply a G-invariant measure on the homogeneous space G/H and the space $\mathcal{E}(G,H)$ coincides with $C_c(G/H)$. We can write then the Haar integral on G as a double integral over H and the quotient space G/H :

$$\int_G f(g)dg = \oint_{G/H} \left(\int_H f(gh)\Delta_{H,G}(h)^{-1}dh \right) d\nu_{G/H}(g), f \in C_c(G). \quad (4.1)$$

For details see [Ber-Con].

Let ρ be a unitary representation of H on the Hilbert space $\mathcal{H}_\rho$ and let $C_c(G/H,\rho)$ be the space of continuous functions $\xi : G \to \mathcal{H}_\rho$, which are compactly supported modulo H satisfying

$$\xi(gh) = \Delta_{H,G}(h)^{\frac{1}{2}}\rho(h^{-1})\xi(g), \ h \in H, g \in G.$$

We define an L^2-norm on $C_c(G/H,\rho)$ as follows

$$\|\xi\|_2^2 = \oint_{G/H} \|\xi(g)\|^2_{\mathcal{H}_\rho} d\nu_{G/H}(g).$$

The induced representation $\mathrm{ind}_H^G\rho$ is just the left regular representation of G on the completion $L^2(G/H,\rho)$ of $C_c(G/H,\rho)$ with respect to the norm $\|.\|_2$ defined above.

4.2.3 The Kernel of Induced Representations

The unitary dual $\hat{G}$, i.e., the space of equivalence classes $[\pi]$ of all irreducible unitary representations π of G has been described via the Kirillov-Bernat-Vergne orbit method (see [Lep-Lud]). Every unitary irreducible representation of G is equivalent to an induced representation $\pi_{l,\mathfrak{p}_l} = \mathrm{ind}_{P_l}^G\chi_l$ for some $l \in \mathfrak{g}^*$ and a Pukanszky polarization $\mathfrak{p}_l$ at l, where χ_l denotes the unitary character $\chi_l(\exp X) := e^{-il(X)}, X \in \mathfrak{p}_l$ of the closed connected subgroup $P_l := \exp(\mathfrak{p}_l)$. A *polarization* at $l \in \mathfrak{g}^*$ is by definition a subalgebra $\mathfrak{p}_l$ of $\mathfrak{g}$, such that $\langle l, [\mathfrak{p}_l,\mathfrak{p}_l]\rangle = \{0\}$ and such that $\dim(\mathfrak{p}_l) = \frac{1}{2}(\dim(\mathfrak{g}/\mathfrak{g}(l)) + \dim(\mathfrak{g}(l)))$. We say that $\mathfrak{p}_l$ or P_l satisfy *Pukanszky's condition* if $\mathrm{Ad}^*(P_l)l = l + \mathfrak{p}_l^\perp$. The representations $\pi_{l,\mathfrak{p}_l}$ and $\pi_{l',\mathfrak{p}_{l'}}$ are equivalent if and only if l and l' are in the same G-orbit $\mathcal{O}$ and so the mapping

$$\Theta : \mathfrak{g}^*/G \to \hat{G}, \ \mathcal{O}(l) \mapsto [\pi_{\mathcal{O}(l)}] := [\mathrm{ind}_{P_l}^G\chi_l]$$

is a bijection and even a homeomorphism (see [Lep-Lud]). We need the following Lemma (see [Lud] and [Boi] for the description of the kernels of such induced representations).

Lemma 4.2.1. *Let H be a closed subgroup of G. Let ρ be a unitary representation of H on the Hilbert space $\mathcal{H}_\rho$ and let $\pi = ind_H^G \rho$. Then $ker(\pi)$ is the set of all functions $f \in L^1(G)$ such that for all $x \in G$ there exists a set $\mathcal{N}$ of measure 0 in G so that for every $x \in G \setminus \mathcal{N}$, we have a set $\mathcal{N}_x$ of measure 0 in G, such that for all $y \notin \mathcal{N}_x$ the linear operator $f_\rho(x, y)$ defined on $\mathcal{H}_\rho$ by*

$$f_\rho(x,y) := \int_H \Delta_{H,G}(h)^{-\frac{1}{2}} f(xhy^{-1})\rho(h)dh$$

exsits and is 0.

Démonstration. Let $f \in L^1(G)$. Let first ρ_0 be the left regular representation of G on $L^2(G/H, 1)$. Choose a non-negative continuous function $\xi \in L^2(G/H, 1)$, which vanishes nowhere on G. Then there is a set $\mathcal{N}$ of measure 0 in G, such that

$$|f| * \xi(x) = \rho_0(|f|)\xi(x) < \infty, \ x \notin \mathcal{N}.$$

Hence for $x \notin \mathcal{N}$,

$$\begin{aligned}
\infty > |f| * \xi(x) &= \int_G |f(y)|\xi(y^{-1}x)dy = \int_G \Delta_G(y^{-1})|f(xy^{-1})|\xi(y)dy \\
&= \oint_{G/H} \int_H \Delta_{H,G}(h^{-1})\Delta_G(h^{-1}y^{-1})|f(xh^{-1}y^{-1})|\xi(yh)dhdy \\
&= \oint_{G/H} \int_H \Delta_{H,G}^{-1/2}(h)\Delta_G(h^{-1})\Delta_G(y^{-1})|f(xh^{-1}y^{-1})|\xi(y)dhdy \\
&= \oint_{G/H} \int_H \Delta_{H,G}^{1/2}(h)\Delta_G(h)\Delta_G(y^{-1})|f(xhy^{-1})|\Delta_H(h^{-1})\xi(y)dhdy \\
&= \oint_{G/H} \Big(\int_H \Delta_{H,G}(h)^{-\frac{1}{2}}\Delta_G(y^{-1})|f(xhy^{-1})|dh\Big)\xi(y)dy.
\end{aligned}$$

Therefore, by the theorem of Fubini, there exists for every $x \in G \setminus \mathcal{N}$ a set $\mathcal{M}_x \subset G$ of measure 0, such that

$$\int_H \Delta_{H,G}(h)^{-\frac{1}{2}}\Delta_G(y^{-1})|f(xhy^{-1})|dh < \infty$$

for every $y \notin \mathcal{M}_x$ and such that the function $y \to \int_H \Delta_{H,G}(h)^{-\frac{1}{2}}\Delta_G(y^{-1})|f(xhy^{-1})|dh$ is integrable. Whence for $x \notin \mathcal{N}$ and $\eta \in C_c(G/H, \rho)$,

$$\begin{aligned}(\pi(f)\eta)(x) &= \int_G f(y)\eta(y^{-1}x)dy = \int_G \Delta_G(y^{-1})f(xy^{-1})\eta(y)dy \\ &= \oint_{G/H}\int_H \Delta_{H,G}^{-1/2}(h)\Delta_G(h^{-1})\Delta_G(y^{-1})f(xh^{-1}y^{-1})\rho(h^{-1})\eta(y)dhdy \\ &= \oint_{G/H}\Big(\int_H \Delta_{H,G}(h)^{-\frac{1}{2}}\Delta_G(y^{-1})f(xhy^{-1})\rho(h)dh\Big)(\eta(y))dy.\end{aligned} \tag{4.2}$$

We deduce from (4.2) that $f \in \ker(\mathrm{ind}_H^G\rho)$ if and only if for every $x \in G \setminus \mathcal{N}$, there exists a set $\mathcal{N}_x \supset \mathcal{M}_x$ of measure 0 in G such that the linear operator

$$\int_H \Delta_{H,G}(h)^{-\frac{1}{2}}\Delta_G(y^{-1})f(xhy^{-1})\rho(h)dh$$

is 0 for every $y \notin \mathcal{N}_x$. □

4.3 Flat Orbits

In this section we characterize the flat orbits of a completely solvable Lie group of endomorphisms of a finite dimensional real vector space V. Let $\overline{\mathbb{D}} = \exp(\mathbb{D})$ be an exponential Lie group of linear endomophisms of V. We assume that $\overline{\mathbb{D}}$ is completely solvable. This means that the eigenvalues of every $D \in \mathbb{D}$, considered as an endomorphism of the complexification $V_{\mathbb{C}}$ of V, are real numbers. We denote by $< \mathbb{D} >$ the associative hull in the endomorphism ring of the vector space V generated by $\mathbb{D}$. Then the group $\overline{\mathbb{D}}$ is contained in the algebra $\mathbb{R}\mathbb{I}_V+ < \mathbb{D} >$. Note that $< \mathbb{D} >$ is linearly generated by the set $\{D^j \ : \ D \in \mathbb{D}, \ j \in \mathbb{N}\}$. For $l \in V^*$, we define :

$$N_{\mathbb{D}}(l) = \{x \in V \ : < l, D(x) >= 0, \ \forall D \in \mathbb{D}\},$$

$$\mathbb{D}(l) = \{D \in \mathbb{D} \ : \ D^t(l) = 0\},$$

$$A_{\mathbb{D}}(l) = \{x \in V \ : < l, T(x) >= 0, \ \forall T \in < \mathbb{D} > \}.$$

Here D^t denotes the transpose of $D : \langle D^t l, X\rangle := \langle l, D(X)\rangle, X \in V, l \in V^*$. It follows from the definitions, that $A_{\mathbb{D}}(l) \subset N_{\mathbb{D}}(l)$ and that

$$A_{\mathbb{D}}(l) = \{X \in N_{\mathbb{D}}(l) : \ T(X) \in N_{\mathbb{D}}(l) \ \forall T \in \mathbb{D}\}.$$

Definition 4.3.1. We say that an orbit $\mathcal{O}(l) = \overline{\mathbb{D}}^t l \subset V^*$ of the exponential completely solvable group $\overline{\mathbb{D}}$ is *flat*, if the subspace $N_{\mathbb{D}}(l)$ of V (and hence $N_{\mathbb{D}}(q)$ of every element $q \in \mathcal{O}(l)$) is $\mathbb{D}$-invariant.

Theorem 4.3.2. *Let $\overline{\mathbb{D}} = exp(\mathbb{D})$ be an exponential completely solvable Lie group of endomorphisms of the real finite dimensional vector space V. Let $l \in V^*$ and $\mathcal{O} = \mathcal{O}(l) = \overline{\mathbb{D}}^t l$ be the $\overline{\mathbb{D}}$-orbit of l. The following statements are equivalent :*
1) $\mathcal{O}$ is flat, i.e. $N_{\mathbb{D}}(l)$ is $\mathbb{D}$-invariant $\Leftrightarrow N_{\mathbb{D}}(l) = A_{\mathbb{D}}(l)$.
2) $\overline{\mathbb{D}}^t \cdot l_{|N_{\mathbb{D}}(l)} = l_{|N_{\mathbb{D}}(l)}$.
3) There exists an analytic function $P : \mathbb{R} \to \mathbb{R}; P(\xi) = 1+a_2\xi^2+a_3\xi^3+...$ for small ξ, with $a_2 \neq 0$, such that for every q in the orbit $\mathcal{O}(l)$, $P(D^t)q \in \mathcal{O}(l)$ for $D \in \mathbb{D}$ small enough.

Démonstration. 1) $\Rightarrow$ 2) Let $X \in N_{\mathbb{D}}(l) = A_{\mathbb{D}}(l)$. Since $D^j(X) \in N_{\mathbb{D}}(l)$ for every $j \in \mathbb{N}^*$, it follows that

$$\langle l, D^j(X)\rangle = 0, j \in \mathbb{N}^*, X \in \mathbb{N}_{\mathbb{D}}(l), D \in \mathbb{D},$$

and so $\langle \exp(D^t)l, X\rangle = \langle l, X\rangle$.
2) $\Rightarrow$ 1) Let $X \in N_{\mathbb{D}}(l)$. For all $D \in \mathbb{D}, s \in \mathbb{R}$, we have then that

$$\begin{aligned} < l, X > &= < \exp(sD^t)(l), X > \\ &= < \sum_{k\geq 0} \frac{(sD^t)^k}{k!}(l), X > \\ &= < (\mathbb{I}_V + sD^t + \frac{s^2}{2!}(D^t)^2 + ...)(l), X > \\ &= < l, X > +s < D^t(l), X > +\frac{s^2}{2!} < (D^t)^2(l), X > +... \end{aligned}$$

It follows that,

$$s < D^t(l), X > +\frac{s^2}{2!} < (D^t)^2(l), X > +... = 0$$

and therefore for all $j \geq 1$, $< (D^t)^j(l), X >= 0$. Hence, $\langle l, T(X)\rangle = 0$ for all $T \in < \mathbb{D} >$ and thus $N_{\mathbb{D}}(l) \subset A_{\mathbb{D}}(l)$, which completes the proof in this case.
3) $\Rightarrow$ 1) We proceed by induction on $d = \dim(V) + \dim(\mathbb{D})$. The result is obviously true if $d = 1$.
Let $d \geq 2$. We take $V_0 = \ker(l) \cap A_{\mathbb{D}}(l)$. We have to treat the following cases :
Case 1 : $V_0 \neq \{0\}$.
Let $p : V \longrightarrow \tilde{V} = V/V_0$ be the canonical projection and j the transposed map of p. Take $\tilde{l} \in \tilde{V}^*$ such that $j(\tilde{l}) = l$. We define the Lie algebra $\tilde{\mathbb{D}}$ by

$$\tilde{D}(p(x)) = p(D(x)), \quad D \in \mathbb{D} \text{ and } x \in V.$$

As $j\big(P(\tilde{D}^t)(\tilde{q})\big) = P(D^t)(q), q \in \mathcal{O}$ and D small enough, the induction hypothesis applied to $\tilde{V}$ and $\tilde{\mathbb{D}}$ implies that $N_{\tilde{\mathbb{D}}}(\tilde{l}) = A_{\tilde{\mathbb{D}}}(\tilde{l})$. Hence $N_{\mathbb{D}}(l) = p^{-1}(N_{\tilde{\mathbb{D}}}(\tilde{l}))$ is $\mathbb{D}$-invariant.
Case 2 : $V_0 = \{0\}$. This implies that $\dim(A_{\mathbb{D}}(l)) = 0$ or 1.
Subcase 2.1 : $A_{\mathbb{D}}(l) = \mathbb{R}Z$, for some $Z \in V \setminus \{0\}$, $\mathbb{D}(Z) = \{0\}$ and $l(Z) \neq 0$. In this case, there exists a non-zero vector $Y \in V$ and two linear functionals α, $\beta \neq 0$ from $\mathbb{D}$ to $\mathbb{R}$ such that

$$D(Y) = \alpha(D)Y + \beta(D)Z, \quad \forall D \in \mathbb{D},$$

since $\overline{\mathbb{D}}$ is completely solvable. It follows that α is a homomorphism of the Lie algebra $\mathbb{D}$. We can suppose that α and β are linearly independent, if $\alpha \neq 0$. Without loss of generality, we can assume that $l(Y) = 0$ and $l(Z) = 1$. Let $\mathbb{D}_0$ be the kernel of β. Then $\mathbb{D}_0$ is a subalgebra of G. Let $\overline{\mathbb{D}}_0 := \exp(\mathbb{D}_0)$. Then $\overline{\mathbb{D}}_0$ is a closed connected subgroup of $\overline{\mathbb{D}}$.
Assume first that $\alpha \neq 0$. The $\overline{\mathbb{D}}_0$-orbit $\mathcal{O}_0$ of l is given by :

$$\mathcal{O}_0 = \{q \in \mathcal{O}(l) \ : \ q(Y) = 0\}.$$

In fact, there exists $\dot{D} \in \mathbb{D}\backslash\mathbb{D}_0$ such that $\beta(\dot{D}) = 1$, $\alpha(\dot{D}) = 0$ and $\mathbb{D} = \mathbb{D}_0 \oplus \mathbb{R}\dot{D}$. Thus $\dot{D}(Y) = Z$ and $D\dot{D}(Y) = 0$, for all $D \in \mathbb{D}$. So we have

$$\overline{\mathbb{D}} = \overline{\mathbb{D}}_0 \exp(\mathbb{R}\dot{D}).$$

Let $q \in \mathcal{O}(l)$ such that $q(Y) = 0$. There exists $D_0 \in \mathbb{D}_0$, and $s \in \mathbb{R}$ such that $q = \exp(D_0^t)\exp(s\dot{D}^t)(l)$. Since $q(Y) = 0$, we have

$$0 =< \exp(D_0^t)\exp(s\dot{D}^t)(l), Y >$$

$$=< l, \exp(s\dot{D})\exp(D_0)(Y) >$$
$$=< l, \exp(s\dot{D})(\mathrm{e}^{\alpha(D_0)}Y) >$$
$$=< l, \mathrm{e}^{\alpha(D_0)}(Y + sZ) >= s\mathrm{e}^{\alpha(D_0)}.$$

This implies that, $s = 0$ and so $q \in (\overline{\mathbb{D}}_0)l$. Thus $\{q \in \mathcal{O}(l) \ : \ q(Y) = 0\} \subset \mathcal{O}_0$. On the other hand, we evidently have $(\overline{\mathbb{D}}_0^t)l(Y) = 0$.
As $< P(\mathbb{D}_0^t)(q), Y >= \{0\}$ for every $q \in \mathcal{O}_0$, it follows for $D \in \mathbb{D}_0, q \in \mathcal{O}_0$, that $P(D^t)q \in \mathcal{O}_0$ whenever $P(D^t)q \in \mathcal{O}$. We can apply the induction hypothesis to $\mathbb{D}_0$ and $\mathcal{O}_0$. Hence

$$\mathbb{R}Y \oplus N_{\mathbb{D}}(l) = N_{\mathbb{D}_0}(l) = A_{\mathbb{D}_0}(l).$$

We show now that $N_{\mathbb{D}}(l)$ is $\mathbb{D}$-invariant. Let $v \in N_{\mathbb{D}}(l)$. We have $< l, D^2(v) >= 0$, for all $D \in \mathbb{D}$. In fact, for all $D_0 \in \mathbb{D}_0$ and $s \in \mathbb{R}$ small enough,

$$\begin{aligned} < P\big(s(\dot{D}+D_0)^t\big)l, Y > &= < l, Y + a_2 s^2(\dot{D}+D_0)^2(Y) > +o(s^3) \\ &= < l, Y + a_2 s^2(\dot{D}^2 + D_0^2 + \dot{D}D_0 + D_0\dot{D})(Y) > +o(s^3) \\ &= a_2 s^2 < l, \dot{D}D_0(Y) > +o(s^3) \\ &= a_2\alpha(D_0)s^2 + o(s^3) =: Q(s). \end{aligned}$$

On the other hand, we have

$$< \exp\big(-Q(s)\dot{D}^t\big)P\big(s(\dot{D}+D_0)^t\big)l, Y >=< P\big(s(\dot{D}+D_0)^t\big)l, Y - Q(s)Z >$$

$$=< P\big(s(\dot{D}+D_0)^t\big)l, Y > -Q(s) = 0.$$

It follows that for $s \in \mathbb{R}$ small enough,

$$\exp\big(-Q(s)\dot{D}^t\big)P\big(s(\dot{D}+D_0)^t\big)l \in \overline{\mathbb{D}}_0.$$

Since $v \in N_{\mathbb{D}}(l) \subset N_{\mathbb{D}_0}(l) = A_{\mathbb{D}_0}(l)$, we have

$$\begin{aligned} < l, v > &= < \exp\big(-Q(s)\dot{D}^t\big)P\big(s(\dot{D}+D_0)^t\big)l, v > \\ &= < P\big(s(\dot{D}+D_0)^t\big)l, v - Q(s)\dot{D}(v) > +o(s^3) \\ & \quad < l, v - Q(s)\dot{D}(v) + a_2 s^2(\dot{D}+D_0)^2(v) > +o(s^3) \\ &= < l, v\rangle + a_2 s^2(\dot{D}+D_0)^2(v) > +o(s^3). \end{aligned}$$

This implies that $a_2 < l, (\dot{D}+D_0)^2(v) >= 0$. As $a_2 \neq 0$, we have $< l, (\dot{D} + D_0)^2(v) >= 0$. Hence $< l, D^2(v) >= 0$ for all $D \in \mathbb{D}$. Now, for $D_1, D_2 \in \mathbb{D}$ and $v \in N_{\mathbb{D}}(l)$,

$$0 =< l, (D_1+D_2)^2(v) >=< l, (D_1^2+D_2^2+2D_1D_2+[D_1,D_2])(v) >= 2 < l, D_1D_2(v) >$$

(since $[D_1, D_2] \in \mathbb{D}$). This shows that $\mathbb{D}(v) \subset \mathrm{N}_{\mathbb{D}}(l)$ and so $N_{\mathbb{D}}(l)$ is $\mathbb{D}-$invariant. The subcase $\alpha = 0$ is similar.
Subcase 2.2 : $A_{\mathbb{D}}(l) = \{0\}$. Then there exists a non-zero $Y \in V$ and non-zero homomorphism α on $\mathbb{D}$ such that

$$D(Y) = \alpha(D)Y, \quad \forall D \in \mathbb{D}.$$

Without loss of generality, we can assume that $l(Y) = 1$. Let $\mathbb{D}_0$ be the kernel of α and $\overline{\mathbb{D}}_0 := \exp(\mathbb{D}_0)$. There exists $\dot{D} \in \mathbb{D}\backslash\mathbb{D}_0$ such that $\alpha(\dot{D}) = 1$ and $\mathbb{D} = \mathbb{D}_0 \oplus \mathbb{R}\dot{D}$. It is easy to see that

$$\mathcal{O}_0 := \overline{\mathbb{D}}_0^t l = \{q \in \mathcal{O} \ : \ q(Y) = 1\}.$$

On the other hand, for all $s \in \mathbb{R}$ small enough and $D_0 \in \mathbb{D}_0$

$$\begin{aligned} < P\big(s(\dot{D}+D_0)^t\big)(l), Y > &= < l, Y + a_2 s^2(\dot{D}+D_0)^2(Y) + a_3 s^2(\dot{D}+D_0)^3(Y) + ... > \\ &= 1 + a_2 s^2 + a_3 s^3 + ... = 1 + Q(s) > 0. \end{aligned}$$

Then, for $q(s) = \ln(1+Q(s))$ we get $\exp(-q(s)\dot{D})P\big(s(\dot{D}+D_0)^t\big)(l) \in \mathcal{O}_0$ for s small enough in $\mathbb{R}$. In addition, by the same reasoning as above, using the induction hypothesis, we see that $N_{\mathbb{D}_0}(l)$ is $\mathbb{D}_0$-invariant. Let $v \in N_{\mathbb{D}}(l)$, we compute

$$\begin{aligned} < l, v > &= < \exp(-q(s)\dot{D})P\big(s(\dot{D}+D_0)^t\big)(l), v > \\ &= < P\big(s(\dot{D}+D_0)^t\big)(l), v - a_2 s^2 \dot{D}(v) > + o(s^3) \\ &= < l, v - a_2 s^2 \dot{D}(v) + a_2 s^2 (\dot{D}+D_0)^2(v) > + o(s^3) \\ &= < l, v > + a_2 s^2 < (\dot{D}+D_0)^2(v) > + o(s^3). \end{aligned}$$

It follows that

$$a_2 s^2 < l, (\dot{D}+D_0)^2(v) > + \theta(s^3) = 0, \text{ for all } s \in \mathbb{R}.$$

Hence, we have $< l, D^2(v) >= 0$ for all $D \in \mathbb{D}$ and so $< l, D_1 D_2(v) >= 0$ for all $D_1, D_2 \in \mathbb{D}$ i.e. $N_{\mathbb{D}}(l)$ is $\mathbb{D}$-invariant.
1) $\Rightarrow$ 3)
Since now $N_{\mathbb{D}}(l)$ is $\mathbb{D}$-invariant, the $\overline{\mathbb{D}}$-orbit $\mathcal{O}$ of l is contained in $l + N_{\mathbb{D}}(l)^{\perp}$. The dimension of this orbit $\mathcal{O}$ is equal to the dimension of $V/N_{\mathbb{D}}(l)$ because the dimension of $\mathcal{O}$ is equal to the dimension of $\mathbb{D}$ modulo the stabilizer $\mathbb{D}(l) := \{D \in \mathbb{D}, ; D^t(l) = 0\}$ of l and since the bilinear map

$$\mathbb{D}/\mathbb{D}(l) \times V/N_D(l) : (D + \mathbb{D}(l), v + N_{\mathbb{D}}(l)) \to \langle l, D(v) \rangle$$

establishes a duality between the two quotient spaces. Hence $\mathcal{O}$ is an open subset of $l + N_{\mathbb{D}}(l)^{\perp}$. We take the function $P(\xi) := 1 + \xi^2,\ \xi \in \mathbb{R}$. Then the mapping

$$\mathbb{D} \times (l + N_{\mathbb{D}}(l)^{\perp}) \mapsto l + N_{\mathbb{D}}(l)^{\perp}; (D, q) \to P(D)q$$

is continuous and so for every $q \in \mathcal{O}$ we can find a small neighbourhood U of 0 in $\mathbb{D}$, such that $P(D)q \in \mathcal{O}$ for every $D \in U$. □

Corollary 4.3.3. *Let $\overline{\mathbb{D}}$ be an exponential completely solvable Lie Group of endomorphisms of the real finite dimensional vector space V. Let $l \in V^*$ and let $\mathcal{O}(l)$ be the $\overline{\mathbb{D}}$-orbit of l in V^*. If $\mathcal{O}(l)$ is closed, then the following statements are equivalent :*
1) $N_{\mathbb{D}}(l)$ is $\mathbb{D}$-invariant : $N_{\mathbb{D}}(l) = A_{\mathbb{D}}(l)$.

2) $\overline{\mathbb{D}}^t \cdot l_{|N_{\mathbb{D}}(l)} = l_{|N_{\mathbb{D}}(l)}$.
3) a) $\mathcal{O}(l)$ *is affine linear.*
b) $\mathcal{O}(l) = l + A_{\mathbb{D}}(l)^{\perp}$.
4) There exists an analytic function $P : \mathbb{R} \to \mathbb{R}; P(\xi) = 1 + a_2\xi^2 + a_3\xi^3 + ...$ *for small* ξ*, with* $a_2 \neq 0$ *, such that for every* q *in the orbit* $\mathcal{O}(l)$*,* $P(D^t)q \in \mathcal{O}(l)$ *for* $D \in \mathbb{D}$ *small enough.*

Démonstration. It suffices to proof the implications 1) $\Rightarrow$ 3)a) and 3)a) $\Rightarrow$ 3)b).
1) $\Rightarrow$ 3)a) For $X \in N_{\mathbb{D}}(l)$ and $D \in \mathbb{D}$, we have

$$< \exp(D^t)(l), X > = < l, X > + < l, D(X) > + \frac{1}{2!} < l, D^2(X) > + ... = < l, X > .$$

Hence $\mathcal{O}(l) \subset l + N_{\mathbb{D}}(l)^{\perp}$.
On the other hand, reasoning as in the proof of the preceding theorem, we see that $\mathcal{O}(l)$ is open in $l + N_{\mathbb{D}}(l)^{\perp}$. Since by hypothesis it is also closed, it follows that $\mathcal{O} = l + N_{\mathbb{D}}(l)^{\perp}$.
3)a) $\Rightarrow$ 3)b) We evidently have

$$\mathcal{O}(l) \subset l + A_{\mathbb{D}}(l)^{\perp}.$$

On the other hand, let W be a subspace of V such that $\mathcal{O}(l) = l + W^{\perp}$. For all $D \in \mathbb{D}$ and $s \in \mathbb{R}$, we have

$$\frac{1}{s}(\exp(sD^t)(l) - l) \in \mathcal{O}(l) - l \subset W^{\perp}.$$

Hence for $X \in W$,

$$< \frac{1}{s}(\exp(sD^t)(l) - l), X > = 0$$

and so

$$< D^t(l), X > + \frac{s}{2!} < (D^t)^2 l, X > + \frac{s^2}{3!} < (D^t)^3 l, X > + ... = 0$$

and therefore for all $j \geq 1$, $D \in \mathbb{D}, X \in W$, $D^j(X) \in \ker(l)$, $\quad j \geq 1$, i.e. $W \subset A_{\mathbb{D}}(l)$. Whence, $W = A_{\mathbb{D}}(l)$. □

Corollary 4.3.4. *Let* $G = exp(\mathfrak{g})$ *be a completely solvable Lie group and let* $l \in \mathfrak{g}^*$*. If the* G*-orbit* $\mathcal{O}(l)$ *of* l *is closed, then the following statements are equivalent :*
1) $\mathfrak{g}(l)$ *is an ideal in* $\mathfrak{g}$.
2) $Ad^*(G)l_{|\mathfrak{g}(l)} = l_{|\mathfrak{g}(l)}$.
3) $\mathcal{O}(l) = l + \mathfrak{g}(l)^{\perp}$.

4.4 Representations Associated to Flat Orbits

Let $l \in \mathfrak{g}^*$ and $\mathfrak{p}_l$ be a polarization for l satisfying the Pukanszky condition. Let $P_l = \exp(\mathfrak{p}_l)$ and $\pi_l \in \hat{G}$ be the representation $\mathrm{ind}_{P_l}^G \chi_l$, where χ_l is the unitary character of P_l defined by $\chi_l(x) := e^{-i\langle l, \log(x)\rangle}, x \in P_l$. Let as in the subsection 4.2.1 $\mathcal{J} = (\mathfrak{g}_i)_{i=1}^n$ be a Jordan-Hölder sequence and $\mathcal{Z} = \{Z_1, \cdots, Z_n\}$ be a Jordan-Hölder basis of $\mathfrak{g}$ adapted to $\mathcal{J}$. We denote by $I^{\mathfrak{p}_l}$ the index set $I^{\mathfrak{p}_l} := \{i \in \{1, \cdots, n\}; \mathfrak{p}_l \cap \mathfrak{g}_i \neq \mathfrak{p}_l \cap \mathfrak{g}_{i+1}\}$. Then for $i \in I^{\mathfrak{p}_l}$, we can take the vector Z_i in $\mathfrak{p}_l$. Let also $I^{\mathfrak{g}/\mathfrak{p}_l}$ be the the index set $\{1, \cdots, n\} \setminus I^{\mathfrak{p}_l} = \{i \in \{1, \cdots, n\}; \mathfrak{g}_i \cap \mathfrak{p}_l = \mathfrak{g}_{i+1} \cap \mathfrak{p}_l\}$.
We consider the function $\psi_{\mathfrak{p}_l}$ defined on G by

$$\psi_{\mathfrak{p}_l}(x) = \prod_{i \in I^{\mathfrak{g}/\mathfrak{p}_l}} \left| \frac{\rho_i(\log(x))}{e^{\frac{\rho_i(\log(x))}{2}} - e^{-\frac{\rho_i(\log(x))}{2}}} \right|.$$

The function $\psi_{\mathfrak{p}_l}$ is bounded and $\mathrm{A}d(G)$-invariant. For $p \in P_l$ we have the following identity :

$$\Delta_{P_l,G}(p)^{\frac{-1}{2}} \frac{j_{\mathfrak{p}_l}(\log p)}{j_{\mathfrak{g}}(\log p)} = \psi_{\mathfrak{p}_l}(p). \tag{4.3}$$

Indeed,

$$\begin{aligned} \Delta_{P_l,G}(p)^{\frac{-1}{2}} \frac{j_{\mathfrak{p}_l}(\log p)}{j_{\mathfrak{g}}(\log p)} &= \prod_{i \in I^{\mathfrak{g}/\mathfrak{p}_l}} e^{-\rho_i(\log(p))/2} \prod_{i \in I^{\mathfrak{g}/\mathfrak{p}_l}} \left| \frac{\rho_i(\log(p))}{1 - e^{-\rho_i(\log(p))}} \right| \\ &= \prod_{i \in I^{\mathfrak{g}/\mathfrak{p}_l}} \left| \frac{\rho_i(\log(p))}{e^{\frac{\rho_i(\log(p))}{2}} - e^{-\frac{\rho_i(\log(p))}{2}}} \right| \\ &= \psi_{\mathfrak{p}_l}(p). \end{aligned}$$

Let $I(l, \mathfrak{p}_l)$ be the closed subspace of $L^1(G)$, given by

$$I(l, \mathfrak{p}_l) = \Big\{ f \in L^1(G) \ : \ \forall u, v \in G, \ \int_G f(uxv) \psi_{\mathfrak{p}_l}(x) e^{-i<l, \log x>} dx = 0 \Big\}.$$

Then $I(l, \mathfrak{p}_l)$ is in fact a twosided ideal of the algebra $L^1(G)$, since for every $f \in I(l, \mathfrak{p}_l)$ the left and right translates of f are all contained in $I(l, \mathfrak{p}_l)$.

Proposition 4.4.1. $I(l, \mathfrak{p}_l)$ *is contained in* $ker(\pi_l)$.

Démonstration. Let $g \in I(l, \mathfrak{p}_l)$ and $\alpha \in C_c(G)$ and let $f := g * \alpha$. Then $f \in I(l, \mathfrak{p}_l)$ too and the function $p \mapsto f(uxpv)$ is contained in $L^1(P_l)$ for every $x, u, v \in G$ since

$$\begin{aligned}
\int_{P_l} |f(uxpv)| dp &= \int_G \int_{P_l} |g(y)||\alpha(y^{-1}uxpv)| dp dy \\
&= \int_G \int_{P_l} \Delta_G(y^{-1}) |g(uxy^{-1})||\alpha(ypv)| dp dy \\
&= \oint_{G/P_l} \int_{P_l} \Delta_G(yq)^{-1} |g(ux(yq)^{-1})| \int_{P_l} |\alpha(yqpv)| dp \Delta_{P_l,G}(q)^{-1} dq d\mu_{G/P_l}(y) \\
&= \oint_{G/P_l} \int_{P_l} \Delta_G(yq)^{-1} \Delta_{P_l,G}(q)^{-1} |g(ux(yq)^{-1})| \int_{P_l} |\alpha(ypv)| dp dq d\mu_{G/P_l}(y).
\end{aligned}$$

The function $y \mapsto \int_{P_l} |\alpha(ypv)| dp =: \tilde{\alpha}_v(y)$ is uniformly bounded in y and so

$$\begin{aligned}
\int_{P_l} |f(uxpv)| dp &= \oint_{G/P_l} \int_{P_l} \Delta_G(yq)^{-1} \Delta_{P_l,G}(q)^{-1} |g(ux(yq)^{-1})| \tilde{\alpha}_v(y) dq d\mu_{G/P_l}(y) \\
&= \int_G |g(uxy)| \tilde{\alpha}_v(y^{-1}) dy < \infty.
\end{aligned}$$

Now, for all $u, v, x \in G$ and all $p \in P_l$ we have that

$$\begin{aligned}
0 &= \int_G f(upxp^{-1}v) \psi_{\mathfrak{p}_l}(x) e^{-i<l, \log x>} dx \\
&= \int_G f(upxp^{-1}v) \psi_{\mathfrak{p}_l}(pxp^{-1}) e^{-i<l, \log x>} dx \\
&= \Delta_G(p) \int_G f(uxv) \psi_{\mathfrak{p}_l}(x) e^{-i<Ad^*(p)l, \log x>} dx \\
&= \Delta_G(p) \int_{\mathfrak{g}} f(u\exp(Y)v) \psi_{\mathfrak{p}_l}(\exp Y) e^{-i<Ad^*(p)l, Y>} j_{\mathfrak{g}}(Y) dY.
\end{aligned}$$

As $Ad^*(P_l)l = l + \mathfrak{p}_l^{\perp}$, we get for $u, v, x \in G, q \in \mathfrak{p}_l^{\perp}$, that :

$$\begin{aligned}
0 &= \int_{\mathfrak{g}} f(u\exp(Y)v) \psi_{\mathfrak{p}_l}(\exp(Y)) e^{-i<l+q, Y>} j_{\mathfrak{g}}(Y) dY \\
&= \int_{\mathfrak{g}/\mathfrak{p}_l} e^{-i\langle q+l, Y\rangle} \int_{\mathfrak{p}_l} f(u\exp(Y+U)v) \psi_{\mathfrak{p}_l}(\exp(Y+U)) e^{-i<l, U>} j_{\mathfrak{g}}(Y+U) dU d\dot{Y}.
\end{aligned}$$

Hence, for every $Y \in \mathfrak{g}, u, v \in G$,

$$0 = \int_{\mathfrak{p}_l} f(u\exp(Y+U)v) \psi_{\mathfrak{p}_l}(\exp(Y+U)) e^{-i<l, U>} j_{\mathfrak{g}}(Y+U) dU.$$

Therefore, for $Y = 0, u, v \in G$,

$$0 = \int_{\mathfrak{p}_l} f(u\exp(U)v)\psi_{\mathfrak{p}_l}(\exp(U))e^{-i<l,U>}j_{\mathfrak{g}}(U)dU.$$

Hence, by (4.3), for all $u, v \in G$,

$$\begin{aligned} 0 &= \int_{\mathfrak{p}_l} f(u\exp(U)v)\Delta_{P_l,G}(\exp(U))^{\frac{-1}{2}} j_{\mathfrak{p}_l}(U)e^{-i<l,U>}dU. \\ &= \int_{P_l} f(upv)\Delta_{P_l,G}(p)^{\frac{-1}{2}} e^{-i<l,\log(p)>}dp. \end{aligned}$$

Thus, by Lemma 4.2.1, $f \in \ker(\pi_l)$ and finally $g \in \ker(\pi_l)$. □

Theorem 4.4.2. *Let $G = exp(\mathfrak{g})$ be a completely solvable Lie group. Let $l \in \mathfrak{g}^*$ such that the G-orbit $\mathcal{O}(l)$ is closed. If $\mathcal{O}(l)$ is affine linear then*

$$ker(\pi_{\mathcal{O}(l)}) = \{f \in L^1(G) : [(f \circ exp)j_{\mathfrak{g}}]\hat{}(\mathcal{O}(l)) = \{0\}\}.$$

Démonstration. Let $\tau_l := \mathrm{ind}_{G(l)}^G \chi_l$, where $G(l) := \exp(\mathfrak{g}(l))$. As $\mathcal{O}(l)$ is affine linear, $\mathcal{O}(l) = l + \mathfrak{g}(l)^{\perp}$ and $\mathfrak{g}(l)$ is an ideal of $\mathfrak{g}$ (by corollary 4.3.4). Hence $G(l)$ is a closed connected normal subgroup of G. Furthermore, we have

$$\ker(\tau_l) = \bigcap_{q \in l+\mathfrak{g}(l)^{\perp}} \ker(\pi_q) = \ker(\pi_l)$$

(see [Lep-Lud]). We show that $\ker(\tau_l) = \{f \in L^1(G) : [(f \circ \exp)j_{\mathfrak{g}}]\hat{}(\mathcal{O}(l)) = \{0\}\}$. Let $f \in C_c(G) * \ker(\tau_l) * C_c(G)$. Then, by Lemma 4.2.1 we get

$$\int_{G(l)} f(sh)\chi_l(h)dh = \int_{\mathfrak{g}(l)} f \circ \exp(s + \varphi_s(h))\chi_l(h)j_{\mathfrak{g}(l)}(h)dh = 0, \qquad (4.4)$$

for all $s \in G$, where

$$\begin{aligned} \varphi_s(h) &= s \cdot_{\mathfrak{g}} h - s \\ &= h + \frac{1}{2}[s,h] + \frac{1}{12}[s,[s,h]] + \frac{1}{12}[h,[h,s]] + \cdots \text{ for small } s \in \mathfrak{g}, h \in \mathfrak{g}(l). \end{aligned}$$

We see that the mapping $\varphi_s : \mathfrak{g}(l) \to \mathfrak{g}(l)$ is a diffeomorphism, whose inverse ψ_s is given by :

$$\psi_s(h) = (-s) \cdot_{\mathfrak{g}} (h + s), \ h \in \mathfrak{g}(l), (s \in G).$$

On the other hand, for all $f \in L^1(G)$ we have

$$\begin{aligned}
&\int_{\mathfrak{g}/\mathfrak{g}(l)}\int_{\mathfrak{g}(l)} f\circ\exp(s+h)j_{\mathfrak{g}}(s+h)dhds \\
= &\int_G f(g)dg = \int_{G/G(l)}\int_{G(l)} f(sh)dh \\
= &\int_{\mathfrak{g}/\mathfrak{g}(l)}\int_{\mathfrak{g}(l)} f\circ\exp(s+\varphi_s(h))j_{\mathfrak{g}(l)}(h)j_{\mathfrak{g}/\mathfrak{g}(l)}(s)dhds \\
= &\int_{\mathfrak{g}/\mathfrak{g}(l)}\int_{\mathfrak{g}(l)} f\circ\exp(s+h)j_{\mathfrak{g}(l)}(\psi_s(h))j_{\mathfrak{g}/\mathfrak{g}(l)}(s)Jac(\psi_s)(h)dhds.
\end{aligned}$$

This proves that

$$Jac(\psi_s)(h) = \frac{j_{\mathfrak{g}}(s+h)}{j_{\mathfrak{g}(l)}(\psi_s(h))j_{\mathfrak{g}/\mathfrak{g}(l)}(s)}.$$

We deduce from equation (4.4) that

$$\int_{\mathfrak{g}(l)} f\circ\exp(s+h)j_{\mathfrak{g}}(s+h)\chi_l(h)dh = 0, s\in\mathfrak{g}.$$

since $< l, \psi_s(h) >=< l, h >$ for all $h \in \mathfrak{g}(l)$, because $\mathfrak{g}(l)$ is an ideal of $\mathfrak{g}$. Therefore

$$\int_{\mathfrak{g}} f\circ\exp(Y)j_{\mathfrak{g}}(Y)e^{-i\langle l+q,Y\rangle}dY = 0, q\in\mathfrak{g}(l)^{\perp}.$$

As $C_c(G) * \ker(\tau_l) * C_c(G)$ is dense in $\ker(\tau_l)$, it follows that $\ker(\tau_l) \subset \{f \in L^1(G) : [(f\circ\exp)j_{\mathfrak{g}}]\widehat{\ }(l+\mathfrak{g}(l)^{\perp}) = 0\}$. Let now $f \in L^1(G)$ such that $[(f\circ\exp)j_{\mathfrak{g}}]\widehat{\ }(l+\mathfrak{g}(l)^{\perp}) = 0$, then by the same computation as above, we can show that $\int_{G(l)} f(sh)\chi_l(tht^{-1})dh = 0$ for all $s, t \in G$. That means by Lemma 4.2.1 that $f \in \ker(\tau_l)$ and thus

$$\ker(\tau_l) = \{f \in L^1(G) : [(f\circ\exp)j_{\mathfrak{g}}]\widehat{\ }(\mathcal{O}(l)) = 0\}.$$

□

We show now the converse direction. We take an exponential solvable Lie group $G = \exp(\mathfrak{g})$ and an exponential completely solvable Lie group $\overline{\mathbb{D}} = \exp(\mathbb{D})$ of automorphisms of $\mathfrak{g}$ containing the group $\mathrm{A}d(G)$. We also suppose that there is an analytic mapping $P : \mathbb{D}\times\mathfrak{g} \to \mathfrak{g}$ such that for small (D, X)

$$P(D,X) = X + aD^2(X) + \sum_{\sum ki+nj\geq 2} a_{\binom{k_1\ldots k_r}{n_1\ldots n_r}}\Big(D^{k_1}\mathrm{ad}^{n_1}(X)\ldots D^{k_r}\mathrm{ad}^{n_r}(X)D^{k_{r+1}}\Big)(X),$$

with $a \neq 0$. We write $P(D) : \mathfrak{g} \to \mathfrak{g}$ for the mapping : $P(D)(X) = P(D,X)$, $(D \in \mathbb{D})$ and we suppose that $P(D)$ is a diffeomorphism of $\mathfrak{g}$ for every $D \in \mathbb{D}$. Define for $D \in \mathbb{D}$ the linear bijection $\check{P}(D)$ of $L^1(\mathfrak{g})$ defined by

$$\check{P}(D)f(X) := f(P(D)X)J_{P(D)}(X), X \in \mathfrak{g},$$

where $J_{P(D)}(X)$ denotes the Jacobian of $P(D)$ at $X \in \mathfrak{g}$.

Definition 4.4.3. For an ideal I in the algebra $L^1(\mathfrak{g})$, let $h(I)$ be the set of characters

$$h(I) := \{q \in \mathfrak{g}^*, 0 = \chi_q(f) = \int_{\mathfrak{g}} f(Y)e^{-i\langle q,Y\rangle}dY,\ f \in I\}.$$

Then $h(I)$ is a closed (possibly empty) subset of $\mathfrak{g}^*$.

Lemma 4.4.4. *Let $\mathfrak{g}$, $\overline{\mathbb{D}}$ and P as above. Let $l \in \mathfrak{g}^*$ and let I be a closed ideal in $L^1(\mathfrak{g})$, so that $h(I)$ is the closure $\overline{\mathcal{O}(l)}$ of the $\overline{\mathbb{D}}$-orbit $\mathcal{O}(l)$ of l. If I is invariant under the maps $\check{P}(D), D \in \mathbb{D}$, then $N_{\mathbb{D}}(l)$ is $\mathbb{D}$-invariant.*

Démonstration. We proceed by induction on the number $d := \dim(\mathbb{D}) + \dim(\mathfrak{g})$. If $d = 1$, the result is obviously true. Suppose now that $d \geq 2$. Let $\mathfrak{g}_0 = \ker(l) \cap A_{\mathbb{D}}(l)$. Then $\mathfrak{g}_0$ is $\mathbb{D}$-invariant. We first assume that $\mathfrak{g}_0 \neq \{0\}$. Let p be the canonical projection of $\mathfrak{g}$ onto $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{g}_0$ and let j be the transpose of p. Let $\tilde{l} \in \tilde{\mathfrak{g}}^*$ be defined by $j(\tilde{l}) = l$. We define a Lie algebra of exponential derivations on $\tilde{\mathfrak{g}}$ in the following way : for all $D \in \mathbb{D}$, let $\tilde{D}$ be defined by $\tilde{D}(p(x)) = p(D(x)), x \in \mathfrak{g}$. So we have evidently $\overline{\mathcal{O}(l)} = j\Big(\overline{(\exp\tilde{\mathbb{D}})\tilde{l}}\Big)$ and $ad\tilde{\mathfrak{g}} \subset \tilde{\mathbb{D}}$.
Let $\tilde{I} = \pi(I)$, where $\pi : L^1(\mathfrak{g}) \to L^1(\tilde{\mathfrak{g}})$ is the canonical surjection :

$$\pi(f)(\tilde{x}) := \int_{\mathfrak{g}_0} f(x+z)dz, f \in L^1(\mathfrak{g}), \tilde{x} = x + \mathfrak{g}_0.$$

Thus $\tilde{I}$ is a closed ideal in $L^1(\tilde{\mathfrak{g}})$ (see [Reiter], page 177) and the hull of $\tilde{I}$ is $h(\tilde{I}) = \overline{(\exp\tilde{\mathbb{D}})\tilde{l}}$.
Define the maps $P(\tilde{D})$ $(\tilde{D} \in \tilde{\mathbb{D}})$ by :

$$\begin{aligned} P(\tilde{D})(\tilde{x}) &= P(D)x + \mathfrak{g}_0,\ \tilde{x} = x + \mathfrak{g}_0, \\ &= \tilde{x} + a\tilde{D}^2(\tilde{x}) + \ldots,\ \text{for small } \tilde{x} \in \tilde{\mathfrak{g}}. \end{aligned}$$

Then

$$P(\tilde{D})(p(x)) = p(P(D)(x)),\ \text{for all } x \in \mathfrak{g}.$$

It is easy to see that $\tilde{I}$ is $\check{P}(\tilde{\mathbb{D}})$-invariant ; Indeed, for all $\tilde{f} = \pi(f) \in \tilde{I}$, $\tilde{D} \in \tilde{\mathbb{D}}$ and $\tilde{x} = p(x)$

$$\begin{aligned} \check{P}(\tilde{D})\tilde{f}(\tilde{x}) &= \tilde{f}(p(P(D)(x)))J_{P(\tilde{D})}(p(x)) \\ &= \int_{\mathfrak{g}_0} f(P(D)(x+h))J_{P(D)}(x+h)dh \\ &= \int_{\mathfrak{g}_0} f(P(D)(x)+h)\frac{J_{P(D)}(x+Q(D,x)^{-1}(h))}{J_{Q(D,x)}}dh, \end{aligned}$$

where $Q(D,x)$ is a diffeomorphism of $\mathfrak{g}_0$ given by

$$Q(D,x)(h) = P(D)(x+h) - P(D)(x) \ (h \in \mathfrak{g}_0, x \in \mathfrak{g}, D \in \mathbb{D}).$$

On the other hand for all $\varphi \in L^1(\mathfrak{g})$, we have

$$\begin{aligned} \int_{\mathfrak{g}} \varphi(x)dx &= \int_{\mathfrak{g}/\mathfrak{g}_0}\int_{\mathfrak{g}_0} \varphi(P(D)(x+h))J_{P(D)}(x+h)dhd\tilde{x} \\ &= \int_{\mathfrak{g}/\mathfrak{g}_0}\int_{\mathfrak{g}_0} \varphi(P(D)(x)+h)\frac{J_{P(D)}(x+Q(D,x)^{-1}(h))}{J_{Q(D,x)(h)}}dhd\tilde{x}, \end{aligned}$$

and

$$\int_{\mathfrak{g}} \varphi(x)dx = \int_{\mathfrak{g}/\mathfrak{g}_0}\int_{\mathfrak{g}_0} \varphi(x+h)dhd\tilde{x} = \int_{\mathfrak{g}/\mathfrak{g}_0}\int_{\mathfrak{g}_0} \varphi(P(D)(x)+h)dhJ_{P(\tilde{D})}(\tilde{x})d\tilde{x}.$$

This implies that

$$J_{P(\tilde{D})}(\tilde{x}) = \frac{J_{P(D)}(x+Q(D,x)^{-1}(h))}{J_{Q(D,x)(h)}}, x \in \mathfrak{g}, h \in \mathfrak{g}_0.$$

We obtain thus that $\check{P}(\tilde{D})\tilde{f}(\tilde{x}) = \pi(\check{P}(D)f(x))$.
By the induction hypothesis we get $N_{\tilde{\mathbb{D}}}(\tilde{l}) = A_{\tilde{\mathbb{D}}}(\tilde{l})$ and thus $N_{\mathbb{D}}(l)$ is $\mathbb{D}$-invariant.
Suppose now that $\mathfrak{g}_0 = \{0\}$. Then $\dim(A_{\mathbb{D}}(l)) = 0$ or 1.
case 1 : $A_{\mathbb{D}}(l) = \mathbb{R}Z$, for some non zero Z in $\mathfrak{g}$. Since $A_{\mathbb{D}}(l)$ is $\mathbb{D}$-invariant and $\langle l, Z\rangle \neq 0$, it follows that $D(Z) = 0$ for all $D \in \mathbb{D}$. In this case, there exists a non-zero $Y \in \mathfrak{g}$ and two linear functionals α, β on $\mathbb{D}$ such that

$$D(Y) = \alpha(D)Y + \beta(D)Z, \ \ \forall D \in \mathbb{D}$$

since $\mathbb{D}$ is completely solvable with $\beta \neq 0$. If $\alpha \neq 0$, then we can assume that α and β are linearly independent. The linear functional α is a homomorphism

of $\mathbb{D}$. Without loss of generality, we can assume that $l(Y) = 0$ and $l(Z) = 1$. We can find an element $\dot{D} \in \ker(\alpha)$, such that $\beta(\dot{D}) = 1$.
We apply now the technique of restriction of an ideal to an invariant subgroup developed in [Hau-Lud]. Let $\mathfrak{g}_1$ be any $\mathbb{D}$-invariant subspace of $\mathfrak{g}$ containing $A_{\mathbb{D}}(l)$, such that $\dim(\mathfrak{g}/\mathfrak{g}_1) = 1$. Such a $\mathfrak{g}_1$ always exists since $\mathbb{D}$ is completely solvable. Let $l_0 \in \mathfrak{g}_1^{\perp}$ with $l_0 \neq 0$ and let χ_t be the character of $\mathfrak{g}$ defined by : $\chi_t(v) = e^{i<tl_0,v>}$, $v \in \mathfrak{g}$. The ideal $J = \bigcap_{t\in\mathbb{R}} \chi_t I$ is $\check{P}(\mathbb{D})$-invariant and $h(J) = h(I) + \mathbb{R}l_0$. Define for $x \in \mathfrak{g}$ and a function $f : \mathfrak{g} \to \mathbb{C}$ the function ${}_xf$ by ${}_xf(y) := f(x + y), y \in \mathfrak{g}$. Let J' be the set of all functions $f \in J$, so that ${}_xf_{|\mathfrak{g}_1} \in L^1(\mathfrak{g}_1)$ for all $x \in \mathfrak{g}$ and so that the maps $x \mapsto {}_xf_{|\mathfrak{g}_1}$ from $\mathfrak{g}_1$ to $L^1(\mathfrak{g}_1)$ are continuous. J' is dense in J and $\check{P}(\mathbb{D})$-invariant, since I is a $\check{P}(\mathbb{D})$-invariant ideal of $L^1(\mathfrak{g})$. We define the ideal I_1 of $L^1(\mathfrak{g}_1)$ as the closure in $L^1(\mathfrak{g}_1)$ of the functions $f_{|\mathfrak{g}_1}$ $f \in J'$. Let $\mathbb{D}_1$ be the restriction of the $\mathbb{D}$ on $\mathfrak{g}_1$ and let $l_1 = l_{|\mathfrak{g}_1}$. The hull of I_1 is exactly the restriction of the hull of J on $\mathfrak{g}_1$. Thus $h(I_1) = \overline{(\exp\mathbb{D}_1{}^t)l_1}$. We still have $ad\mathfrak{g}_1 \subset \mathbb{D}_1$. The ideal I_1 is $\check{P}(\mathbb{D}_1)$-invariant. Indeed, since $P(D)$ maps $\mathfrak{g}_1$ into $\mathfrak{g}_1$, we have that $J_{P(D)}(h)$ is a constant times $J_{P(D_1)}(h), h \in \mathfrak{g}_1$.
The induction hypothesis applied to I_1, l_1, $\mathbb{D}_1$ implies that $N_{\mathbb{D}}(l) \supset N_{\mathbb{D}}(l) \cap \mathfrak{g}_1 = N_{\mathbb{D}_1}(l_1) = A_{\mathbb{D}_1}(l_1) = A_{\mathbb{D}}(l)$. Hence we obtain that either $A_{\mathbb{D}}(l) = N_{\mathbb{D}}(l)$ (if $N_{\mathbb{D}}(l) \subset \mathfrak{g}_1$) or $\dim(A_{\mathbb{D}}(l)) + 1 = \dim(N_{\mathbb{D}}(l)) \leq 2$.
Let now $\mathbb{D}_0$ be the kernel of β. Then $\mathbb{D}_0$ is a subalgebra of $\mathbb{D}$. Let $\overline{\mathbb{D}}_0 = \exp(\mathbb{D}_0)$. We have that $N_{\mathbb{D}_0}(l) = \mathbb{R}Y \oplus N_{\mathbb{D}}(l)$ and the closure of the $\overline{\mathbb{D}}_0$-orbit of l is given by :

$$\overline{(\overline{\mathbb{D}}_0^t)l} = \{q \in \overline{(\overline{\mathbb{D}}^t)l} \ : \ q(Y) = 0\}.$$

We look at the ideal I_1 defined to be the closure in $L^1(\mathfrak{g})$ of the sum of the ideal I and the kernel K_Y of the surjective homomorphism

$$\pi_Y : L^1(\mathfrak{g}) \to L^1(\mathfrak{g}/\mathbb{R}Y), \pi(f)(x + \mathbb{R}Y) := \int_{\mathbb{R}} f(x + yY)dy.$$

It is easy to check that K_Y is $\check{P}(D)$-invariant for every $D \in \mathbb{D}_0$, since $P(D)(\mathbb{R}Y) \subset \mathbb{R}Y, D \in \mathbb{D}_0$. The hull of I_1 is the subset $h(I) \cap \{q \in \mathfrak{g}^*; \langle q, Y\rangle = 0\} = \overline{\overline{\mathbb{D}}_0^t l} =: \overline{\mathcal{O}_0(l)}$. By the induction hypothesis for $I_1, \mathbb{D}_0$ and $\mathfrak{g}$, we get

$$N_{\mathbb{D}_0}(l) = A_{\mathbb{D}_0}(l)$$

and so $N_{\mathbb{D}_0}(l)$ is $\mathbb{D}_0$-invariant. This implies that $\dim(N_{\mathbb{D}_0}(l)) \leq 3$ since $\dim(N_{\mathbb{D}_0}(l)) = \dim(N_{\mathbb{D}}(l)) + 1$. If $\dim(N_{\mathbb{D}_0}(l)) = 2$ then $N_{\mathbb{D}_0}(l) = A_{\mathbb{D}_0}(l) = \mathbb{R}Y + \mathbb{R}Z$. Whence $N_{\mathbb{D}}(l) = A_{\mathbb{D}}(l)$.
We prove now that the case $\dim(N_{\mathbb{D}_0}(l)) = 3$ can not happen. Suppose otherwise. If we have a $\mathbb{D}$-invariant subspace $\mathfrak{g}_1$ of co-dimension 1 containing $N_{\mathbb{D}_0}(l) \supset N_{\mathbb{D}}(l)$, then we have seen that, $A_{\mathbb{D}}(l) = N_{\mathbb{D}}(l)$ and so $N_{\mathbb{D}_0}(l)$ is

of dimension 2. Hence no $\mathbb{D}$-invariant subspace can contain $N_{\mathbb{D}_0}(l)$. In other terms, either $\mathfrak{g} = A_{\mathbb{D}_0}(l)$ or the smallest $\mathbb{D}$-invariant subspace of $\mathfrak{g}$ containing $A_{\mathbb{D}_0}(l)$ equals $\mathfrak{g}$.
We show that in the two cases $\mathfrak{g}$ is abelian. If $\mathfrak{g} = A_{\mathbb{D}_0}(l) = \mathbb{R}Y_0 + \mathbb{R}Y + \mathbb{R}Z$, then for a $\dot{D} \in \mathbb{D}$ with $\beta(\dot{D}) = 1$ and $\alpha(\dot{D}) = 0$, we have that

$$0 = \langle l, [\exp s\dot{D}(\mathfrak{g}), \exp s\dot{D}(\mathfrak{g})]\rangle =< (\exp s\dot{D}^t)l, [\mathfrak{g}, \mathfrak{g}] >$$

for all $s \in \mathbb{R}$. It follows that if we write $[Y_0, Y] = cY$ for some c, then

$$0 = \langle l, \exp s\dot{D}[Y_0, Y]\rangle = \langle l, c\exp s\dot{D}Y\rangle = c\langle l, Y + sZ\rangle = cs$$

and so $[Y_0, Y] = 0$ and $\mathfrak{g}$ is abelian.
In the second case take again $\dot{D} \in \mathbb{D} \setminus \mathbb{D}_0$ and let

$$Y_1 = \dot{D}Y_0, \ldots, Y_k = \dot{D}^k Y_0 \qquad (k = 2, 3, \ldots, n),$$

where n is the largest integer such that the set $\{Y_1, \cdots, Y_n\}$ is linearly independent modulo span$\{Y, Z\}$. The subspace $\mathfrak{h}$ of $\mathfrak{g}$, spanned by $Y_0, Y_1, \ldots, Y_n, Y$ and Z is by definition $\dot{D}$-invariant. It is also $\mathbb{D}_0$-invariant. Indeed $A_{\mathbb{D}_0}(l)$ is $\mathbb{D}_0$-invariant, it follows that $\mathbb{D}_0(Y_0) \subset A_{\mathbb{D}_0}(l) \subset \mathfrak{h}$. If the functional $\alpha = 0$, then $\mathbb{D}_0$ is an ideal in $\mathbb{D}$ and so we see that inductively on $k = 1, \cdots$, for $D \in \mathbb{D}_0$,

$$D(Y_k) = D(\dot{D}(Y_{k-1})) = \dot{D}(Y_{k-1}) + [D, \dot{D}](Y_{k-1}) \in \mathfrak{h}.$$

If $\alpha \neq 0$, we can take $\dot{D}$ in $\ker(\alpha) \cap [\mathbb{D}, \mathbb{D}]$. In particular $\dot{D}$ is a nilpotent endomorphism. Take now $\mathbb{D}_{00} = \ker(\alpha) \cap \ker(\beta)$, which is an ideal of $\mathbb{D}$ contained in $\mathbb{D}_0$. The subspace $\mathfrak{h}$ is therefore $\mathbb{D}_{00}$-invariant by the argument above. There exists an element $\dot{D}_0 \in \mathbb{D}_0$, such that $\alpha(\dot{D}_0) = 1$. Again, by induction on k, as $[\dot{D}_0, \dot{D}] = -\dot{D}$ modulo $\mathbb{D}_{00}$, we have that

$$\dot{D}_0 Y_k = [\dot{D}_0, \dot{D}]Y_{k-1} + \dot{D}\dot{D}_0 Y_{k-1} \in \mathfrak{h},$$

$k = 1, 2, \ldots, n$. Thus $\mathfrak{h}$ is $\mathbb{D}$-invariant and so $\mathfrak{h} = \mathfrak{g}$.
We show first now that Y is central in $\mathfrak{g}$; we have that, since $\dot{D}$ is a derivation of $\mathfrak{g}$ and since $Y \in A_{\mathbb{D}_0}(l)$,

$$0 =< l, [Y_1, Y] >= \langle l, [\dot{D}(Y_0), Y]\rangle =< l, \dot{D}([Y_0, Y]) - \underbrace{[Y_0, \dot{D}(Y)]}_{=0} >= \alpha(\mathrm{a}d(Y_0)).$$

It follows that $[Y_0, Y] = 0$ and so by induction on k,

$$\begin{aligned} [Y_k, Y] &= [\dot{D}(Y_{k-1}), Y] \\ &= \dot{D}([Y_{k-1}, Y]) - [Y_{k-1}, \dot{D}(Y)] \\ &= 0 - [Y_{k-1}, Z] = 0. \end{aligned}$$

Hence Y is central in $\mathfrak{g}$.
We prove now that $[Y_0, \mathfrak{g}] = 0$. We remark that for all $j \geq 1$, $ad(Y_j)$ is nilpotent, since for these j's, $ad(Y_j) \in [\mathbb{D}, \mathbb{D}]$. This implies that $[Y_0, Y_j] = a_j Y \in \mathbb{R}Y$ for some $a_j \in \mathbb{R}$, because $Y_0 \in A_{\mathbb{D}_0}(l)$ and so $[Y_j, Y_0] \in \mathbb{R}Y$.
On the other hand, by induction on k, we can check that

$$[Y_k, Y_\ell] = \sum_{j=0}^{k} (-1)^j C_k^j \dot{D}^{k-j}[Y_0, Y_{\ell+j}], \forall k, \ell = 1, 2, \cdots, n.$$

Using the formula

$$\begin{aligned} 0 &= [Y_k, Y_k] \\ &= \sum_{j=0}^{k} (-1)^j C_k^j \dot{D}^{k-j}[Y_0, Y_{k+j}] \\ &= (-1)^k [Y_0, Y_{2k}] - (-1)^{k-1} \dot{D}([Y_0, Y_{2k-1}]) \\ &= (-1)^k a_{2k} Y - (-1)^{k-1} a_{2k-1} Z, \ k = 1, \cdots, n, \end{aligned}$$

we deduce that for any $k = 1, 2, \ldots, n$, $a_k = 0$, and hence $ad(Y_0) = 0$. Whence Y_0 is contained in the center of $\mathfrak{g}$ and so is then $Y_1 = \dot{D}(Y_0)$ and inductively all the Y_k's, $k = 2, \cdots, n$. Finally $\mathfrak{g}$ is abelian. Then the polynomial maps $P(D)$, $D \in \mathbb{D}$, are reduced to the linear maps given by

$$P(D)(x) = x + aD^2(x) + \sum_{k \geq 0} b_k D^{2+k}(x)$$

for some $b_k \in \mathbb{R}$ and for $D \in \mathbb{D}$. As I is invariant under these linear maps, the hull $h(I)$ of I is invariant under the corresponding linear maps $P(D)^t$, which have the form

$$P(D)^t = 1 + aD^2 + \sum_{k \geq 0} b_k D^{2+k}, \ D \in \mathbb{D} \text{ small}.$$

Since the orbit $\mathcal{O}(l)$ is open in its closure (see [Ber-Con]), we have that for every $q \in \mathcal{O}(l)$, $P(D)^t(q) \in \mathcal{O}(l)$ for D small enough. Applying now Theorem 4.3.2 we have that $N_{\mathbb{D}}(l) = A_{\mathbb{D}}(l)$, but this contradicts the assumption that $\dim(N_{\mathbb{D}_0}(l)) = 3$.
case 2 : $\dim(A_{\mathbb{D}}(l)) = 0$. In this case, there exists a non-zero $Y \in \mathfrak{g}$ and a homomorphism $\alpha \neq 0$ on $\mathbb{D}$ such that

$$D(Y) = \alpha(D)Y, \quad \forall D \in \mathbb{D}.$$

Without loss of generality, we can assume that $l(Y) = 1$. Let $\mathbb{D}_0$ be the kernel of α and suppose that $ad(\mathfrak{g}) \not\subset \mathbb{D}_0$. There exists $X \in \mathfrak{g}$ so that $[X, Y] = Y$.

Then, the $\overline{\mathbb{D}}$-orbit is saturated with respect to the $\mathbb{D}$-invariant subspace $\mathfrak{g}_1 = \{U \in \mathfrak{g} : [U, Y] = 0\}$. Let $\mathbb{D}_1$ be the restriction of $\mathbb{D}$ on $\mathfrak{g}_1$ and

$$I_1 = \overline{\{h * f_{|\mathfrak{g}_1} : f \in I, h \in C_c(G)\}}^{\|\cdot\|_1}.$$

Then the ideal I_1 is $\check{P}(\mathbb{D}_1)$-invariant and $h(I_1) = h(I)_{|\mathfrak{g}_1}$. Furthermore $h(I_1) = \overline{(\exp \mathbb{D}_1^t) l_{|\mathfrak{g}_1}}$ and by the induction hypothesis for $I_1, \mathbb{D}_1, \mathfrak{g}_1$, we have that

$$N_{\mathbb{D}}(l) = N_{\mathbb{D}_1}(l_{|\mathfrak{g}_1}) = A_{\mathbb{D}_1}(l_{|\mathfrak{g}_1}) = A_{\mathbb{D}}(l) = \{0\}.$$

Assume now that $\alpha(\mathrm{a}d\mathfrak{g}) = 0$. In this case Y is central in $\mathfrak{g}$ and we have for $\mathbb{D}_0 := \ker(\alpha)$

$$\overline{(\exp \mathbb{D}_0^t) l} = \{q \in \overline{\mathcal{O}(l)} : q(Y) = 1\}.$$

Let $\mathfrak{g}_1$ be a $\mathbb{D}$-invariant subspace of $\mathfrak{g}$ of co-dimension 1. We define

$$J = \bigcap_{q \in \mathfrak{g}_1^\perp} \chi_q I \text{ and } I_1 = \overline{\{h * f_{|\mathfrak{g}_1} : f \in J, h \in C_c(G)\}}^{\|\cdot\|_1}.$$

Then I_1 is $\check{P}(\mathbb{D}_{|\mathfrak{g}_1})$-invariant, $h(J) = h(I) + \mathfrak{g}_1^\perp$ and $h(I_1) = \overline{\overline{\mathbb{D}}^t l_1}$. Hence, by the induction hypothesis applied to $I_1, \mathbb{D}_1, \mathfrak{g}_1$ we obtain :

$$N_{\mathbb{D}}(l) \cap \mathfrak{g}_1 = N_{\mathbb{D}_{|\mathfrak{g}_1}}(l_{|\mathfrak{g}_1}) = A_{\mathbb{D}_{|\mathfrak{g}_1}}(l_{|\mathfrak{g}_1}) = \{0\}.$$

Let now $\mathfrak{y} := \mathbb{R}Y$ and let $K := \{f \in L^1(\mathfrak{g}) : \int_{\mathfrak{y}} f(u+y)dy = 0, u \text{ almost everywhere}\}$. Then the hull of the ideal K is the affine subspace $l + \mathfrak{y}^\perp$ and K is $\check{P}(D_0)$-invariant for every $D_0 \in \mathbb{D}_0$ small enough, since we can write for $u \in \mathfrak{g}$ and $y \in \mathfrak{y}$:

$$P(D)(u + y) = u_D + P(D)y = u_D + q(D)y$$

for some $u_D \in \mathfrak{g}$ depending only on u and D and some real number $q(D)$. Hence the closure J_0 of the ideal $I + K$ is also $\check{P}(\mathbb{D}_0)$-invariant and its hull is equal to the closure of the $\overline{\mathbb{D}_0}$-orbit of l. Applying the induction hypothesis to $\mathbb{D}_0$ and J_0, we see that $N_{\mathbb{D}}(l) + \mathbb{R}Y = N_{\mathbb{D}_0}(l) = A_{\mathbb{D}_0}(l)$. We have seen above that for any $\mathbb{D}$-invariant co-one dimensional subspace $\mathfrak{g}_1$ of $\mathfrak{g}$ the dimension of $N_{\mathbb{D}}(l) \cap \mathfrak{g}_1 = 1$. Hence the dimension of $N_{\mathbb{D}_0}(l)$ is less or equal to 2. If this dimension is one, then $N_{\mathbb{D}}(l) = \{0\}$. If $N_{\mathbb{D}}(l)$ is contained in a proper $\mathbb{D}$-invariant subspace, then we have also finished by the argument above. It remains the case where $N_{\mathbb{D}_0}(l)$ is of dimension 2 and contained in no $\mathbb{D}$-invariant proper subspace. We can write $N_{\mathbb{D}_0}(l) = \mathbb{R}U + \mathbb{R}Y$, where $l(U) = 0$. Since $U \in N_{\mathbb{D}_0}(l)$ and $N_{\mathbb{D}_0}(l)$ is $\mathbb{D}_0$-invariant, it follows that $\mathbb{R}U$ must be itself $\mathbb{D}_0$-invariant. Hence there exists a character γ of $\mathfrak{g}$, such that $[T, U] =$

$\gamma(T)U$, $T \in \mathfrak{g}$. Hence $N_{\mathbb{D}_0}(l)$ is contained in the nilradical of $\mathfrak{g}$. But then $\mathfrak{g}$ itself is nilpotent, since the smallest $\mathbb{D}$-invariant subspace containing U and Y is equal to $\mathfrak{g}$ and the elements of $\mathbb{D}$ are derivations of $\mathfrak{g}$. Then necessarily $\alpha = 0$ and so $N_{\mathbb{D}_0}(l)$ is contained in the center of $\mathfrak{g}$ and finally $\mathfrak{g}$ itself is abelian. Since the hull of I is the closure of a $\overline{\mathbb{D}}$-orbit, which is $P(D)^t$-invariant for small D in $\mathbb{D}$, we can now apply as before Theorem 4.3.2 and we have that $N_{\mathbb{D}}(l)$ is $\mathbb{D}$-invariant. □

Theorem 4.4.5. *Let $G = exp(\mathfrak{g})$ be a completely solvable Lie group and let $l \in \mathfrak{g}^*$. Suppose that the coadjoint orbit $\mathcal{O}(l)$ of l is closed in $\mathfrak{g}^*$. Let $\pi_l \in \hat{G}$ be associated to $\mathcal{O}(l)$. The following statments are equivalent :*
1) $ker(\pi_l) = \{f \in L^1(G) : [(f \circ exp)j_{\mathfrak{g}}\check{]}(\mathcal{O}(l)) = 0\}$,
2) The orbit $\mathcal{O}(l)$ is affine linear.

Démonstration. 1⇒2) It is clear that $I_l = \{(f \circ \exp)j_{\mathfrak{g}} : f \in \ker(\pi_l)\}$ is invariant under the linear maps $\check{P}(ad(X))$, $X \in \mathfrak{g}$, defined by :

$$P(ad(X))(Y) = X \cdot_{\mathfrak{g}} Y \cdot_{\mathfrak{g}} X - 2X = Y + \frac{1}{6}\mathrm{ad}(X)^2 Y + \cdots \text{ higher brackets in } X,\ Y \in \mathfrak{g},$$

since $\ker(\pi_l)$ is translation-invariant by elements of G and $I_l \subset L^1(\mathfrak{g})$ is translation-invariant by elements of $\mathfrak{g}$. Furthermore we have that

$$\int_{\mathfrak{g}} f(P(\mathrm{ad}(X)Y)) \frac{j_{\mathfrak{g}}(Y)}{j_{\mathfrak{g}}(Y+2X)} \Delta_G(\exp X) dY = \int_{\mathfrak{g}} f(Y)dY, X \in \mathfrak{g}, f \in L^1(\mathfrak{g})$$

and that the hull $h(I_l) = \mathcal{O}(l)$ by definition. Hence, by Lemma 4.4.4, $\mathcal{O}(l)$ is an affine linear orbit (we take $\mathbb{D} = \mathrm{ad}\mathfrak{g}$).
2)⇒1) (Theorem 4.4.2). □

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