UNIVERSITÉ PAUL VERLAINE METZ

École Doctorale
IAEM Lorraine

UNIVERSITÄT PADERBORN

Fakultät für Elektrotechnik,
Informatik und Mathematik

THÈSE DE DOCTORAT

Discipline : Mathématiques

présentée par

DISSERTATION

im Fach Mathematik

vorgelegt von

Carsten BALLEIER

pour obtenir le grade de

Docteur de l'Université Paul Verlaine-Metz

zur Erlangung des Doktorgrades

der Universität Paderborn

Geometry and Quantization of Howe Pairs of Symplectic Actions

Soutenance publique
le 1er Juillet 2009

Öffentliche Verteidigung
am 1. Juli 2009

Rapporteurs: **Gutachter:**

Prof. Alan T. HUCKLEBERRY, Ruhr-Universität Bochum
Juan-Pablo ORTEGA, Université de Franche-Comté à Besançon

Jury: **Promotionskommission:**

M. Helge GLÖCKNER	**Professor, Universität Paderborn**
Mme Simone GUTT	**Professeur, Université de Metz**
M. Sönke HANSEN	**Professor, Universität Paderborn**
M. Joachim HILGERT	**Professor, Universität Paderborn**
M. Alan T. HUCKLEBERRY	**Professor, Ruhr-Universität Bochum**
M. Martin OLBRICH	**Professor, Universität Luxemburg**
M. Marcus SLUPINSKI	**Maître de Conférences, Université de Strasbourg**
M. Tilmann WURZBACHER	**Professeur, Université de Metz**

Laboratoire de Mathématiques et Applications de Metz, Ile du Saulcy, F-57045 Metz Cedex 1

Institut für Mathematik, Fakultät EIM, Universität Paderborn, Warburger Str. 100, D-33098 Paderborn

Abstract

Motivated by the representation-theoretic notion of Howe duality, we seek an analogous construction in symplectic geometry in the sense that its geometric quantization decomposes in a Howe dual fashion.

We find that in the symplectic context, the correct setting is given by two Lie groups acting on a symplectic manifold when these two actions commute and satisfy the *symplectic Howe condition*, i. e., these actions are Hamiltonian and their collective functions are their mutual centralizers in the Poisson algebra of smooth functions on the symplectic manifold. Once this condition is satisfied, we can describe the orbit structure in detail. In particular, there is a bijection between the coadjoint orbits in one moment image and those in the other moment image – this bijection is what we call the *coadjoint orbit correspondence*.

We study the coadjoint orbit correspondence further and show, if the acting Lie groups are compact and the symplectic manifold is prequantizable, that it preserves integrality of the coadjoint orbits, so to both coadjoint orbits in the correspondence an irreducible representation can be associated. We thus have a bijection between certain parts of the unitary duals of both Lie groups acting on the symplectic manifold. Applying known results about the interchangeability of quantization and reduction, we see that for a Kähler manifold, its quantization (as a representation of the product of both groups acting on the manifold) decomposes into a multiplicity-free direct sum of tensor products of irreducibles of the individual groups, the pairs being given by the bijection obtained before – as one would expect according to Howe duality.

This main result is accompanied by a study of the local structure of a manifold carrying two commuting Hamiltonian action which proves a local version of the orbit correspondence and by a discussion about the relation of the coadjoint orbit correspondence to the generalized symplectic leaf correspondence in singular dual pairs.

Zusammenfassung

Motiviert durch den darstellungstheoretischen Begriff der Howe-Dualität, suchen wir eine analoge Konstruktion in der symplektischen Geometrie. Analog bedeutet hierbei, dass die geometrische Quantisierung eine Zerlegung mit Howe-Dualität besitzen soll.

Wir stellen fest, dass die im symplektischen Kontext korrekte Situation gegeben ist durch zwei Lie-Gruppen, die auf derselben symplektischen Mannigfaltigkeit wirken, wenn diese Wirkungen kommutieren und die symplektische Howe-Bedingung erfüllen, d. h. beide Wirkungen sind Hamiltonsch und die kollektiven Funktionen beider Wirkungen sind gegenseitig ihre Zentralisatoren in der Poisson-Algebra der glatten Funktionen auf der symplektischen Mannigfaltigkeit. Ist diese Bedingung erfüllt, dann sind wir in der Lage, die Bahnenstruktur detailliert zu beschreiben und zu zeigen, dass eine Bijektion zwischen den koadjungierten Bahnen im Bild der ersten Impulsabbildung und denen im Bild der zweiten Impulsabbildung existiert – es ist diese Bijektion, die wir im folgenden als *Korrespondenz koadjungierter Bahnen* bezeichnen.

Wir setzen die Untersuchung der Korrespondenz koadjungierter Bahnen fort und zeigen, dass für Wirkungen kompakter Lie-Gruppen auf präquantisierbaren symplektischen Mannigfaltigkeiten die Integralität der koadjungierten Bahnen erhalten bleibt, und daher beiden koadjungierten Bahnen gleichzeitig irreduzible Darstellungen zugeordnet werden können. Somit besteht eine Bijektion zwischen bestimmten Teilmengen der unitären Duale beider auf der symplektischen Mannigfaltigkeit wirkenden Lie-Gruppen. Wendet man nun bekannte Resultate über die Vertauschbarkeit von Quantisierung und symplektischer Reduktion an, dann erkennen wir, dass die Quantisierung einer Kähler-Mannigfaltigkeit (betrachtet als Darstellung des Produktes beider auf der Mannigfaltigkeit wirkender Gruppen) in eine multiplizitätenfreie direkte Summe von Tensorprodukten der irreduziblen Darstellungen beider Gruppen zerfällt, wobei die Paare durch die zuvor beschriebene Bijektion gegeben sind – wie man es im Sinne der Howe-Dualität erwartet.

Dieses Hauptresultat wird begleitet von der Untersuchung der lokalen Struktur einer Mannigfaltigkeit, auf der zwei Hamiltonsche Wirkungen gegeben sind, die eine lokale Version der Bahnenkorrespondenz liefert, sowie von einer Betrachtung der Beziehung der Korrespondenz koadjungierter Bahnen zur Korrespondenz verallgemeinerter symplektischer Blätter in singulären dualen Paaren.

Résumé

Motivé par la dualité de Howe dans la théorie des représentations de groupes de Lie, on cherche une construction analogue en géométrie symplectique, c'est-à-dire on souhaite que sa quantification géométrique décompose de manière Howe-duale.

On trouve que dans le contexte symplectique, le cadre correct est donné par deux groupes de Lie agissant sur la même variété symplectique si ces actions commutent et satisfont la *condition de Howe symplectique*, i. e., ces actions sont hamiltoniennes et leurs fonctions collectives sont leurs centralisateurs mutuelles dans l'algèbre de Poisson des fonctions lisses sur la variété symplectique. Une fois cette condition est remplie, nous pouvons décrire la structure d'orbites en détail. En particulier, il y a une bijection entre les orbites coadjointes dans une image d'application moment et celles dans l'image de l'autre application moment – or, il est cette bijection que nous appelerons la *correspondance d'orbites coadjointes*.

On poursuit l'étude de la correspondance d'orbites coadjointes et on montre que, si les groupes de Lie qui agissent sont compacts et la variété symplectique est préquantifiable, l'integralité est préservée par la correspondance. Ainsi, il est possible d'associer en même temps des représentations irréductibles aux deux orbites de la correspondance. Donc, nous avons une bijection entre certaines parties des duaux unitaires des deux groupes de Lie qui agissent sur la variété symplectique. En appliquant des résultats connus qui assurent que la quantification et la réduction commutent, nous constatons que la quantification d'une variété kählerienne (vue comme une représentation du produit des deux groupes qui agissent sur la variété) admet une décomposition en somme direct sans multiplicités de produits tensoriels des représentations irréductibles des deux groupes, les paires étant données par la bijection obtenue précédemment – parfaitement en accord avec la dualité de Howe.

Ce résultat principal est accompagné par l'étude de la structure locale d'une variété avec deux actions hamiltoniennes qui commutent, ce qui donne une version locale de la correspondance d'orbites, ainsi que par des réflexions sur la relation entre la correspondance d'orbites coadjointes et la correspondance de feuilles symplectiques généralisées dans des paires duales singulières.

Danksagung

Diese Dissertation wäre nicht entstanden ohne den Einsatz meines *directeur de thèse* für das IRTG im allgemeinen und ohne seine mathematische und nicht-mathematische Unterstützung meiner Arbeit im besonderen, weshalb ich ihm, Tilmann Wurzbacher, zu besonderem Dank verpflichtet bin. Ebenso danke ich Joachim Hilgert als meinem Ko-Betreuer und Sprecher des IRTG in Paderborn.

My thanks go to Alan T. Huckleberry and Juan-Pablo Ortega for accepting to be referees for this thesis, and to the other members of the jury as well.

Für das freundschaftliche und motivierende Umfeld, in dem diese Arbeit entstand, danke ich allen, die das IRTG mit Leben gefüllt und ihm seinen internationalen Charakter gegeben haben.

Dafür, in dieser wechselvollen Zeit meine Launen ertragen und zu mir gehalten zu haben, danke ich Elodie, danke ich meinen Eltern – und meinen Großeltern, deren Rückhalt mich bis zum heutigen Tag fortwährend begleitet hat.

Merci!

Carsten Balleier
11. Mai 2009

Contents

(revised version of 18th August 2009)

1 Introduction

Howe's duality [How89] is a well-studied notion in the theory of reductive Lie groups. It is based on the notion of Howe dual pairs, which specialize the concept of a double commutant to Lie group theory.

Definition 1.1. Given a real reductive Lie group G. A *(Howe) dual pair* in G is a pair (G_1, G_2) of two real reductive Lie subgroups of G which are their mutual centralizers in G, i. e.,

$$Z_G(G_1) = G_2 \text{ and } Z_G(G_2) = G_1.$$

For certain of these pairs and certain representations (ϱ, V) of the large group G there is a decomposition of the following type:

$$V \cong \bigoplus_{[V_\alpha]\in\mathcal{D}} V_\alpha \otimes W_\alpha,$$

where V_α represents a class $[V_\alpha]$ in $\mathcal{D}$ and W_α represents the class $\Lambda([V_\alpha])$, $\mathcal{D}$ is the set of occurring irreducible representations of G_1, and the occurring irreducible representations of G_1 and G_2 are in bijection via an injective map $\Lambda : \mathcal{D} \to \Lambda(\mathcal{D}) \subseteq \widehat{G_2}$, this bijection being called *Howe duality*. However, it is not known in general which representations of which groups admit such a decomposition. For real reductive groups, results have been obtained on a case-by-case basis. The most prominent dual pair is $(Sp(n,\mathbb{R}), O(k))$ in $Sp(nk,\mathbb{R})$ together with the Howe dual decomposition of the metaplectic (or oscillator) representation. Examples for this notion arise from physics [How85].

In this thesis, the question is studied how Howe dual representations emerge via geometric quantization. The starting point (in section 3) is a symplectic manifold (M,ω) on which two Lie group actions (of G_1 and G_2) are given that commute with each other. If both actions are Hamiltonian with equivariant moment maps $\Phi_i : M \to \mathfrak{g}_i^*$, $i = 1, 2$, their moment components Poisson-commute, and thus

$$\Phi_i^* C^\infty(\mathfrak{g}_i^*) \subseteq Z_{C^\infty(M)}(\Phi_j^* C^\infty(\mathfrak{g}_j^*))$$

for $i + j = 3$.

Besides the very important technical requirement that both actions need to be proper, we impose at first the following condition on the orbits:

$$(\mathfrak{g}_i \cdot z)^{\angle} = \mathfrak{g}_j \cdot z \qquad (i + j = 3),$$

i. e., in a point $z \in M$ of the manifold, the tangent spaces of both orbits are mutually their symplectic complement. For points satisfying this condition, the symplectic slice theorem can be specialized, hence we can describe very explicitly local models of the orbits and the normal forms of the moment maps. Based on this explicit description, one observes a bijection between the coadjoint orbits lying in the image of one moment map in normal form and those lying in the image of the other moment map in normal form (see Lemma 3.11 and comments thereafter).

In order to obtain a global correspondence, we invoke (in section 3.2) the classical (non-singular) dual pair notions of symplectic geometry. We will remove all (implicit) genericity assumptions in these notions and use but the essential conditions in there. In particular, the symplectic Howe condition

$$Z_{C^\infty(M)}(\Phi_j^* C^\infty(\mathfrak{g}_j^*)) = \Phi_i^* C^\infty(\mathfrak{g}_i^*) \qquad (i+j=3)$$

will be used and allows, if satisfied, to describe explicitly all level sets of both moment maps. The preceding condtion is the most natural candidate for the "dequantization" of the double commutant condition in representation theory (see our Lemma 5.3). From the form of the level sets, we deduce our central result, the *correspondence theorem for coadjoint orbits* (Thm. 3.26). We do not only obtain a bijection between the coadjoint orbits in the moment images, i. e., $\Lambda : \Phi_1(M)/G_1 \to \Phi_2(M)/G_2$, but also symplectomorphims between the reduced spaces constructed using one moment map and the coadjoint orbits in the image of the other moment map, i. e., $M_{\alpha_1} \cong \Lambda(\mathcal{O}_{\alpha_1})$ and $M_{\alpha_2} \cong \Lambda^{-1}(\mathcal{O}_{\alpha_2})$.

We have obtained our coadjoint orbit correspondence from the symplectic Howe condition without special genericity considerations, in contrast to the symplectic leaf correspondence for classical dual pairs [OR04, Thm. 11.1.9]. This is due to the fact that we dispose of far more structure in our setting of proper Hamiltonian actions compared to the setting in which the symplectic leaf correspondence is shown. Alternatively, our correspondence can also be seen as a consequence of the generalized symplectic leaf correspondence for singular dual pairs (see Cor. 4.16.1). This is described in detail in section 4.

Eventually, in section 5, we come to quantizing the $(G_1 \times G_2)$-action on (M, ω) (which makes us assume from here on that ω lies in an integral de Rham class and that both groups are compact). Our first result in this section is that the orbit correspondence preserves integrality of the coadjoint orbits, so it makes sense to quantize (M, ω) and to ask for a decomposition of the so-obtained $(G_1 \times G_2)$-representation into the irreducibles of G_1 and G_2, thought of as the quantizations of the coadjoint orbits in the moment images. In Thm. 5.17, we actually get a Howe dual decomposition of the geometric quantization of (M, ω) in case that M is kählerian. Precisely, we obtain that

$$\Gamma_{\mathrm{hol}}(M, L) \cong \bigoplus_{\alpha_1 \in \Phi_1(M) \cap \mathfrak{t}_{\mathbb{Z}}^+} \Gamma_{\mathrm{hol}}(\mathcal{O}_{\alpha_1}, L_{\alpha_1}) \otimes \Gamma_{\mathrm{hol}}(\mathcal{O}_{\alpha_2}, L_{\alpha_2}),$$

where $\mathcal{O}_{\alpha_2} = \Lambda(\mathcal{O}_{\alpha_1})$ and $\mathfrak{t}_{\mathbb{Z}}^+$ is the set of integral points of a fixed Weyl chamber of $\mathfrak{g}_1$. In order to show this, we apply results about the interchangeability of quantization and reduction.

The remaining sections illustrate the orbit correspondence on natural examples of commuting Hamiltonian actions of Lie groups.

2 Reminder on Symplectic Geometry and Hamiltonian Actions

This section collects standard facts on differentiable manifolds, foliations, symplectic and Poisson geometry, proper group actions, standard moment maps, slices and normal forms, and the preservation of slices by a moment map, which will be used throughout the remainder of this thesis.

2.1 Differential Geometric Conventions

For the convenience of the reader and to avoid ambiguities about certain conventions, some definitions are fixed in this section. Notation follows often, but not always, those which are used in [OR04].

Unlike in [OR04], the term *manifold* stands for differentiable manifolds which are always assumed to be smooth (C^∞), finite-dimensional, Hausdorff and paracompact without stating this every time.

Let M and N be manifolds. Any smooth map $f : M \to N$ admits a derivative which is a bundle map, defined between the tangent bundles, $Tf : TM \to TN$. Point-wise, one can consider the dual $T_z^* f : T_{f(z)}^* N \to T_z^* M$. Explicitly, these maps are given as follows: Take a vector $X \in T_z M$ at a point $z \in M$ which is represented by a curve $c_X : (-T, T) \to M$ with $T \in \mathbb{R}_+$, $c_X(0) = z$ and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 c_X = X$. Then define

$$T_z f : T_z M \to T_{f(z)} N, \qquad T_z f(X) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 f(c_X(t)),$$

and the dual map ($\alpha \in T_{f(z)}^* N$)

$$T_z^* f : T_{f(z)}^* N \to T_z^* M, \qquad \langle T_z^* f(\alpha), X \rangle = \langle \alpha, T_z f(X) \rangle.$$

A map $f : M \to N$ between two manifolds is called a *submersion* or *immersion* at a point $z \in M$ if $T_z f$, its tangent map in that point, is surjective or injective, respectively. If the respective property holds everywhere, the specification of the point is omitted. The dimension of the image of $T_z f$ is called the *rank* of the map f in the point z. Maps of constant rank have convenient properties which generalize submersions and immersions.[1] For later use, we record the following theorem (see, e. g., [Hel78, Thm. 15.5]).

Theorem 2.1. *Let M and N be manifolds of dimensions m and n, and $f : M \to N$ be a smooth map. Suppose that f has constant rank k in a neighbourhood of a point $z \in M$. Then there exist local charts $\xi : U_z \to \mathbb{R}^m$ ($z \in U_z \subseteq M$) and $\eta : U_{f(z)} \to \mathbb{R}^n$ ($f(z) \in U_{f(z)} \subseteq N$) such that $\xi(z) = 0$ and $\eta(f(z)) = 0$; further*

$$\eta \circ f \circ \xi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0).$$

In particular, $f(U_z)$ is a k-dimensional submanifold of N.

[1] Maps of constant rank are also called *subimmersions* in [AMR88, Ch. 3.5].

Let $\mathcal{X}(M) = \Gamma^\infty(TM)$ denote the smooth vector fields on a manifold M and $\Omega^k(M) = \Gamma^\infty(\bigwedge^k(T^*M))$ the smooth k-forms on M. The symbol $\lrcorner$ is used to denote the contraction of a vector field with a form: $X \lrcorner\, \omega = \omega(X, \ldots)$ for $X \in \mathcal{X}(M)$ and $\omega \in \Omega^k(M)$ for any $k \in \mathbb{N}$.

Another notation that will be occasionally used but which is not completely standard is G^0 to denote the connected component of a Lie group G which contains the identity.

Foliations and distributions. Let M be a manifold. A smooth vector subbundle E of the tangent bundle TM is called a *distribution*. One calls E *involutive* if for any open subset $U \subseteq M$ and any vector fields $X, Y \in \Gamma^\infty(U, E)$, their commutator $[X, Y]$ lies again in $\Gamma^\infty(U, E)$. It is called *integrable* if for every point $z \in M$ there exists a local submanifold $N \subseteq M$ containing z such that $TN = E_{|N}$, a *local integral manifold of E at z*. By the (local) Frobenius theorem, E is involutive if and only if it is integrable.

Let $\mathcal{L} = \{\mathcal{L}_\alpha\}_{\alpha \in A}$ be a partition of M into *leaves*, i. e., $M = \bigcup_{\alpha \in A} \mathcal{L}_\alpha$ and $\mathcal{L}_\alpha \cap \mathcal{L}_\beta = \emptyset \, \forall \alpha, \beta \in A, \alpha \neq \beta$. A partition into leaves $\mathcal{L}$ is called a *foliation* if for any point $z \in M$, there exists a chart (U, φ_U), $z \in U$, with

$$\varphi_U : U \to U' \times V' \subseteq E \times F$$

such that for any $\alpha \in A$, the connected components $(U \cap \mathcal{L}_\alpha)^i$ of $U \cap \mathcal{L}_\alpha$ are given by $\varphi_U((U \cap \mathcal{L}_\alpha)^i) = U' \times \{c^i_\alpha\}$, where $c^i_\alpha \in V' \subseteq F$ are constants and E and F appropriate vector spaces.

Define the tangent bundle of a foliation to be $T(M, \mathcal{L}) = \bigcup_{\alpha \in A} \bigcup_{z \in \mathcal{L}_\alpha} T_z\mathcal{L}_\alpha$. The global version of Frobenius theorem says that a distribution E is involutive if and only if there exists a foliation $\mathcal{L}$ on M such that $E = T(M, \mathcal{L})$. We denote by $M/\mathcal{L}$ the space of leaves with its quotient topology.

The framework we have just established requires all leaves of the distribution to be of the same dimension – this being too restrictive, in general, the notion of a *generalized foliation* will be introduced. Given a partition $\mathcal{L}$ of a manifold into leaves (indexed by $\alpha \in A$), $\mathcal{L}$ is called foliated in a general sense if at any point $z \in M$, there exists a chart (U, φ_U) around z ($\varphi_U : U \to W \subseteq \mathbb{R}^m$) such that for any $\alpha \in A$, there is a positive integer $n_\alpha \leq m$ (the dimension of the leaf $\mathcal{L}_\alpha$) and a subset $A_\alpha \subseteq \mathbb{R}^{m-n_\alpha}$ satisfying

$$\varphi_U(U \cap \mathcal{L}_\alpha) = \{(z_1, \ldots, z_m) \in W \mid (z_{n_\alpha+1}, \ldots, z_m) \in A_\alpha\}.$$

Each $(z^i_{n_\alpha+1}, \ldots, z^i_m) \in A_\alpha$ determines a connected component $(U \cap \mathcal{L}_\alpha)^i$ of $U \cap \mathcal{L}_\alpha$.

A leaf $\mathcal{L}_\alpha$ of a generalized foliation is called *regular* if there exists an open neighbourhood of $\mathcal{L}_\alpha$ intersecting only leaves of the same dimension and *singular* otherwise. The set of points belonging to regular leaves is open and dense in M.

The same idea which lead us to generalized foliations justifies the definition of *generalized distributions* as a subset $D \subseteq TM$ such that at any point $z \in M$ the intersection $D(z) \cap T_zM$ is a linear subspace of T_zM. D is called differentiable if for any $z \in M$ and any $v \in D(z)$, there exists an open neighbourhood U of z and a section $X \in \Gamma^\infty(U, TM)$ taking its values in D such that $v = X_z$. Further, D is called *completely integrable* if for any $z \in M$, there is an integral manifold of D containing z which is of maximal dimension. If D is invariant under the local flows associated to the differentiable sections of D, it is called *involutive*. Finite compositions of these flows starting at $z \in M$ define the *accessible set* of D through

z, which are at the same time the *maximal integral manifolds*. They constitute a generalized foliation.

2.2 Hamiltonian Actions on Symplectic Manifolds

This section recalls the basic notions of symplectic geometry (mainly based on [OR04, Ch. 4.1]).

Definition 2.2. A *symplectic vector space* is a pair (V, ω) where V is a vector space and ω a non-degenerate antisymmetric bilinear form on V. A *symplectic manifold* is a pair (M, ω) where M is a manifold and ω a *symplectic* form, i. e., ω is a closed ($\mathrm{d}\omega = 0$) non-degenerate 2-form on M. For every $z \in M$, the tangent space at z, together with the symplectic form at z, (T_zM, ω_z), is a symplectic vector space. A symplectic manifold is called *exact* if there is a 1-form ϑ such that $\omega = -\mathrm{d}\vartheta$.

For a linear subspace W of a symplectic vector space (V, ω), one commonly uses the following defintions.

Definition 2.3. The *symplectic complement* of W in V is the subspace

$$W^{\angle} = \{v \in V \mid \omega(v, w) = 0 \,\forall w \in W\}.$$

Remark 2.4. The term *complement* is justified by the fact that for the dimension of the symplectic complement $W^{\angle}$ of W in V, one has the relation

$$\dim W + \dim W^{\angle} = \dim V.$$

However, unlike for the orthogonal complement (i. e., a complement taken w. r. t. a symmetric positive-definite bilinear form), we do in general not have $W \cap W^{\angle} = \{0\}$.[2]

Definition 2.5. The subspace W is called *isotropic* if $W \subseteq W^{\angle}$. It is called *coisotropic* if $W^{\angle} \subseteq W$. A subspace which is isotropic and coisotropic is called *Lagrangian* (i. e., $W = W^{\angle}$). A subspace W is called *symplectic* if the restricted form $\omega_{|W \times W}$ is non-degenerate.

These notions carry over to submanifolds by requiring all tangent spaces to have the corresponding property as a subspace.

For two subspaces with complementary dimension, one can prove:

Lemma 2.6. *Let (V, ω) be a symplectic vector space containing the linear subspaces W_1, W_2. Suppose $W_1 \subseteq W_2^{\angle}$ and $\dim W_1 + \dim W_2 = \dim V$. This already implies $W_1 = W_2^{\angle}$.*

Proof. Suppose we have a strict inclusion $W_1 \subset W_2^{\angle}$, hence $\dim W_1 < \dim W_2^{\angle}$. Then by $\dim W_1 = \dim V - \dim W_2 = \dim W_2^{\angle}$, we have a contradiction. □

Note that by its non-degeneracy, the symplectic form on V defines an isomorphism between V and its dual space V^*, which justifies the following definition on a symplectic manifold (M, ω).

[2] At this point, notions vary in the literature: The symplectic complement is also called skew-complement, symplectic orthogonal complement, ω-perpendicular space, etc.

Definition 2.7. A vector field $X \in \mathcal{X}(M)$ is called *Hamiltonian* if the form $X \lrcorner\, \omega$ is exact, i. e., there exists a function $f \in C^\infty(M)$ such that $X \lrcorner\, \omega = \mathrm{d}f$. Conversely, given a function $f \in C^\infty(M)$, there is the *Hamiltonian vector field* X_f *of* f, which is uniquely determined by $X_f \lrcorner\, \omega = \mathrm{d}f$. A vector field X is called *symplectic* or *locally Hamiltonian* if $\mathcal{L}_X\omega = 0$ or, equivalently, $X \lrcorner\, \omega$ is closed. The set of Hamiltonian vector fields on (M, ω) will be denoted by $\mathcal{X}_{\mathrm{H}}(M, \omega)$, the set of locally Hamiltonian vector fields by $\mathcal{X}_{\mathrm{LH}}(M, \omega)$ (ω may be omitted if clear from the context).

One has the property $[\mathcal{X}_{\mathrm{LH}}(M, \omega), \mathcal{X}_{\mathrm{LH}}(M, \omega)] = \mathcal{X}_{\mathrm{H}}(M, \omega)$, since $[X, Y] \lrcorner\, \omega = \mathrm{d}(\omega(X, Y))$ for $X, Y \in \mathcal{X}_{\mathrm{LH}}(M, \omega)$.

On a symplectic manifold (M, ω), consider now a smooth action $\Psi : G \times M \to M, (g, z) \mapsto \Psi_g(z)$ of a Lie group G. If there is no risk of confusion, we will also write $g \cdot z$ for $\Psi_g(z)$ and consequently, $G \cdot z$ for the whole G-orbit under this action. The fundamental vector fields of the G-action are given, for any $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$, by $\xi^M_z = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \exp(t\xi) \cdot z$ (sometimes denoted differently to be clear about the action, e. g., ξ^Ψ); the space of fundamental vectors for this action at z will be denoted by $\mathfrak{g} \cdot z = \{\xi^M_z \mid \xi \in \mathfrak{g}\}$. Two special actions to occur are the left and right action of G on itself, induced by $L_h : G \to G, g \mapsto hg$ and $R_h : G \to G, g \mapsto gh$.

Definition 2.8. An action is called *symplectic* if it preserves the symplectic form, i. e., for all $g \in G$, $\Psi^*_g\omega = \omega$ holds. A symplectic action is called *Hamiltonian* if all fundamental vector fields are Hamiltonian, i. e., $\xi^M \lrcorner\, \omega = \mathrm{d}\Phi^\xi$, for some function Φ^ξ. For a Hamiltonian action, a smooth map $\Phi : M \to \mathfrak{g}^*$ exists which is given by $\langle \Phi, \xi \rangle = \Phi^\xi$ for all $\xi \in \mathfrak{g}$; it is called a *(standard) moment map.* A moment map is called *equivariant* if $\mathrm{Ad}^*(g)\Phi = \Phi \circ \Psi_g$ holds for all $g \in G$.

Remark. Here, Ad^* denotes the coadjoint action of G on $\mathfrak{g}^*$, the dual of its Lie algebra, i. e., $\mathrm{Ad}^*(g) = [\mathrm{Ad}(g^{-1})]^*$. With this choice, we differ from the conventions in [OR04].

Note that for an exact symplectic manifold with a 1-form ϑ that is invariant under the group action, $\mathrm{d}\Phi^\xi = \xi^M \lrcorner\, \omega = -\xi^M \lrcorner\, \mathrm{d}\vartheta = \mathrm{d}(\xi^M \lrcorner\, \vartheta)$ (by invariance of ω under the group action), and thus, up to a constant, $\Phi^\xi = \xi^M \lrcorner\, \vartheta$ is (up to constant) the moment map for the G-action on this manifold.

Observe that if an action is Hamiltonian, there exists a $\mathbb{R}$-linear map $\lambda : \mathfrak{g} \to C^\infty(M)$ which makes the following diagram commute (note that the upper line would be an exact sequence of Lie algebra homomorphisms, for $C^\infty(M)$ equipped with its Poisson-Lie structure, if ϱ was defined to map $f \mapsto -X_f$; see also the next subsection):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^\infty(M) & \xrightarrow{\varrho} & \mathcal{X}_H(M, \omega) \\ & & & & & \nwarrow^{\lambda} & \uparrow \tau \\ & & & & & & \mathfrak{g} \end{array} \tag{2.1}$$

Here, $\varrho : f \mapsto X_f$ assigns to f its Hamiltonian vector field and $\tau : \xi \mapsto \xi^M$ gives the fundamental vector field. So, we have $\tau = \varrho \circ \lambda$. Explicitly, λ maps ξ to the corresponding momentum component Φ^ξ.

In general, the map λ is not a Lie algebra homomorphism, but

$$\{\lambda(\xi_1), \lambda(\xi_2)\} = \lambda([\xi_1, \xi_2]) - \Sigma(\xi_1, \xi_2) \qquad \forall \xi_1, \xi_2 \in \mathfrak{g}, \tag{2.2}$$

where $\Sigma \in Z^2(\mathfrak{g}, \mathbb{R})$ is a two-cocycle. This is proved, together with further identities satisfied by Σ, in [OR04, Thm. 4.5.25].

In order to state basic properties of the moment map, one needs two further definitions.

Definition 2.9. Let $\mathfrak{h}$ be a Lie subalgebra of a Lie algebra $\mathfrak{g}$. The *annihilator* of $\mathfrak{h}$ in $\mathfrak{g}^*$ is the subspace

$$\mathfrak{h}^\circ = \{\alpha \in \mathfrak{g}^* \mid \alpha(\xi) = 0\, \forall \xi \in \mathfrak{h}\}.$$

Definition 2.10. For a smooth action of a Lie group G on a smooth manifold M, the *stabilizer* of a point $z \in M$ under the action of G is given by

$$\mathrm{Stab}_G(z) = \{g \in G \mid g \cdot z = z\},$$

often simply denoted by G_z. The stabilizer is a Lie subgroup of G. Its Lie algebra is given by

$$\mathrm{Lie}(\mathrm{Stab}_G(z)) = \mathrm{Stab}_{\mathfrak{g}}(z) = \mathfrak{g}_z = \big\{\xi \in \mathfrak{g} \mid \xi_z^M = 0\big\}.$$

Proposition 2.11. *Let $\Phi : M \to \mathfrak{g}^*$ be a moment map of the Hamiltonian G-action on the symplectic manifold (M, ω). Then*

$$\ker T_z\Phi = (T_z(G \cdot z))^{\angle}$$

is the symplectic complement of $T_z(G \cdot z)$ in T_zM and

$$\mathrm{im}\, T_z\Phi = (\mathrm{Lie}(\mathrm{Stab}_G(z)))^\circ$$

is the annihilator of the stabilizer Lie algebra of z in $\mathfrak{g}$, for any $z \in M$.

Proof. From the definition of a moment map, one sees

$$\ker T_z\Phi = \big\{X \in T_zM \mid \omega(\xi_z^M, X) = 0\, \forall \xi \in \mathfrak{g}\big\}.$$

The tangent space of the G-orbit at z is $T_z(G \cdot z) = \mathfrak{g} \cdot z = \big\{\xi_z^M \mid \xi \in \mathfrak{g}\big\}$, thus its symplectic complement in T_zM is the kernel of $T_z\Phi$.

The proof of the second statement is immediate, too. See also [OR04, Prop. 4.5.12] for a more general result. □

2.3 Some Facts from Poisson Geometry

Another structure, which exists on any symplectic manifold (M, ω), is its Poisson structure. However, we will in some cases need to deal with manifolds that only have a Poisson structure. Thus the definition will be given in general.

Definition 2.12. A pair $(M, \{\cdot,\cdot\})$ consisting of a manifold M and a bilinear map (the *Poisson bracket*) $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is called a Poisson manifold if it makes $C^\infty(M)$ into a Lie algebra and is a derivation, i. e.,

$$\{fg, h\} = f\{g, h\} + \{f, h\}g \quad \forall f, g, h \in C^\infty(M).$$

The canonical Poisson structure on a symplectic manifold (M, ω) is given by

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M), \quad (f, g) \mapsto \omega(X_f, X_g).$$

In the Poisson context, a Hamiltonian vector field X_f is assigned to a function $f \in C^\infty(M)$ via

$$X_f = \{\cdot, f\}.$$

This assignment is a Lie algebra antihomomorpism, i. e.,

$$X_{\{f,g\}} = -[X_f, X_g] \quad \forall f, g \in C^\infty(M).$$

The Poisson bracket depends on its arguments, say f and g, only through $\mathrm{d}f$ and $\mathrm{d}g$, hence one can define the *Poisson tensor field* B by

$$B(\mathrm{d}f, \mathrm{d}g) = \{f, g\},$$

which induces further a vector bundle map $B^\sharp : T^*M \to TM$ given by $B_z(\alpha_z, \beta_z) = \left\langle \alpha_z, B^\sharp_z(\beta_z) \right\rangle$, for any $\alpha, \beta \in T^*M$ and using the natural pairing of TM and T^*M (written here $\langle \cdot, \cdot \rangle$). The image $D = B^\sharp(T^*M)$ of this map is a generalized distribution, its dimension $\dim D_z$ at any point $z \in M$ as a linear subspace of T_zM is called the *rank* of $(M, \{\cdot, \cdot\})$ at z.

An important feature of Poisson manifolds is the fact that they admit a decomposition into *symplectic leaves*, i. e., into submanifolds carrying a symplectic structure induced from the Poisson structure via the characteristic distribution D. This is made precise in the following theorem (see, e. g., [OR04, Thm. 4.1.28]).

Theorem 2.13. *Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and D its characteristic distribution. Then D is a smooth and integrable generalized distribution and its maximal integral leaves form a generalized foliation decomposing M into symplectic submanifolds $\mathcal{L}$, whose symplectic structure is the unique one which makes the inclusion $i : \mathcal{L} \to M$ into a Poisson map.*

A *Poisson map* (or *canonical map*) is a smooth map which preserves the Poisson structures, i. e., $\varphi : M \to N$ such that $\varphi^*\{f, g\}_N = \{\varphi^* f, \varphi^* g\}_M$ for all $f, g \in C^\infty(N)$.

Example 2.14. For our topic, one example of a Poisson manifold will play a central role: the dual $\mathfrak{g}^*$ of a Lie algebra. The Poisson structure on $\mathfrak{g}^*$ is given by either of

$$\{f, g\}(\alpha) = \pm \langle \alpha, [\delta_\alpha f, \delta_\alpha g] \rangle$$

at $\alpha \in \mathfrak{g}^*$ for $f, g \in C^\infty(\mathfrak{g}^*)$. Here, to f has been assigned the element $\delta_\alpha f \in \mathfrak{g}$ by $\langle \beta, \delta_\alpha f \rangle = T_\alpha f(\beta)$ for any $\beta \in \mathfrak{g}^*$ and identifying $T_\alpha \mathfrak{g}^* \cong \mathfrak{g}^*$.

We will always use the Poisson bracket defined with $+$.

One shows that Hamiltonian vector fields with respect to this Poisson structure are always tangent to coadjoint orbits, thus the symplectic leaves of $\mathfrak{g}^*$ are the connected components of the coadjoint orbits with the so-called KKS symplectic structure [OR04, Thm. 4.5.31].

2.4 Symplectic Slice Theorem

This section will review the Witt-Artin decomposition, the symplectic slice theorem and the Marle-Guillemin-Sternberg normal form of the moment map, following [OR04, Ch. 7].

Proper Actions. We start by recalling the notions of a proper Lie group action and of slices and tubes for such an action, before actually discussing them in the symplectic context. This will only include the essential statements which are needed afterwards; more extensive treatments are in [OR04, Ch. 2.3] or [DK00, Ch. 2]. One has to keep in mind that in the literature, the definitions of a proper action may differ from the one we use; these differences are discussed in [Bil04].

Definition 2.15. Let G be a Lie group acting on the manifold M via the map $\Psi : G \times M \to M$. We call Ψ a *proper action* whenever the map $G \times M \ni (g, z) \mapsto (z, \Psi_g(z)) \in M \times M$ is proper. This is equivalent to: For any two convergent sequences $\{z_n\}$ and $\{g_n \cdot z_n\}$ in M, there exists a convergent subsequence $\{g_{n_k}\}$ in G. The action is called *proper at* $z \in M$ if this holds for all sequences $\{z_n\}$ and $\{g_n \cdot z_n\}$ converging to z.

Actions of a compact group G are always proper because every sequence in G has a convergent subsequence.

A first impression of the importance of proper actions is the following proposition which lists some of their features (details can be found in [OR04, Prop. 2.3.8] or [DK00, Prop. 2.5.2], among others).

Proposition 2.16. *Let G be Lie group which acts properly on the manifold M. Then:*

(i) For any $z \in M$, the stabilizer G_z is compact.

(ii) The orbit space M/G is a Hausdorff topological space.

(iii) Let N be any G-invariant subset of M and let $f \in C^\infty(M)$ be such that the restriction $f_{|N}$ is constant on each G-orbit. Then there is a smooth G-invariant function $F \in C^\infty(M)^G$ satisfying $F_{|N} = f_{|N}$.

(iv) M admits a smooth G-invariant Riemannian structure.

For a proper symplectic G-action on (M, ω), the G-invariant Riemannian structure may be chosen in a way which is compatible with the symplectic structure.

Proposition 2.17. *Let (M, ω) be a symplectic manifold with a proper symplectic G-action. Then there exists an almost complex structure $J : TM \to TM$ (i. e., $J^2 = -\mathrm{id}$) which is G-equivariant and compatible with the symplectic form ω, i. e., there is a G-invariant Riemannian metric given by $\langle X, Y\rangle = \omega(X, JY)\, \forall X, Y \in \mathcal{X}(M)$.*

Proof. For any manifold with proper action, there exists a smooth G-invariant Riemannian metric by the preceding proposition. Choose an arbitrary G-invariant metric on M and denote it by $\langle\cdot,\cdot\rangle'$. By the non-degeneracy of the metric and the symplectic form ω, there is further a map $A : TM \to TM$, invertible for any $z \in M$, such that

$$\langle X, Y\rangle' = \omega(X, AY) \quad \forall X, Y \in \mathcal{X}(M).$$

Additionally, A is smooth and G-equivariant resulting from the corresponding properties of the metric and the symplectic form: On the one hand,

$$\omega_{g\cdot z}(T_z\Psi_g(\cdot), T_z\Psi_g \circ A_z(\cdot)) = \omega_z(\cdot, A_z\cdot)$$

and on the other,

$$\omega_{g\cdot z}(T_z\Psi_g(\cdot), A_z \circ T_z\Psi_g(\cdot)) = \langle T_z\Psi_g(\cdot), T_z\Psi_g(\cdot)\rangle'_{g\cdot z} = \langle\cdot,\cdot\rangle'_z = \omega_z(\cdot, A_z\cdot),$$

hence $A_{g\cdot z} \circ T_z\Psi_g = T_z\Psi_g \circ A_z$.

Fix a point $z \in M$, then $A_z \in GL(T_zM)$ and we can apply polar decomposition (w. r. t. $\langle\cdot,\cdot\rangle'_z$), i. e., we have smooth maps $u : GL(T_zM) \to GL(T_zM)$ and $s : GL(T_zM) \to GL(T_zM)$ with the following properties: The values of u are unitary maps with respect to the chosen metric on T_zM, the values of s are positive definite linear maps, and for all $A_z \in GL(T_zM)$, the identity $A_z = u(A_z) \circ s(A_z)$ holds. Moreover, these maps are uniquely determined by these properties [Pfl01, Thm. A.2.1].

Now we have to check whether $u(A_z)$ and $s(A_z)$ are still G-equivariant. But $A_{g\cdot z} \circ T_z\Psi_g = T_z\Psi_g \circ A_z$ implies

$$u(A_{g\cdot z}) \circ s(A_{g\cdot z}) \circ T_z\Psi_g = T_z\Psi_g \circ u(A_z) \circ s(A_z),$$

which gives, by the uniqueness of u and s,

$$u(A_{g\cdot z}) \circ T_z\Psi_g = T_z\Psi_g \circ u(A_z) \text{ and } T_{g\cdot z}\Psi_{g^{-1}} \circ s(A_{g\cdot z}) \circ T_z\Psi_g = s(A_z).$$

Now, like in the proof of the non-equivariant version of this statement (see, e. g., [Bla02, Thm. 4.3]), one puts $J = u(A)$ and defines a new metric $\langle\cdot,\cdot\rangle$ through the positive definite map $s(A)$. One verifies that $J^2 = -\,\mathrm{id}$ holds and the compatibility between $\langle\cdot,\cdot\rangle$ and ω is satisfied. Note that u and s depend smoothly on the point $z \in M$. □

Slices and Tubes. As a means to describe the structure of a G-manifold locally, slices and tubes are introduced.

Definition 2.18. Let M be a manifold and G a Lie group acting on M. Let $z \in M$ such that the orbit $G \cdot z$ is closed in M. A *tube* around $G \cdot z$ is a G-equivariant diffeomorphism

$$\varphi : G \times_{G_z} S \to U,$$

where U is a G-invariant neighbourhood of $G \cdot z$ in M and S is some manifold on which G_z acts. If S is a submanifold of M containing z such that $G_z \cdot S = S$, then it is called a *slice at* z.

The importance of proper actions lies in the fact that for them, slices do always exist [Pal61, Thm. 2.3.3] (the formulation below taken from [OR04, Thm. 2.3.31]).

Theorem 2.19 (Slice Theorem). *Let M be a manifold and G a Lie group acting properly on M at the point $z \in M$. Then there exists a slice for the G-action at z.*

Remark 2.20. A linear action of a non-compact reductive group on a vector space is not proper as the stabilizer of zero is the whole group and thus not compact. In particular, the coadjoint action is not proper, in general. However, slices may exist under certain conditions, which will be explained at the end of this section in order to relate slices in M to those in the image of a moment map.

For the remainder of the section, let (M, ω) be a symplectic manifold and G a Lie group which acts properly and symplectically on M.

Witt-Artin decomposition. Having established slices in general, the next step is to find slices which are adapted to the symplectic structure. As slices are transverse to the orbits, the restriction of the symplectic form to the tangent spaces of the orbits is considered. Fix a point $z \in M$; for this point, we define the following linear subspace of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$:

$$\mathfrak{k}(z) = \left\{ \xi \in \mathfrak{g} \mid \xi_z^M \in (\mathfrak{g} \cdot z)^{\angle} \right\} \subseteq \mathfrak{g},$$

which contains the Lie algebra $\mathfrak{g}_z = \mathrm{Stab}_{\mathfrak{g}}(z)$. The subspace $\mathfrak{k}(z) \cdot z$ of $\mathfrak{g} \cdot z$ spanned by the fundamental vector fields corresponding to $\mathfrak{k}(z)$ is the degenerate part of the orbit tangent space; indeed, $\mathfrak{k}(z) \cdot z = (\mathfrak{g} \cdot z) \cap (\mathfrak{g} \cdot z)^{\angle}$. We actually have (compare [OR04, Thm. 7.1.1(i)]):

Lemma 2.21. *$\mathfrak{k}(z)$ is a $\mathrm{Ad}(G_z)$-invariant Lie subalgebra of $\mathfrak{g}$.*

Proof. As the subspace property is obvious, it remains to show that $\mathfrak{k}(z)$ is closed under the Lie bracket. By the G-invariance of the symplectic form ω, we have for any $\xi, \eta, \zeta \in \mathfrak{g}$ that

$$0 = (\mathcal{L}_{\xi^M}\omega)(\eta^M, \zeta^M) = \xi^M(\omega(\eta^M, \zeta^M)) - \omega([\xi^M, \eta^M], \zeta^M) - \omega(\eta^M, [\xi^M, \zeta^M]).$$

Inserting this into the definition of the exterior derivative, one obtains

$$\mathrm{d}\omega(\xi^M, \eta^M, \zeta^M) = -\omega([\xi^M, \eta^M], \zeta^M) - \omega([\eta^M, \zeta^M], \xi^M) - \omega([\zeta^M, \xi^M], \eta^M) = 0,$$

the zero being due to the closedness of ω. Applying $[\xi^M, \eta^M] = [\eta, \xi]^M$ and specializing to $\xi, \eta \in \mathfrak{k}(z)$, one sees

$$\omega_z([\xi, \eta]_z^M, \zeta_z^M) = -\omega_z([\eta, \zeta]_z^M, \xi_z^M) - \omega_z([\zeta, \xi]_z^M, \eta_z^M) = 0,$$

the whole expression vanishing because $\xi_z^M, \eta_z^M \in (\mathfrak{g} \cdot z)^{\angle}$ by the definition of $\mathfrak{k}(z)$. Now $\zeta \in \mathfrak{g}$ was arbitrary, so this implies $[\xi, \eta] \in \mathfrak{k}(z)$.

For the invariance, take $g \in G_z, \xi \in \mathfrak{k}(z), \eta \in \mathfrak{g}$ and note

$$\omega_z([\mathrm{Ad}(g)\xi]_z^M, \eta_z^M) = \omega_z(T_z\Psi_g(\xi_z^M), \eta_z^M) = \omega_z(\xi_z^M, [\mathrm{Ad}(g^{-1})\eta]_z^M) = 0,$$

hence $[\mathrm{Ad}(g)\xi]_z^M \in (\mathfrak{g} \cdot z)^{\angle}$. □

With this preparation, we state a modified form of the Witt-Artin decomposition given in [OR04, Thm. 7.1.1].

Theorem 2.22. *Let (M, ω) be a symplectic manifold with a proper and symplectic action of a Lie group G. On M, choose a G-equivariant compatible almost-complex structure $J : TM \to TM$ and the associated G-invariant Riemannian metric $\langle \cdot, \cdot \rangle$. Then for any $z \in M$, there is a direct decomposition of the tangent space at z which is orthogonal w. r. t. this metric:*

$$T_zM = \mathfrak{k}(z) \cdot z \oplus Q_z \oplus V_z \oplus W_z,$$

where the summands are described by the following definitions and properties.

(i) $\mathfrak{k}(z) \cdot z$ and $W_z = J_z(\mathfrak{k}(z) \cdot z)$ are isotropic subspaces of (T_zM, ω_z). W_z is contained in the orthogonal complement of $\mathfrak{g} \cdot z$ in T_zM.

(ii) *Q_z, given as the orthogonal complement of $\mathfrak{k}(z) \cdot z$ in $\mathfrak{g} \cdot z$, is a symplectic subspace of (T_zM, ω_z) and stable under J_z.*

(iii) *Let V_z be the orthogonal complement of W_z in $(\mathfrak{g} \cdot z)^{\perp}$. The linear subspace V_z is a symplectic subspace of (T_zM, ω_z) and stable under J_z. We then have*

$$(\mathfrak{g} \cdot z)^{\angle} = \mathfrak{k}(z) \cdot z \oplus V_z,$$

the direct sum being orthogonal.

(iv) *Choose a $\mathrm{Ad}(G_z)$-invariant linear complement of $\mathfrak{g}_z$ in $\mathfrak{k}(z)$; call it $\mathfrak{m}(z)$. Then $\mathfrak{m}(z)$ and W_z^* are G_z-equivariantly isomorphic.*

Remark. Let us underline that all summands in the decomposition of T_zM are G_z-invariant, which follows immediately from the invariance of $\mathfrak{k}(z) \cdot z$ and $\mathfrak{g} \cdot z$.

Proof. (i) $\mathfrak{k}(z) \cdot z$ is isotropic by its definition. With $\omega_z(J_zX, J_zY) = \omega_z(X, Y)$ for any $X, Y \in T_zM$, this implies that $J_z(\mathfrak{k}(z) \cdot z) = W_z$ is isotropic, too. Take $J_z\eta_z^M \in J_z(\mathfrak{k}(z) \cdot z) = W_z$ and $\xi \in \mathfrak{g}$. Then $\langle J_z\eta_z^M, \xi_z^M \rangle = \omega(\eta_z^M, \xi_z^M) = 0$ because $\eta_z^M \in \mathfrak{k}(z) \cdot z \subseteq (\mathfrak{g} \cdot z)^{\angle}$, hence $W_z \subseteq (\mathfrak{g} \cdot z)^{\perp}$. In particular, W_z is orthogonal to $\mathfrak{k}(z) \cdot z$.

(ii) By definition, the degeneracy of ω on the orbit is $\mathfrak{k}(z) \cdot z$, hence Q_z is symplectic.

From (i), we conclude $(W_z)^{\perp} \supseteq \mathfrak{g} \cdot z \supseteq Q_z$; thus $J_zQ_z \subseteq (\mathfrak{k}(z) \cdot z)^{\perp}$ holds. We may write $Q_z = (\mathfrak{k}(z) \cdot z)^{\perp} \cap \mathfrak{g} \cdot z$. In order to show that $J_zQ_z \subseteq Q_z$, we check that $J_zQ_z \subseteq \mathfrak{g} \cdot z$: Note first that $\mathfrak{k}(z) \cdot z \subseteq (\mathfrak{g} \cdot z)^{\angle}$ implies $(\mathfrak{k}(z) \cdot z)^{\perp} \supseteq ((\mathfrak{g} \cdot z)^{\angle})^{\perp}$, therefore,

$$\begin{aligned} J_zQ_z \cap \mathfrak{g} \cdot z &= J_z\Big((\mathfrak{k}(z) \cdot z)^{\perp} \cap \mathfrak{g} \cdot z\Big) \cap \mathfrak{g} \cdot z \\ &= J_z\Big((\mathfrak{k}(z) \cdot z)^{\perp} \cap \mathfrak{g} \cdot z \cap ((\mathfrak{g} \cdot z)^{\angle})^{\perp}\Big) \\ &= J_z\Big((\mathfrak{k}(z) \cdot z)^{\perp} \cap \mathfrak{g} \cdot z\Big) = J_zQ_z, \end{aligned}$$

and since J_z is an isomorphism on T_zM, this gives us $J_zQ_z = Q_z$.

(iii) Now $V_z = (\mathfrak{g} \cdot z \oplus W_z)^{\perp} = (Q_z \oplus \mathfrak{k} \cdot z \oplus J_z(\mathfrak{k} \cdot z))^{\perp}$ is the orthogonal complement of a J_z-stable linear subspace, hence V_z is stable itself. As a vector space with a complex structure, V_z is symplectic.

$\mathfrak{k}(z) \cdot z$ and V_z are disjoint by definition; they are orthogonal as a consequence of V_z and W_z being orthogonal. Further, $(\mathfrak{g} \cdot z)^{\angle} = J_z((\mathfrak{g} \cdot z)^{\perp}) = J_z(W_z \oplus V_z) = \mathfrak{k}(z) \cdot z \oplus V_z$.

(iv) Note that $\mathfrak{m}(z) \cdot z = \mathfrak{k}(z) \cdot z$ since $\mathfrak{g}_z \cdot z = \{0\}$. By restriction to $\mathfrak{m}(z)$, we have an isomorphism $\tau_{z|\mathfrak{m}(z)} : \mathfrak{m}(z) \to \mathfrak{m}(z) \cdot z$, which is G_z-equivariant since for $g \in G_z$,

$$\begin{aligned} \tau(\mathrm{Ad}(g)\xi)_z = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \Psi_{\exp(t\,\mathrm{Ad}(g)\xi)}(z) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \Psi_{g\exp(t\xi)g^{-1}}(z) \\ &= T_{g^{-1}.z}\Psi_g(\tau(\xi)_{g^{-1}.z}) = T_z\Psi_g(\tau(\xi)_z). \end{aligned}$$

Further, the symplectic form is non-degenerate and therefore induces an isomorphism $\omega^\sharp : \mathfrak{k}(z) \cdot z \to W_z^*, \xi_z^M \mapsto \omega(\xi_z^M, \cdot) = \langle J_z \xi_z^M, \cdot \rangle$, which is equivariant as G_z acts symplectically.

As with $\mathfrak{m}(z)$, one can choose a linear complement $\mathfrak{q}(z)$ of $\mathfrak{k}(z)$ in $\mathfrak{g}$ such that $\mathfrak{q}(z) \cdot z = Q_z$; but unlike $\mathfrak{m}(z)$, it has no further particular property. □

Notation. If a point $z \in M$ is fixed, the explicit mention of the point z in the notation $\mathfrak{k}(z)$, $\mathfrak{m}(z)$, V_z, etc. will be omitted in the sequel.

Remark 2.23. We have decomposed the Lie algebra $\mathfrak{g}$ into a direct sum $\mathfrak{g}_z \oplus \mathfrak{m} \oplus \mathfrak{q}$. However, we have not yet defined a scalar product on $\mathfrak{g}$, hence have no orthogonality. Recall that $\mathfrak{m} \cdot z = \mathfrak{k} \cdot z \cong \mathfrak{m}$ and $\mathfrak{q} \cdot z = Q_z \cong \mathfrak{q}$. Thus on $\mathfrak{m} \oplus \mathfrak{q}$ we can use the G-invariant scalar product $\langle \cdot, \cdot \rangle$ (which is defined on $T_z M$; here, we restrict it to $\mathfrak{k} \cdot z \oplus Q_z$). On $\mathfrak{g}_z$ we choose an arbitrary G_z-invariant inner product, and we declare $\mathfrak{g}_z$ to be orthogonal to $\mathfrak{m} \oplus \mathfrak{q}$. As $\mathfrak{g}_z$ and the scalar product are G_z-invariant, $\mathfrak{m}$ is the (unique) orthogonal complement of $\mathfrak{g}_z$ in $\mathfrak{k}$ w. r. t. this scalar product, in particular, $\mathfrak{m}$ is G_z-invariant. Analogously, $\mathfrak{q}$ is the G_z-invariant orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$.

A useful property of the subalgebra $\mathfrak{k}$ in the decomposition (for $z \in M$) is recorded in the following lemma (let $\mathfrak{g}_\alpha = \mathrm{Stab}_{\mathfrak{g}}(\alpha)$ for $\alpha \in \mathfrak{g}^*$ under the coadjoint action).

Lemma 2.24. *Take (M, ω) and G as before, and let the proper G-action be Hamiltonian with equivariant moment map $\Phi : M \to \mathfrak{g}^*$. Then $\mathfrak{g}_\alpha = \mathfrak{k}$ for $\alpha = \Phi(z)$.*

Proof. By its definition, $\mathfrak{g}_\alpha = \{\xi \in \mathfrak{g} \mid \mathrm{ad}^*(\xi)\alpha = 0\}$. By the equivariance of Φ, this may be written as $\mathfrak{g}_\alpha = \{\xi \in \mathfrak{g} \mid (T_z\Phi)(\xi_z^M) = 0\}$. Then Prop. 2.11 about the kernel of $T_z\Phi$ yields $\mathfrak{g}_\alpha = \{\xi \in \mathfrak{g} \mid \xi \in T_z(G \cdot z)^{\angle}\} = \mathfrak{k}$. □

Remark 2.25. Note that by Thm. 26.5 and the corollary to Thm. 26.4 of [GS90], the inclusion $[\mathfrak{g}_{\Phi(z)}, \mathfrak{g}_{\Phi(z)}] \subseteq \mathfrak{g}_z$ holds at all points $z \in M$ for which the dimension of the coadjoint G-orbit through $\Phi(z)$ is maximal among all coadjoint orbits in the image of the moment map.

Symplectic slice theorem. With the help of the Witt-Artin decomposition, we will obtain the local structure of a symplectic manifold with proper symplectic action.

Definition 2.26. Let (M, ω) be a symplectic manifold and G a Lie group acting properly and symplectically on it. Let $z \in M$ and let $V = V_z$ be a symplectic vector space carrying a G_z-action as in Thm. 2.22(iii). Any such space will be called a *symplectic normal space* at z. Since the G_z-action on $(V, \omega_{z|V \times V})$ is linear and symplectic, it has an associated moment map to be denoted by $\Phi_V : V \to \mathfrak{g}_z^*$ and given by $\langle \Phi_V(v), \xi \rangle = \frac{1}{2}\omega_z(\xi_v^V, v)$ (for $v \in V, \xi \in \mathfrak{g}_z$). Here, the fundamental vector field for the G_z-action on V is $\xi_v^V = \xi \cdot v$, where $\mathfrak{g}_z$ acts via the derivative of the linear G_z-action on V.

Proposition 2.27. *Let (M, ω) be a symplectic manifold and G a Lie group acting properly and symplectically on it. Let $z \in M$, let V be a symplectic normal space at z, and $\mathfrak{m} \subseteq \mathfrak{g}$ be the subspace chosen in the Witt-Artin decomposition (Thm. 2.22). Then there exist open G_z-invariant neighbourhoods $\mathfrak{m}^*_{\mathrm{res}}$ and V_{res} of the origin in $\mathfrak{m}^*$ and V, respectively, such that the twisted product*

$$Y_z = G \times_{G_z} (\mathfrak{m}^*_{\mathrm{res}} \times V_{\mathrm{res}})$$

is a symplectic manifold with symplectic form ω^{Y_z} given by

$$\omega^{Y_z}_{[g,\varrho,v]}(T_{(g,\varrho,v)}\pi(T_eL_g(\xi_1),\alpha_1,u_1),T_{(g,\varrho,v)}\pi(T_eL_g(\xi_2),\alpha_2,u_2)) = \langle\alpha_2+T_v\Phi_V(u_2),\xi_1\rangle - \langle\alpha_1+T_v\Phi_V(u_1),\xi_2\rangle + \langle\varrho+\Phi_V(v),[\xi_1,\xi_2]\rangle + \omega_z(\xi_1^M,\xi_2^M)+\omega_z(u_1,u_2),$$

where $\pi : G\times(\mathfrak{m}^*_{\text{res}}\times V_{\text{res}}) \to G\times_{G_z}(\mathfrak{m}^*_{\text{res}}\times V_{\text{res}})$ is the projection, $[g,\varrho,v]\in Y_z$, $\xi_1,\xi_2\in\mathfrak{g}$, $\alpha_1,\alpha_2\in\mathfrak{m}^*$ and $u_1,u_2\in V$.

The Lie group G acts symplectically on (Y_z,ω^{Y_z}) by $g\cdot[h,\eta,v]=[gh,\eta,v]$ for any $g\in G$ and $[h,\eta,v]\in Y_z$.

The symplectic manifold (Y_z,ω^{Y_z}) will be called the symplectic tube *of (M,ω) at $z\in M$.*

This proposition is proved in [OR04, Prop. 7.2.2] and used to formulate the symplectic slice theorem [OR04, Thm. 7.4.1].

Theorem 2.28 (Symplectic Slice Theorem)**.** *Let (M,ω) be a symplectic manifold and let G be a Lie group acting properly and symplectically on M. Let $z\in M$ and let (Y_z,ω^{Y_z}) be the symplectic tube at z as in the preceding proposition. Then there is an open G-invariant neighbourhood U of z in M and a G-equivariant symplectic diffeomorphism $\varphi : U\to Y_z$ satisfying $\varphi(z)=[e,0,0]$.*

Marle-Guillemin-Sternberg normal form. Now, the local structure of the symplectic manifold is known, so it is natural to ask for an adapted form of the moment map. An answer is given by the following theorem [OR04, Thm. 7.5.5].

Theorem 2.29. *Let (M,ω) be a connected symplectic manifold acted symplectically and properly upon by a Lie group G. Suppose that this action has an associated G-equivariant moment map $\Phi : M\to\mathfrak{g}^*$. Let (Y_z,ω^{Y_z}) be the symplectic tube at a point $z\in M$ that models a G-invariant open neighbourhood U of the orbit $G\cdot z$ via a G-equivariant symplectic diffeomorphism $\varphi : (U,\omega_{|U})\to(Y_z,\omega^{Y_z})$ as in Thm. 2.28. Then the symplectic left G-action on (Y_z,ω^{Y_z}) admits a moment map $\Phi_{Y_z} : Y_z\to\mathfrak{g}^*$ given by*

$$\Phi_{Y_z} : [g,\varrho,v]\mapsto \operatorname{Ad}^*(g)(\Phi(z)+\varrho+\Phi_V(v)).$$

The map $\Phi_{Y_z}\circ\varphi$ is a moment map for the symplectic G-action on $(U,\omega_{|U})$. Moreover, if the group G is connected, this moment map satisfies $\Phi_{|U}=\Phi_{Y_z}\circ\varphi$.

In particular, $\Phi_{Y_z}(\varphi(z))=\Phi_{Y_z}([e,0,0])=\Phi(z)$.

Orbits and Slices in $\Phi(M)\subseteq\mathfrak{g}^*$. The normal form of the moment map permits to relate the orbit structure on $\mathfrak{g}^*$ to that on M, especially to show that the moment map sends slices on M to slices in its image. Therefore, we recall a theorem about slices of the adjoint action on a reductive Lie algebra [Var77, 1., Thm. 20].

Theorem 2.30. *Let G be a real reductive Lie group, $\mathfrak{g}$ be its Lie algebra and $\xi\in\mathfrak{g}$ a semisimple element*[3]*. Let $G_\xi=\{g\in G\mid \operatorname{Ad}(g)\xi=\xi\}$ be its stabilizer under the adjoint action and $\mathfrak{g}_\xi$*

[3] An element of a Lie algebra is called semisimple if its matrix representation is a matrix which is diagonalizable over $\mathbb{C}$. This is independent of the chosen basis. In particular, all elements of Lie algebras of compact Lie groups are semisimple.

the corresponding Lie algebra. Denote by $Z(\mathfrak{g}_\xi)$ *the centre of* $\mathfrak{g}_\xi$*. Define* $[\mathfrak{g}_\xi, \mathfrak{g}_\xi](t) = \{\eta \in [\mathfrak{g}_\xi, \mathfrak{g}_\xi] \mid |\lambda| < t \ \forall\lambda \text{ which are eigenvalues of } \mathrm{ad}(\eta)\}$ *as an invariant*[4] *neighbourhood of the origin of the semisimple part of* $\mathfrak{g}_\xi$*. Then there exists an open neighbourhood* V *of* ξ *in* $Z(\mathfrak{g}_\xi)$ *and a number* $t > 0$ *such that* $V \times [\mathfrak{g}_\xi, \mathfrak{g}_\xi](t)$ *is a slice at* ξ *for the adjoint* G*-action on* $\mathfrak{g}$*.*

As the adjoint and coadjoint action of a reductive Lie group can be identified using a non-degenerate invariant inner product, this theorem holds analogously in the coadjoint case.

Note that the preimage of $G_z \times_{G_z} (\mathfrak{m}^*_{\text{res}} \times V_{\text{res}})$ under the equivariant symplectomorphism $\varphi : U \to Y_z$ of Thm. 2.28 is a symplectic slice for the G-action on M. We will now study the image of $G_z \times_{G_z} (\mathfrak{m}^*_{\text{res}} \times V_{\text{res}})$ under the normal form of the moment map, Φ_{Y_z}, described in Thm. 2.29. Take $[g, \varrho, v] \in G_z \times_{G_z} (\mathfrak{m}^*_{\text{res}} \times V_{\text{res}})$. Then

$$\Phi_{Y_z}([g, \varrho, v]) = \mathrm{Ad}^*(g)\Phi(z) + \mathrm{Ad}^*(g)\Phi_V(v) + \mathrm{Ad}^*(g)\varrho.$$

Due to $G_z \subseteq G_{\Phi(z)}$, we see that $\mathrm{Ad}^*(g)\Phi(z) = \Phi(z)$. The image of Φ_V is contained in $\mathfrak{g}^*_z$ as it is a moment map of the G_z-action on V, thus $\mathrm{Ad}^*(g)\Phi_V(v) \in \mathfrak{g}^*_z$. As explained in Rem. 2.23, $\mathfrak{m}$ is invariant, thus also $\mathfrak{m}^*$, so that $\mathrm{Ad}^*(g)\varrho \in \mathfrak{m}^*$.

Suppose now that $\Phi(z)$ is semisimple. Hence $\Phi_{Y_z}([g, \varrho, v]) \in \Phi(z) + \mathfrak{g}^*_z + \mathfrak{m}^* = \Phi(z) + \mathfrak{g}^*_{\Phi(z)}$, so after possibly shrinking $\mathfrak{m}^*_{\text{res}}$ and V_{res}, we may apply Thm. 2.30 and see that the image of a (sufficiently small) slice under Φ_{Y_z} is contained in a slice for the coadjoint action.

[4] The invariance is shown in [Var77, Lemma 18].

3 Orbit Correspondence for Commuting Hamiltonian Actions

In this section, the general results which we have summarized before will be applied to the particular situation of two Lie groups acting on the same manifold. We assume that these actions commute. Then, we are able to show relations between their stabilizers, their moment maps, and to adapt the local model to this situation. From the local model, we obtain a first natural bijection between coadjoint orbits in the images of the normal forms of both moment maps.

We also prove a global correspondence theorem. Starting by analyzing the classical dual pair definitions of symplectic geometry, we see that they already allow to describe a certain singular behaviour. More precisely, we get a bijective correspondence between the coadjoint orbits in the moment images of two commuting proper Hamiltonian actions including all orbit types. Further, we are able to describe explicitly reduced spaces in this setting.

3.1 General Properties of Commuting Hamiltonian Actions, Local Models and Local Correspondence

Now we are going to consider a situation where we have two Hamiltonian actions on one symplectic manifold, and we suppose that they commute. This will permit us to say something about the invariance properties of the moment maps, and applying the results of the preceding section, about the local structure of these actions. First note the following obvious fact for commuting actions.

Lemma 3.1. *Given two Lie groups G and H acting on a set M. If these actions commute, the following holds for any point $z \in M$:*

$$\mathrm{Stab}_H(z') = \mathrm{Stab}_H(z) \quad \forall z' \in G \cdot z.$$

The analogous statement for G and H interchanged is also true.

Proof. Write the action of G on the left and H on the right. Then for any $z' = g \cdot z$, one has $(g \cdot z) \cdot h = g \cdot z \Leftrightarrow z \cdot h = z$ by multiplication with g^{-1}. □

Now assume that one has symplectic actions of two Lie groups G_1 and G_2 on the symplectic manifold (M, ω). Denote the fundamental vector fields by $\xi^{(1)}$ for $\xi \in \mathfrak{g}_1$ and $\eta^{(2)}$ for $\eta \in \mathfrak{g}_2$. As the actions are symplectic, they are locally Hamiltonian and

$$[\xi^{(1)}, \eta^{(2)}] \lrcorner\, \omega = \mathrm{d}(\omega(\xi^{(1)}, \eta^{(2)}))$$

holds (see explanations after (4.1.5) in [OR04]). One concludes that, if both actions commute ($[\xi^{(1)}, \eta^{(2)}] = 0$), the "symplectic angle" $\omega(\xi^{(1)}, \eta^{(2)})$ between the orbits is constant on the connected components of M. We can say a little more for Hamiltonian actions.

Lemma 3.2. *Let (M, ω) be a symplectic manifold. Let the Lie groups G_1 and G_2 act symplectically on M, admitting equivariant moment maps Φ_1 and Φ_2. Assume these actions commute. Then for any $\xi \in \mathfrak{g}_1$ and any $\eta \in \mathfrak{g}_2$ the Poisson bracket of the moment components vanishes, i. e.,*

$$\left\{\Phi_1^\xi, \Phi_2^\eta\right\} = 0.$$

Proof. Let $\lambda_i : \mathfrak{g}_i \to C^\infty(M)$ be the Poisson homomorphisms corresponding to the Hamiltonian action of the G_i. Denote by $\lambda = \lambda_1 + \lambda_2 : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to C^\infty(M)$ the combined map for the joint action. Then for any $\xi_1 \in \mathfrak{g}_1$ and $\eta_2 \in \mathfrak{g}_2$ (and thus $\xi_1 \oplus 0, 0 \oplus \eta_2 \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$), we have

$$\{\Phi_1^{\xi_1}, \Phi_2^{\eta_2}\} = \{\lambda_1(\xi_1), \lambda_2(\eta_2)\} = \\ \{\lambda(\xi_1 \oplus 0), \lambda(0 \oplus \eta_2)\} = \lambda([\xi_1 \oplus 0, 0 \oplus \eta_2]) = \lambda([\xi_1, 0] \oplus [0, \eta_2]) = 0.$$

□

By

$$0 = \{\Phi_1^\xi, \Phi_2^\eta\} = \omega(\xi^{(1)}, \eta^{(2)}) = -\xi^{(1)}(\Phi_2^\eta) = \eta^{(2)}(\Phi_1^\xi),$$

one sees that not only the components of the moment maps Φ_1 and Φ_2 commute but that the orbits of both actions are symplectically orthogonal to each other ($\mathfrak{g}_1 \cdot z \subseteq (\mathfrak{g}_2 \cdot z)^\angle \, \forall\, z \in M$) and the moment components of one action are constant on the connected components of the orbits of the other action, i. e., for connected groups (or, at least, connected orbits of) G_1 and G_2, Φ_1 is G_2-invariant and Φ_2 is G_1-invariant.

For the remainder of section 3.1, the objects of study will be two Lie groups G_1, G_2, a connected symplectic manifold (M, ω), and two Hamiltonian actions $\Psi_1 : G_1 \times M \to M$ and $\Psi_2 : G_2 \times M \to M$ which commute. Furthermore, the moment maps $\Phi_1 : M \to \mathfrak{g}_1^*$ and $\Phi_2 : M \to \mathfrak{g}_2^*$ will always be G_1- and G_2-equivariant, respectively. One additional relation between these commuting actions will be important.

Definition 3.3. Let (M, ω) be a symplectic manifold. Two commuting actions on M by Lie groups G_1 and G_2 will be called *symplectically complementary* if

$$(\mathfrak{g}_1 \cdot z)^\angle = \mathfrak{g}_2 \cdot z \qquad \forall z \in M.$$

Two easy lemmas will be stated, the first one giving a simple tool to check for symplectic complementarity and the second one giving an immediate consequence of this property.

Lemma 3.4. *If the actions of G_1 and G_2 on (M, ω) are symplectically orthogonal, i. e., $\mathfrak{g}_1 \cdot z \subseteq (\mathfrak{g}_2 \cdot z)^\angle \, \forall z \in M$, and the dimensions of their orbits are complementary, i. e., $\dim \mathfrak{g}_1 \cdot z + \dim \mathfrak{g}_2 \cdot z = \dim M$, they are symplectically complementary.*

Proof. This lemma is an immediate consequence of Lemma 2.6, which states the same for two linear subspaces of a symplectic vector space. □

Lemma 3.5. *If the actions of G_1 and G_2 on (M, ω) are symplectically complementary, the tangent spaces of the orbits of the simultaneous action of $G_1 \times G_2$ are coisotropic subspaces of the tangent spaces T_zM, i. e.,*

$$T_z((G_1 \times G_2) \cdot z)^\angle \subseteq T_z((G_1 \times G_2) \cdot z) \qquad \forall z \in M.$$

Proof. For every point $z \in M$, we can calculate

$$\begin{aligned} T_z((G_1 \times G_2) \cdot z)^\angle &= [T_z(G_1 \cdot z) + T_z(G_2 \cdot z)]^\angle \\ &= T_z(G_1 \cdot z)^\angle \cap T_z(G_2 \cdot z)^\angle = T_z(G_2 \cdot z) \cap T_z(G_1 \cdot z) \\ &\subseteq T_z((G_1 \times G_2) \cdot z), \end{aligned}$$

which is the definition of a coisotropic subspace. □

The next step is to make explicit the Witt-Artin decomposition of the tangent spaces of M for both the individual actions of G_1, G_2 and for the simultaneous action of $G_1 \times G_2$. It turns out that T_zM has a very particular structure if the individual actions are symplectically complementary.

Again, we need – as in section 2.4 – the Lie subalgebras corresponding to the degenerate parts of the tangent spaces of the group orbits. Denote them by $\mathfrak{k}_1$, $\mathfrak{k}_2$ und $\mathfrak{k}_{12}$ for $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_{12}$ (again fixing a point $z \in M$ and omitting it in the notation).

Proposition 3.6. *Let (M,ω) be a symplectic manifold with two symplectic and symplectically complementary actions of the Lie groups G_1 and G_2. Suppose the $(G_1 \times G_2)$-action is proper.*[5] *Choose a $(G_1 \times G_2)$-equivariant compatible almost-complex structure J and the $(G_1 \times G_2)$-invariant Riemannian structure $\langle\cdot,\cdot\rangle = \omega(\cdot, J\cdot)$. Then for any $z \in M$, the tangent space decomposes orthogonally as*

$$T_zM = \mathfrak{k}_1 \cdot z \oplus Q_1 \oplus Q_2 \oplus W,$$

where the summands are described by the following properties:

(i) $\mathfrak{k}_1 \cdot z = \mathfrak{k}_2 \cdot z = \mathfrak{k}_{12} \cdot z = T_z(G_1 \cdot z) \cap T_z(G_2 \cdot z)$ *and* $\mathfrak{k}_{12} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$,

(ii) the symplectic part of the $(G_1 \times G_2)$-orbit is given by $Q_{12} = Q_1 \oplus Q_2$,

(iii) for any choice of $G_{i,z}$-invariant complements $\mathfrak{m}_i$ of $\mathfrak{g}_{i,z}$ in $\mathfrak{k}_i$ ($i \in \{1,2,12\}$), one has $\mathfrak{m}_1 \cong \mathfrak{m}_2 \cong \mathfrak{m}_{12}$; and $W \cong \mathfrak{m}_i^*$ $G_{i,z}$-equivariantly.

Remark. The fact that the degenerate parts of the orbits coincide reminds one of the common centre of the groups G_1 and G_2 in a Howe pair (G_1, G_2) in G (see App. B for this fact).

Proof. By Thm. 2.22, we may decompose T_zM under the simultaneous $(G_1 \times G_2)$-action as

$$T_zM = \mathfrak{k}_{12} \cdot z \oplus Q_{12} \oplus V_{12} \oplus W_{12},$$

the decomposition being orthogonal w. r. t. $\langle\cdot,\cdot\rangle$.

(i) By the symplectic complementarity of the orbits, i. e., $\mathfrak{g}_1 \cdot z = (\mathfrak{g}_2 \cdot z)^{\angle}$, we write $\mathfrak{k}_1 = \left\{\xi \in \mathfrak{g}_1 \mid \xi_z^{(1)} \in \mathfrak{g}_2 \cdot z\right\}$ and $\mathfrak{k}_2 = \left\{\xi \in \mathfrak{g}_2 \mid \xi_z^{(2)} \in \mathfrak{g}_1 \cdot z\right\}$. Therefore, $\mathfrak{k}_1 \cdot z = \{v \in \mathfrak{g}_1 \cdot z \mid v \in \mathfrak{g}_2 \cdot z\}$, hence $\mathfrak{k}_1 \cdot z = \mathfrak{k}_2 \cdot z = \mathfrak{g}_1 \cdot z \cap \mathfrak{g}_2 \cdot z$. Also for the joint action, $\mathfrak{k}_{12} = \left\{\xi \in \mathfrak{g}_1 \oplus \mathfrak{g}_2 \mid \xi_z^M \in \mathfrak{g}_1 \cdot z \cap \mathfrak{g}_2 \cdot z\right\}$, i. e., $\mathfrak{k}_{12} \cdot z = \mathfrak{g}_1 \cdot z \cap \mathfrak{g}_2 \cdot z$. From this follows that $W_i = J_z(\mathfrak{k}_i \cdot z)$ is independent of $i \in \{1,2,12\}$, so we call it W.

The inclusion $\mathfrak{k}_1 \oplus \mathfrak{k}_2 \subseteq \mathfrak{k}_{12}$ is clear from the definition. Let $\xi = \eta_1 + \eta_2 \in \mathfrak{k}_{12}$. Then $\eta_{1,z}^{(1)} + \eta_{2,z}^{(2)} \in \mathfrak{g}_1 \cdot z \cap \mathfrak{g}_2 \cdot z \subseteq \mathfrak{g}_1 \cdot z$. Because $\eta_{1,z}^{(1)} \in \mathfrak{g}_1 \cdot z$, we have

$$\eta_{2,z}^{(2)} = (\eta_{1,z}^{(1)} + \eta_{2,z}^{(2)}) - \eta_{1,z}^{(1)} \in \mathfrak{g}_1 \cdot z,$$

hence $\eta_2 \in \mathfrak{k}_2$, and in the same manner we obtain $\eta_1 \in \mathfrak{k}_1$; so, the reversed inclusion also holds, thus $\mathfrak{k}_1 \oplus \mathfrak{k}_2 = \mathfrak{k}_{12}$.

[5] Recall that this implies that the individual actions of G_1 and G_2 on M are proper. However, the converse implication is not true, see e. g., the example of T^*G with left and right G-action treated in section 7.

(ii) Another immediate observation is that the symplectic normal space for the simultaneous action is trivial, $V_{12} = \{0\}$, because

$$\mathfrak{k}_{12} \cdot z \oplus V_{12} = [(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cdot z]^{\angle} = \mathfrak{g}_1 \cdot z \cap \mathfrak{g}_2 \cdot z = \mathfrak{k}_{12} \cdot z.$$

From $(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cdot z = \mathfrak{g}_1 \cdot z + \mathfrak{g}_2 \cdot z = (\mathfrak{g}_1 \cdot z \cap \mathfrak{g}_2 \cdot z) \oplus Q_1 \oplus Q_2 = \mathfrak{k}_{12} \cdot z \oplus Q_1 \oplus Q_2$ conclude $Q_{12} = Q_1 \oplus Q_2$. Here, Q_1 and Q_2 are the symplectic parts of the orbit tangent spaces of the individual actions. As Q_1 and Q_2 are symplectically perpendicular and both J_z-stable, they are also orthogonal w. r. t. $\langle \cdot, \cdot \rangle$.

(iii) Note that the maps $\tau_{i,z} : \mathfrak{g}_i \to T_zM$ ($i \in \{1, 2, 12\}$), which assign the fundamental vector fields evaluated in z to the elements of the Lie algebra, have kernel $\mathfrak{g}_{i,z}$ and are thus isomorphisms when restricted to $\mathfrak{m}_i$, i. e., $\tau_{i,z|\mathfrak{m}_i} : \mathfrak{m}_i \to \mathfrak{m}_i \cdot z$. Using $\mathfrak{m}_i \cdot z = \mathfrak{k}_i \cdot z$ and (i), this proves the first claim. The isomorphism with W and its equivariance are shown as in Thm. 2.22.

The symplectic complementarity further allows to observe $\mathfrak{g}_1 \cdot z = \mathfrak{k}_1 \cdot z \oplus Q_1 = (\mathfrak{g}_2 \cdot z)^{\angle} = \mathfrak{k}_2 \cdot z \oplus V_2$. Using the chosen Riemannian metric, $\mathfrak{k}_1 \cdot z$ and Q_1 are orthogonal. Applying (i), and that the symplectic normal space V_2 is the orthogonal complement to $\mathfrak{k}_2 \cdot z$ in $(\mathfrak{g}_2 \cdot z)^{\angle}$, it follows that $V_2 = Q_1$; analogously, we have $V_1 = Q_2$. □

At first glance, the complements $\mathfrak{m}_i$ (for $i \in \{1, 2, 12\}$) seem to be unrelated. In the following two lemmas, we show their relationship.

Lemma 3.7. *Define linear subspaces of* $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ *by*

$$\mathfrak{m}_{12}^{\pm} = \left\{ \xi + \eta \in \mathfrak{m}_1 \oplus \mathfrak{m}_2 \mid \xi_z^{(1)} = \pm \eta_z^{(2)} \right\}.$$

Both subspaces are $G_{12,z}$*-invariant.*

Proof. Note first that for $(g_1, g_2) \in G_{12,z}$, one has $(g_1, g_2) \cdot z = z$, hence $g_1^{-1} \cdot z = g_2 \cdot z$. Take $\xi + \eta \in \mathfrak{m}_{12}^{\pm}$. Then, using the fact that the actions $\Psi^{(1)}$ and $\Psi^{(2)}$ commute, one calculates

$$\begin{aligned} [\mathrm{Ad}(g_1)\xi]_z^{(1)} &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \exp(t\,\mathrm{Ad}(g_1)\xi) \cdot z = \left(T_{g_1^{-1} \cdot z} \Psi_{g_1}^{(1)}\right) \xi_{g_1^{-1} \cdot z}^{(1)} \\ &= \left(T_{g_2 \cdot z} \Psi_{g_1}^{(1)}\right) \xi_{g_2 \cdot z}^{(1)} = \left(T_{g_2 \cdot z} \Psi_{g_1}^{(1)}\right) \left(T_z \Psi_{g_2}^{(2)}\right) \xi_z^{(1)} \\ &= \pm \left(T_{g_2 \cdot z} \Psi_{g_1}^{(1)}\right) \left(T_z \Psi_{g_2}^{(2)}\right) \eta_z^{(2)} = \pm \left(T_{g_2 \cdot z} \Psi_{g_1}^{(1)}\right) [\mathrm{Ad}(g_2)\eta]_{g_1^{-1} \cdot z}^{(2)} = \pm [\mathrm{Ad}(g_2)\eta]_z^{(2)}. \end{aligned}$$

Thus we have shown that $\mathrm{Ad}((g_1, g_2))(\xi + \eta) \in \mathfrak{m}_{12}^{\pm}$. □

Remark 3.8. The $G_{12,z}$-invariance of $\mathfrak{m}_{12}^{\pm}$ does not rely on the compactness of $G_{12,z}$, hence not on the properness of the $(G_1 \times G_2)$-action.

As we already know that $\mathfrak{m}_1 \cong \mathfrak{m}_2$, we can show similar isomorphisms involving the spaces $\mathfrak{m}_{12}^{\pm}$. In particular, in the next lemma we see that the $G_{12,z}$-invariant space $\mathfrak{m}_{12}^{+}$ is actually a complement adapted to the decomposition in Prop. 3.6. All notation is used as before.

Lemma 3.9. *The subspace $\mathfrak{m}_{12}^+$ is an admissible choice for $\mathfrak{m}_{12}$, i. e., it is a $G_{12,z}$-invariant linear complement to $\mathfrak{g}_{12,z}$ in $\mathfrak{k}_{12}$. Further hold $\mathfrak{g}_{12,z} = (\mathfrak{g}_{1,z} \oplus \mathfrak{g}_{2,z}) \oplus \mathfrak{m}_{12}^-$, $\mathfrak{m}_{12}^- \cong \mathfrak{m}_{12}^+$, $\mathfrak{m}_1 \oplus \mathfrak{m}_2 = \mathfrak{m}_{12}^- \oplus \mathfrak{m}_{12}^+$, and $\mathfrak{m}_{12}^+ \cong \mathfrak{m}_1 \cong \mathfrak{m}_2$.*

Proof. First, we prove that $\mathfrak{g}_{12,z} = (\mathfrak{g}_{1,z} \oplus \mathfrak{g}_{2,z}) \oplus \mathfrak{m}_{12}^-$. For $\xi + \eta \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$ to lie in the stabilizer of the simultaneous action at $z \in M$, $\xi_z^{(1)} + \eta_z^{(2)} = 0$ must hold. So, either both summands are zero (and thus $\xi + \eta \in \mathfrak{g}_{1,z} \oplus \mathfrak{g}_{2,z}$) or $\xi_z^{(1)} = -\eta_z^{(2)} \neq 0$ (and thus $\xi + \eta \in \mathfrak{m}_{12}^-$).

The isomorphism $\mathfrak{m}_{12}^- \cong \mathfrak{m}_{12}^+$ is given by $\xi + \eta \mapsto \xi - \eta$. The intersection $\mathfrak{m}_{12}^+ \cap \mathfrak{m}_{12}^- = \{0\}$ is trivial because $\xi_z^{(1)} - \eta_z^{(2)} = 0$ and $\xi_z^{(1)} + \eta_z^{(2)} = 0$ together implies $\xi_z^{(1)} = \eta_z^{(2)} = 0$. As $\mathfrak{m}_{12}^+ \oplus \mathfrak{m}_{12}^- \subseteq \mathfrak{m}_1 \oplus \mathfrak{m}_2$ is clear, we now show that any element of $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ can be decomposed into the sum of an element of $\mathfrak{m}_{12}^+$ and one of $\mathfrak{m}_{12}^-$.

Recall the fact that $\tau_{i,z|\mathfrak{m}_i}$ is an isomorphism for $i = 1, 2$. Take $\xi + \eta \in \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Then one can form the vector $v = \frac{1}{2}(\xi_z^{(1)} + \eta_z^{(2)})$ as well as $w = \frac{1}{2}(\xi_z^{(1)} - \eta_z^{(2)})$, and verify that $\tau_{1,z}^{-1}(v) + \tau_{2,z}^{-1}(v) \in \mathfrak{m}_{12}^+$ and $\tau_{1,z}^{-1}(w) - \tau_{2,z}^{-1}(w) \in \mathfrak{m}_{12}^-$.

From the preceding calculations, one sees $\mathfrak{k}_{12} = \mathfrak{g}_{12,z} \oplus \mathfrak{m}_{12}^+$, hence $\mathfrak{m}_{12}^+$ is an appropriate choice for $\mathfrak{m}_{12}$; also $\mathfrak{m}_{12}^+ \cong \mathfrak{m}_1 \cong \mathfrak{m}_2$. □

Remark 3.10. The isomorphisms $\mathfrak{m}_{12}^{\pm} \cong \mathfrak{m}_i$ can be realized as follows: Let $p_i : \mathfrak{m}_1 \oplus \mathfrak{m}_2 \to \mathfrak{m}_i$ be the projection. Then its restriction $p_{i|\mathfrak{m}_{12}^{\pm}} : \mathfrak{m}_{12}^{\pm} \to \mathfrak{m}_i$ is an isomorphism as it is clearly linear; surjectivity and injectivity follow from the fact that for any $\xi \in \mathfrak{m}_i$, there is a unique $\eta \in \mathfrak{m}_j$ $(i + j = 3)$ such that $\xi_z^{(i)} = \pm\eta_z^{(j)}$. Furthermore, recall that on the $\mathfrak{g}_i$, we have introduced a scalar product compatible with $\langle\cdot,\cdot\rangle$ on T_zM and yielding an orthogonal direct decomposition $\mathfrak{g}_i = \mathfrak{g}_{i,z} \oplus \mathfrak{m}_i \oplus \mathfrak{q}_i$. This induces a scalar product on $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, which we will also denote by $\langle\cdot,\cdot\rangle$. Take now $\xi + \eta \in \mathfrak{m}_{12}^+$ (i. e., $\xi_z^{(1)} = \eta_z^{(2)}$) and $\xi' + \eta' \in \mathfrak{m}_{12}^-$ (i. e., $\xi_z'^{(1)} = -\eta_z'^{(2)}$), so $\langle \xi + \eta, \xi' + \eta' \rangle = \langle \xi, \xi' \rangle + \langle \eta, \eta' \rangle = \left\langle \xi_z^{(1)}, \xi_z'^{(1)} \right\rangle + \left\langle \eta_z^{(2)}, \eta_z'^{(2)} \right\rangle = -\left\langle \eta_z^{(2)}, \eta_z'^{(2)} \right\rangle + \left\langle \eta_z^{(2)}, \eta_z'^{(2)} \right\rangle = 0$. Thus $\mathfrak{m}_{12}^+$ and $\mathfrak{m}_{12}^-$ are orthogonal to each other.

Having this information about the tangent spaces at hand, we proceed to describe the symplectic tubes, according to Prop. 2.27, for the actions we consider. For G_1, the tube takes the form $Y_{1,z} = G_1 \times_{G_{1,z}} (\mathfrak{m}_1^* \times \mathfrak{q}_2 \cdot z)_{\mathrm{res}}$; and for G_2 analogously, $Y_{2,z} = G_2 \times_{G_{2,z}} (\mathfrak{m}_2^* \times \mathfrak{q}_1 \cdot z)_{\mathrm{res}}$. The tube of the simultaneous action of $G_1 \times G_2$ is $Y_{12,z} = (G_1 \times G_2) \times_{G_{12,z}} \mathfrak{m}_{12,\mathrm{res}}^*$, where we assume that $\mathfrak{m}_{12}$ is chosen to be $\mathfrak{m}_{12}^+$ (recall that the subscript res is used to denote neighbourhoods of zero in these vector spaces).

The next step is to give the Marle-Guillemin-Sternberg normal forms of the moment maps for our setting, applying Thm. 2.29. In this context, terms of the form $\omega_z(\xi_v^V, v)$ for $v \in V$ and $\xi \in \mathfrak{g}_z$, the Lie algebra of the stabilizer group of z, occur. In our case, we look at $V_1 = \mathfrak{q}_2 \cdot z$ where $v = \eta_z^{(2)} \in \mathfrak{q}_2 \cdot z$. Then for $\xi \in \mathfrak{g}_{1,z}$, one obtains

$$\begin{aligned}\xi_v^{V_1} &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 T_z\Psi^{(1)}_{\exp(t\xi)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_0 \Psi^{(2)}_{\exp(s\eta)}(z)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_0 \Psi^{(1)}_{\exp(t\xi)}(\Psi^{(2)}_{\exp(s\eta)}(z)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_0 \Psi^{(2)}_{\exp(s\eta)}(\Psi^{(1)}_{\exp(t\xi)}(z)) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_0 \Psi^{(2)}_{\exp(s\eta)}(z) = 0,\end{aligned}$$

where the fact that $\Psi^{(1)}$ and $\Psi^{(2)}$ commute has been used, as well as ξ stabilizing z. So, the terms of this type vanish, and the normal forms of the moment maps are

$$\begin{aligned} \Phi_{1,Y_{1,z}} &: [g_1, \varrho_1, v_1] \mapsto \mathrm{Ad}^*(g_1)(\Phi_1(z) + \varrho_1), \\ \Phi_{2,Y_{2,z}} &: [g_2, \varrho_2, v_2] \mapsto \mathrm{Ad}^*(g_2)(\Phi_2(z) + \varrho_2), \\ \Phi_{12,Y_{12,z}} &: [(g_1, g_2), \varrho] \mapsto \mathrm{Ad}^*((g_1, g_2))(\Phi_1(z) + \Phi_2(z) + \varrho), \end{aligned}$$

for the two individual actions and the simultaneous action. Here, $g_i \in G_i$, $\varrho \in \mathfrak{m}^*_{12,\mathrm{res}}$, and $v_i \in V_i$. Observe that, given ϱ, upon setting $\varrho_i = \varrho_{|\mathfrak{m}_i} \in \mathfrak{m}_i^*$, one has

$$\Phi_{12,Y_{12,z}}([(g_1, g_2), \varrho]) = \Phi_{1,Y_{1,z}}([g_1, \varrho_1, v_1]) + \Phi_{2,Y_{2,z}}([g_2, \varrho_2, v_2]) \in \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*,$$

even though these moment maps belong to different models. (A common model does, in general, not exist.)

These explicit formulae permit to make a statement about the relation between the slices on the manifold and in the moment images.

Lemma 3.11. *Let Ω be the intersection of the principal orbit type submanifolds of the G_1- and G_2-action (both proper). Let $z \in \Omega$ be a point where the orbits of G_1 and G_2 have symplectically complementary tangent spaces. Assume further that $\Phi_i(z)$ is semisimple (for $i = 1, 2$). Then the moment map of the G_i-action maps a slice in z for the $(G_1 \times G_2)$-action on M to a slice in $\Phi_i(z)$ for the coadjoint G_i-action on $\Phi_i(\Omega) \subseteq \mathfrak{g}_i^*$.*

Proof. Around z, the manifold is modelled by $Y_{12,z}$ as above. A slice for the $(G_1 \times G_2)$-action corresponds to $G_{12,z} \times_{G_{12,z}} \mathfrak{m}^*_{12,\mathrm{res}}$. Take $(g_1, g_2) \in G_{12,z}$ and $\varrho \in \mathfrak{m}^*_{12,\mathrm{res}}$. By Lemma 3.7, one has that $\Phi_{12,Y_{12,z}}([(g_1, g_2), \varrho]) = \Phi_1(z) + \Phi_2(z) + \mathrm{Ad}^*((g_1, g_2))\varrho \in \Phi_1(z) + \Phi_2(z) + \mathfrak{m}^*_{12,\mathrm{res}}$. Restricting to either of the $\mathfrak{g}_i$, this yields slices of the form $\Phi_i(z) + \mathfrak{m}^*_{i,\mathrm{res}}$ in $\mathfrak{g}_i^*$ if $\Phi_i(z)$ is semisimple, according to Thm. 2.30 and because

$$\begin{aligned} \dim \mathfrak{m}_i &= \dim \mathfrak{g}_{i,\Phi_i(z)} - \dim \mathfrak{g}_{i,z} \\ &= \mathrm{rk}\, T_z\Phi_i - (\dim \mathfrak{g}_i - \mathfrak{g}_{i,\Phi_i(z)}) = \dim \Phi_i(\Omega) - \dim \mathrm{Ad}^*(G_i)\Phi_i(z) \end{aligned}$$

has the correct dimension of a slice. □

So, locally, this yields a correspondence between the orbit through $\Phi_1(z) + \varrho_1 \in \mathfrak{g}_1^*$ and the one through $\Phi_2(z) + \varrho_2 \in \mathfrak{g}_2^*$, i.e., $M/(G_1 \times G_2)$ and the $\Phi_i(M)/G_i$ are, close to the respective orbit through z, parametrized by neighbourhoods of zero in the isomorphic vector spaces $\mathfrak{m}_{12}^+ \cong \mathfrak{m}_1 \cong \mathfrak{m}_2$.

3.2 Symplectic Geometry of Dual Pairs Arising from Group Actions

The next step in our work is to establish a global correspondence between coadjoint orbits in the images of two moment maps corresponding to two commuting Hamiltonian actions on the same manifold. This is achieved by adapting the theory of symplectic dual pairs to our setting and studying the consequences for the orbit structure of our group actions.

We start by citing the (classical) dual pair definitions as in [OR04, Ch. 11], and state some basic properties.

Let (M, ω) be a symplectic manifold and $(P_i, \{\cdot,\cdot\}_{P_i})$ with $i = 1, 2$ two Poisson manifolds such that in the diagram

$$\begin{array}{ccc} & (M,\omega) & \\ {\scriptstyle\pi_1} \swarrow & & \searrow {\scriptstyle\pi_2} \\ (P_1, \{\cdot,\cdot\}_{P_1}) & & (P_2, \{\cdot,\cdot\}_{P_2}) \end{array} \tag{3.1}$$

π_1 and π_2 are surjective submersions preserving the Poisson structures. We then call the diagram a *symplectic dual pair diagram.*

Definition 3.12. A diagram of this type is called a *symplectic Howe dual pair* if the Poisson subalgebras $\pi_1^* C^\infty(P_1)$ and $\pi_2^* C^\infty(P_2)$ centralize each other in $(C^\infty(M), \{\cdot,\cdot\}_\omega)$, i. e.,

$$Z_{C^\infty(M)}(\pi_1^* C^\infty(P_1)) = \pi_2^* C^\infty(P_2)$$

and vice versa.

Definition 3.13. A diagram of this type is called a *Lie-Weinstein dual pair* if

$$\forall z \in M : (\ker T_z\pi_1)^\angle = \ker T_z\pi_2.$$

The Lie-Weinstein condition implies the relation $\dim M = \dim P_1 + \dim P_2$ for the dimensions, which avoids singular behaviour. Both notions of dual pairs are related by the following proposition [OR04, Prop. 11.1.3].

Proposition 3.14. *(i) Given a Lie-Weinstein dual pair whose maps π_1 and π_2 have connected fibres. Then this pair is also a symplectic Howe dual pair.*

(ii) Given a symplectic Howe dual pair where $\dim M = \dim P_1 + \dim P_2$ holds. Then this pair is also a Lie-Weinstein dual pair.

Now, the aim is to interpret diagrams of the form $\mathfrak{g}_1^* \xleftarrow{\Phi_1} M \xrightarrow{\Phi_2} \mathfrak{g}_2^*$ as symplectic dual pairs according to the preceding definitions. We place ourselves in a setting similar to that of section 3.1, in particular Lemma 3.5, and find immediately:

Lemma 3.15. *Given commuting Hamiltonian free actions of the Lie groups G_1 and G_2 on the symplectic manifold (M, ω) which are symplectically complementary, the moment maps being Φ_1 and Φ_2, respectively. Then the diagram $\Phi_1(M) \xleftarrow{\Phi_1} M \xrightarrow{\Phi_2} \Phi_2(M)$ is a Lie-Weinstein dual pair. The converse also holds.*

Proof.
$$\ker T_z\Phi_1 = T_z(G_1 \cdot z)^\angle = T_z(G_2 \cdot z) = (\ker T_z\Phi_2)^\angle \qquad \forall z \in M$$
□

The basic motivation for introducing symplectic notions of dual pairs is the close relationship between the spaces P_1 and P_2 that one can prove for a dual pair. In particular, the spaces of symplectic leaves of P_1 and P_2 are in natural bijection, as described by the following theorem ([OR04, Thm. 11.1.9], notation used as in the reference, see also [Bla01, App. E]).

Theorem 3.16 (Symplectic Leaf Correspondence)**.** *Let a Lie-Weinstein dual pair be given,*

$$(P_1, \{\cdot,\cdot\}_{P_1}) \xleftarrow{\pi_1} (M,\omega) \xrightarrow{\pi_2} (P_2, \{\cdot,\cdot\}_{P_2}) \,,$$

where moreover π_1 and π_2 have connected fibres. Then:

(i) The generalized distribution $D_z = \ker T_z\pi_1 + \ker T_z\pi_2$ ($z \in M$) is smooth and integrable;

(ii) There are bijections between the leaf spaces M/D and $P_1/\{\cdot,\cdot\}_{P_1}$ as well as M/D and $P_2/\{\cdot,\cdot\}_{P_2}$, given by $\mathcal{L} \in M/D \mapsto \pi_1(\mathcal{L})$ and $\mathcal{L} \in M/D \mapsto \pi_2(\mathcal{L})$, respectively. Consequently, $\pi_2 \circ \pi_1^{-1} : P_1/\{\cdot,\cdot\}_{P_1} \to P_2/\{\cdot,\cdot\}_{P_2}, \mathcal{L}_1 \mapsto \pi_2(\pi_1^{-1}(\mathcal{L}_1))$ gives a bijection between the spaces of symplectic leaves of P_1 and P_2.[6]

(iii) Let $\mathcal{L}_1 \subseteq P_1$ and $\mathcal{L}_2 \subseteq P_2$ be two symplectic leaves in correspondence and $\mathcal{L}$ the corresponding leaf in M. If $i_\mathcal{L} : \mathcal{L} \to M$ is the inclusion, then

$$(i_\mathcal{L})^*\omega = (\pi_{1|\mathcal{L}})^*\omega_1 + (\pi_{2|\mathcal{L}})^*\omega_2,$$

ω_1 and ω_2 being the symplectic forms on the symplectic leaves $\mathcal{L}_1$ and $\mathcal{L}_2$, resp.

It would be possible to apply this theorem directly to our situation of two Hamiltonian actions which commute and find a natural correspondence between the symplectic leaves, i. e., the coadjoint orbits, in $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$. However, assuming the moment maps to be surjective submersions restricts the applicability of such a result. In the next section we will therefore introduce the notion of singular dual pairs. Nevertheless, relaxing the non-singular definitions given above will permit us to establish a correspondence theorem that applies to realistic group action settings.

Definition 3.17. Let (M,ω) be a symplectic manifold on which mutually commuting Hamiltonian actions of Lie groups G_1 and G_2 are given. Denote the equivariant moment maps by $\Phi_i : M \to \mathfrak{g}_i^*$ ($i = 1, 2$). The *symplectic Howe condition* is said to be satisfied if

$$Z_{C^\infty(M)}(\Phi_i^* C^\infty(\mathfrak{g}_i^*)) = \Phi_j^* C^\infty(\mathfrak{g}_j^*)$$

for $i + j = 3$. We then also speak of a *Howe pair of symplectic actions.*

Definition 3.18. Let (M,ω) be a symplectic manifold on which mutually commuting Hamiltonian actions of Lie groups G_1 and G_2 are given. Denote the equivariant moment maps by $\Phi_i : M \to \mathfrak{g}_i^*$ ($i = 1, 2$). Suppose there is a dense open set $\Omega \subseteq M$ such that for any $z \in \Omega$

$$(\mathfrak{g}_1 \cdot z)^\angle = \mathfrak{g}_2 \cdot z.$$

Then we say that the *Lie-Weinstein condition* is satisfied by these commuting actions. We also speak of a *Lie-Weinstein pair of symplectic actions.*

[6] All leaves are seen both as submanifolds and elements of the quotient spaces. The expression $\pi_1^{-1}(\mathcal{L}_1)$ has to be understood as the union of preimages of the points of the leaf $\mathcal{L}_1$, i. e., $\pi_1^{-1}(\mathcal{L}_1) = \bigcup_{x\in\mathcal{L}_1} \pi_1^{-1}(x)$, and $\pi_2(\pi_1^{-1}(\mathcal{L}_1))$ is then the image of this union of preimages under π_2, which is again a leaf (now in P_2).

Both definitions have been given without assuming the actions to be proper. Adding this assumption, we are able to describe in detail the orbit structure of the commuting Hamiltonian actions. We first cite the following result [KL97, Cor. 2.6], that one checks to hold for proper actions even though it is merely stated for actions of compact groups in the reference (compactness is only used through the closedness of orbits and the slice theorem, which remain available).

Proposition 3.19. *Let (M, ω) be a symplectic manifold acted on properly by a Lie group G which preserves ω. Let Φ be the moment map for this action. Then*

$$Z_{C^\infty(M)}(C^\infty(M)^\Phi_{\mathrm{loc}}) = C^\infty(M)^{G^0} \text{ and } Z_{C^\infty(M)}(C^\infty(M)^{G^0}) = C^\infty(M)^\Phi_{\mathrm{loc}}.$$

Here, the superscript Φ means functions that are constant on all level sets of Φ (merely locally, i. e., on the connected components of the Φ-fibres, with the corresponding subscript loc*).*

The centralizers of $\Phi^* C^\infty(\mathfrak{g}^*) \subseteq C^\infty(M)^\Phi \subseteq C^\infty(M)^\Phi_{\mathrm{loc}}$ are all equal to $C^\infty(M)^{G^0}$; and the centralizers of $C^\infty(M)^G \subseteq C^\infty(M)^{G^0}$ both equal $C^\infty(M)^\Phi_{\mathrm{loc}}$.

Now we are in a position to show a key consequence of the symplectic Howe condtion: In the case of proper actions, together with some connectedness conditions, the level sets of the moment maps are itself orbits of group actions.

Proposition 3.20. *Let commuting Hamiltonian proper actions of the connected Lie groups G_1 and G_2 be given on the symplectic manifold (M, ω).*[7] *Suppose there are equivariant moment maps Φ_1 and Φ_2 whose level sets are connected.*

Then the symplectic Howe condition

$$Z_{C^\infty(M)}(\Phi_i^* C^\infty(\mathfrak{g}_i^*)) = \Phi_j^* C^\infty(\mathfrak{g}_j^*) \qquad (i + j = 3)$$

implies that

$$\forall z \in M : \Phi_i^{-1}(\Phi_i(z)) = G_j \cdot z \qquad (i + j = 3),$$

i. e., the levels sets of the moment map of one action are the orbits of the other action.

Proof. Define $N_z = \Phi_1^{-1}(\Phi_1(z))$ to be the level set of Φ_1 containing the point $z \in M$. Note that the level sets are all closed as they are the preimage of a point under a smooth map.

The level sets N_z are G_2-invariant, thus we may restrict the action of G_2 to them; this action on each N_z is again proper, thus its orbits closed. By the connectedness assumption for the N_z, we have to show that the G_2-orbits are open in their respective level set N_z. Assume now the contrary, i. e., that there is no open G_2-orbit in N_z.

Consequently, there exists a non-constant smooth G_2-invariant function $f : M \to \mathbb{R}$, which separates (at least) two orbits in N_z. By Prop. 3.19, we now see that this is equivalent to $f \in Z_{C^\infty(M)}(\Phi_2^* C^\infty(\mathfrak{g}_2^*))$. Thus, the symplectic Howe condition implies that $f = f' \circ \Phi_1$, and therefore, f is constant on the level set N_z of Φ_1. □

Remark 3.21. (i) All levels sets of a moment map are connected if, e. g., the group acting on M is a connected torus and the moment map is proper [Ati82, Thm. 1][HNP94, Thm. 4.1], or if G is connected and compact and M a smooth affine complex variety [Sja98, Cor. 4.13].

[7] Actually, it is sufficient if all G_1- and G_2-orbits on M are connected.

(ii) It is indeed possible to remove the connectedness condition on the level sets, see [BW09].

For the Lie-Weinstein condition, we cannot show such a strong result as for the symplectic Howe condition. However, the following weakened version holds.

Proposition 3.22. *Let commuting Hamiltonian proper actions of the connected Lie groups G_1 and G_2 be given on the connected symplectic manifold (M, ω). Suppose there are equivariant moment maps Φ_1 and Φ_2 whose level sets are connected. Let Ω be the intersection of the principal orbit type submanifolds of the G_1- and the G_2-action. Further suppose that there exists a point $z_0 \in \Omega$ such that $\mathfrak{g}_1 \cdot z_0 = (\mathfrak{g}_2 \cdot z_0)^\angle$. Then:*

(i) $\forall z \in \Omega \,:\, \mathfrak{g}_1 \cdot z = (\mathfrak{g}_2 \cdot z)^\angle$,

(ii) $\forall z \in \Omega \,:\, \Phi_i^{-1}(\Phi_i(z)) = G_j \cdot z \quad (i + j = 3)$.

Proof. (i) Recall that by Lemma 3.2, $\{\Phi_1^\xi, \Phi_2^\eta\} = 0$ holds for all $\xi \in \mathfrak{g}_1$ and $\eta \in \mathfrak{g}_2$. This means that $\mathfrak{g}_1 \cdot z \subseteq (\mathfrak{g}_2 \cdot z)^\angle$ for any $z \in M$. But in z_0 equality holds, so $\dim M = \dim \mathfrak{g}_1 \cdot z_0 + \dim \mathfrak{g}_2 \cdot z_0$. As on Ω all G_1-orbits are of the same dimension, and the same holds for the G_2-orbits, the claim follows from Lemma 3.4.

(ii) Again by Lemma 3.2, we see that $G_j \cdot z \subseteq \Phi_i^{-1}(\Phi_i(z))$ for any $z \in M$. Moreover, note that for $z \in \Omega$, $\Phi_i^{-1}(\Phi_i(z)) \cap \Omega$ is a submanifold, and one has

$$\mathfrak{g}_j \cdot z \subseteq T_z(\Phi_i^{-1}(\Phi_i(z))) \subseteq \ker T_z \Phi_i = (\mathfrak{g}_i \cdot z)^\angle.$$

On Ω, the moment maps have constant rank, and thus if we take $z \in \Omega$, we know by (i) that $(\mathfrak{g}_i \cdot z)^\angle = \mathfrak{g}_j \cdot z$, so that on Ω, the inclusions above are equalities, in particular, $\mathfrak{g}_j \cdot z = T_z(\Phi_i^{-1}(\Phi_i(z)))$. Therefore, $G_j \cdot z = \Phi_i^{-1}(\Phi_i(z))$ for any $z \in \Omega$.

□

Remark 3.23. (i) Here, we do no longer assume any maps to be submersions. Therefore, the Lie-Weinstein condition is weaker than symplectic complementarity as orbits of smaller dimension (thus violating the implicit dimension condition) are now admitted.

(ii) If two commuting actions satisfy the Lie-Weinstein condition, the joint action of the product of both groups is coisotropic (same proof as Lemma 3.5), where an action is said to be coisotropic if there is an open subset such that all orbits contained in it are coisotropic submanifolds. Details about the definition of coisotropic actions can be found in Def. 2 and the remarks following it of [HW90] where it is shown that this notion is equivalent to multiplicity-freeness in the case of actions of compact connected groups.

Knowing the level sets of the moment maps as precisely as above permits to relate the stabilizers of the actions on M and on the moment images $\Phi_i(M)$.

Lemma 3.24. *Let G_1 and G_2 be Lie groups and (M, ω) be a symplectic manifold. Let Hamiltonian actions of both groups on M be given which commute, and denote the equivariant moment maps by $\Phi_i : M \to \mathfrak{g}_i^*$. Let $G_{12,z} = \{(g_1, g_2) \in G_1 \times G_2 \mid (g_1, g_2) \cdot z = z\}$ be the stabilizer of a point $z \in M$ under the simultaneous action of both groups, write H_1 and H_2 for the projections of $G_{12,z}$ to the groups G_1 and G_2.*

Then

(i) H_1 *is contained in* $G_{1,\Phi_1(z)}$*, the stabilizer of the* Φ_1*-image of* z *under the coadjoint action (even* $G_{1,z} \subseteq H_1 \subseteq G_{1,\Phi_1(z)}$*), and*

(ii) if, furthermore, the level sets of Φ_1 *are actually* G_2*-orbits, then* $H_1 = G_{1,\Phi_1(z)}$.

The same statements hold for 1 *and* 2 *interchanged.*

Proof. (i) Take $g_1 \in H_1$, i. e., there exists $g_2 \in G_2$ such that $(g_1, g_2) \cdot z = z$. From this one obtains

$$\Phi_1(z) = \Phi_1((g_1, g_2) \cdot z) = g_1 \cdot \Phi_1((e, g_2) \cdot z) = g_1 \cdot \Phi_1(z),$$

where G_1-equivariance and G_2-invariance of Φ_1 have been used.

(ii) Now take $g_1 \in G_{1,\Phi_1(z)}$, then $\Phi_1(g_1 \cdot z) = \Phi_1(z)$, i. e., z and $g_1 \cdot z$ lie in a common level set of Φ_1, which is by assumption a G_2-orbit. Thus there exists $g_2 \in G_2$ such that $(g_1, g_2) \cdot z = z$, which was to be shown.

Of course, interchanging both actions does not alter the proof. □

3.3 Orbit Correspondence and Duality between Orbits and Reduced Spaces

The particular orbit structure that we have described allows to define a correspondence between the orbits in the images of the moment maps Φ_1 and Φ_2. Further, one observes a duality between coadjoint orbits and reduced spaces, which we define now.

Definition 3.25. Given a symplectic manifold (M, ω) with a Hamiltonian action of a Lie group G which admits an equivariant moment map $\Phi : M \to \mathfrak{g}^*$, one defines the *point-reduced space* at $\alpha \in \mathfrak{g}^*$ by

$$M_\alpha = \Phi^{-1}(\alpha)/G_\alpha,$$

and the *orbit-reduced space* at $\mathcal{O}_\alpha = \mathrm{Ad}^*(G)\alpha \subseteq \mathfrak{g}^*$ by

$$M_{\mathcal{O}_\alpha} = \Phi^{-1}(\mathcal{O}_\alpha)/G.$$

Historically, reduced spaces are also called *Marsden-Weinstein reductions.*

Of course, these spaces spaces are in general not manifolds but stratified spaces. However, for both definitions, the reduced spaces can be endowed with a symplectic structure and they turn out to be symplectomorphic (see [OR04] for a detailed discussion of the properties of non-singular and singular reduced spaces).

With these definitions, we state:

Theorem 3.26. *Let commuting Hamiltonian proper actions of the connected Lie groups* G_1 *and* G_2 *be given on the symplectic manifold* (M, ω)*. Suppose there are equivariant moment maps* Φ_1 *and* Φ_2 *whose level sets are connected. If the symplectic Howe condition is satisfied, then:*

(i) There is a bijection $\Lambda : \Phi_1(M)/G_1 \to \Phi_2(M)/G_2$*, given by*

$$\Lambda \, : \, \mathcal{O}_\alpha \mapsto \Phi_2(\Phi_1^{-1}(\mathcal{O}_\alpha)),$$

where $\alpha \in \Phi_1(M)$*, and* $\mathcal{O}_\alpha = \mathrm{Ad}^*(G_1)\alpha$ *is seen as an element of* $\Phi_1(M)/G_1$*. To each pair* $(\mathcal{O}_\alpha, \Lambda(\mathcal{O}_\alpha))$ *belongs a unique orbit* $(G_1 \times G_2) \cdot z$ *in* M *given by* $\Phi_1^{-1}(\mathcal{O}_\alpha)$*. We say that orbits* $(G_1 \times G_2) \cdot z$*,* $\mathcal{O}_{\alpha_1}$ *and* $\mathcal{O}_{\alpha_2}$ *are* in correspondence *if* $\mathcal{O}_{\alpha_2} = \Lambda(\mathcal{O}_{\alpha_1})$ *and* $(G_1 \times G_2) \cdot z = \Phi_1^{-1}(\mathcal{O}_{\alpha_1})$*.*

(ii) *If Φ_1 and Φ_2 are open maps, then Λ is a homeomorphism. The same conclusion holds if the groups G_1 and G_2 both are compact and Φ_1 and Φ_2 both are closed maps. Finally, the same conclusion holds if for $i+j=3$, G_i is compact and Φ_i is closed, and Φ_j is open.*

(iii) *Write $M_{\alpha_1} = \Phi_1^{-1}(\alpha_1)/G_{1,\alpha_1}$ and $M_{\alpha_2} = \Phi_2^{-1}(\alpha_2)/G_{2,\alpha_2}$ for the respective point reduced spaces of $\alpha_1 \in \mathfrak{g}_1^*$ and $\alpha_2 \in \mathfrak{g}_2^*$. These spaces can be described as coadjoint orbits of the other action, i. e., there are G_2-equivariant and G_1-equivariant, resp., symplectomorphisms*

$$M_{\alpha_1} \to \Lambda(\mathcal{O}_{\alpha_1}) \quad \text{and} \quad M_{\alpha_2} \to \Lambda^{-1}(\mathcal{O}_{\alpha_2}).$$

(iv) *The reduced space $M_{(\alpha_1,\alpha_2)}$ for the joint action of $G_1 \times G_2$ is either a point (if $\mathcal{O}_{\alpha_1}$ and $\mathcal{O}_{\alpha_2}$ are in correspondence) or empty otherwise.*

Proof. In order to simplify indices, some statements will only be proved for one action if the other case is analogous.

(i) By Prop. 3.20, we know that for any $z \in M$, the level sets of both moment maps are individual orbits: $\Phi_i^{-1}(\Phi_i(z)) = G_j \cdot z$ $(i+j=3)$. This implies that the preimage of any coadjoint orbit in either moment image $\Phi_i(M)$ $(i=1,2)$ is exactly one orbit of the joint action of $G_1 \times G_2$ on M, i. e.,

$$\Phi_i^{-1}(\mathrm{Ad}^*(G_i)\Phi_i(z)) = (G_1 \times G_2) \cdot z,$$

from which follows that Λ is well-defined and bijective.

(ii) Note that all maps in the diagram

$$\Phi_1(M)/G_1 \xleftarrow{\pi_1} \Phi_1(M) \xleftarrow{\Phi_1} M \xrightarrow{\Phi_2} \Phi_2(M) \xrightarrow{\pi_2} \Phi_2(M)/G_2$$

are continuous, and π_1 and π_2 are always open as well.

Let us denote $\Lambda : \Phi_1(M)/G_1 \to \Phi_2(M)/G_2$ by Λ_{12} and its set-theoretic inverse by Λ_{21}. If Φ_1 is open, then for U open in $\Phi_2(M)/G_2$ we have that $\Lambda_{21}(U) = (\pi_1 \circ \Phi_1)(\Phi_2^{-1}(\pi_2^{-1}(U)))$ is an open set, i. e., Λ_{12} is continuous. Similarly, if G_1 is compact and Φ_1 is closed, we have for A closed in $\Phi_2(M)/G_2$ that $\Lambda_{21}(A)$ is closed as well, i. e., again Λ_{12} is continuous.

The conclusions of (ii) follow easily.

(iii) Let $z \in M$ and $\alpha_2 = \Phi_2(z)$ its value under the moment map of the second action. Consider the restricted map $\Phi_{2|G_2\cdot z} : G_2 \cdot z \to \mathcal{O}_{\alpha_2}$, and recall that $G_2 \cdot z = \Phi_1^{-1}(\alpha_1)$ for $\alpha_1 = \Phi_1(z)$. Recall from Lemma 3.24 that $G_{1,\alpha_1} = \{h \in G_1 \mid \exists g \in G_2 : (h,g)\cdot z = z\}$. This group acts on $G_2 \cdot z$ and one has $M_{\alpha_1} = G_2 \cdot z / G_{1,\alpha_1}$. Thus $\Phi_{2|G_2\cdot z}$ induces

$$\tilde{\Phi}_2 : M_{\alpha_1} \to \mathcal{O}_{\alpha_2},$$

which inherits from Φ_2 smoothness and G_2-equivariance. It is clearly surjective and we now show that it is injective: Take $\alpha \in \mathcal{O}_{\alpha_2}$, $\tilde{z}_1, \tilde{z}_2 \in G_2 \cdot z / G_{1,\alpha_1}$ so that $\alpha =$

$\tilde{\Phi}_2(\tilde{z}_1) = \tilde{\Phi}_2(\tilde{z}_2)$. Now fixing preimages $z_1, z_2 \in G_2 \cdot z$ of $\tilde{z}_1, \tilde{z}_2$, they have the property $\Phi_2(z_1) = \Phi_2(z_2)$, and for some $h \in G_1$, $z_2 = h \cdot z_1$ holds because the level sets of Φ_2 are G_1-orbits. However, $z_2 = g \cdot z_1$ for some $g \in G_2$, which implies by part (ii) of Lemma 3.24 that $h \in G_{1,\alpha_1}$. Therefore, $\tilde{z}_1 = \tilde{z}_2$, and $\tilde{\Phi}_2$ is a bijection. Applying Sard's Theorem, one notices that a smooth equivariant bijection between finite-dimensional homogeneous spaces has a smooth inverse.

In remains to show that $\tilde{\Phi}_2$ is a symplectomorphism. Denote by $i_{G_2 \cdot z} : G_2 \cdot z \to M$ the inclusion of $G_2 \cdot z = \Phi_1^{-1}(\alpha_1)$ into the ambient manifold. We observe that the equivariance properties of Φ_2 imply $i^*_{G_2 \cdot z}\omega = (\Phi_{2|G_2 \cdot z})^* \omega^{\mathcal{O}_{\alpha_2}}$, where $\omega^{\mathcal{O}_{\alpha_2}}$ is the KKS symplectic form. But the symplectic form $\omega^{M_{\alpha_1}}$ on M_{α_1} is defined such that it also pulls back to $i^*_{G_2 \cdot z}\omega = \pi^* \omega^{M_{\alpha_1}}$, via the quotient map $\pi : G_2 \cdot z \to G_2 \cdot z / G_{1,\alpha_1}$. As $G_2 \cdot z, \mathcal{O}_{\alpha_2}$ and M_{α_1} are G_2-homogeneous, $\Phi_{2|G_2 \cdot z}$ and π are surjective and G_2-equivariant, the coincidence of the pullbacks (to $G_2 \cdot z$) implies that $\omega^{M_{\alpha_1}} = \tilde{\Phi}_2^* \omega^{\mathcal{O}_{\alpha_2}}$, which was to be shown.

(iv) Let $\Phi = \Phi_1 \oplus \Phi_2$. Take $\alpha_1 \in \mathfrak{g}_1^*$ and $\alpha_2 \in \mathfrak{g}_2^*$. Then

$$\Phi^{-1}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2})$$

is empty if $\mathcal{O}_{\alpha_2} \neq \Lambda(\mathcal{O}_{\alpha_1})$; otherwise, for any $z \in M$ such that $\Phi_1(z) \in \mathcal{O}_{\alpha_1}$ and $\Phi_2(z) \in \mathcal{O}_{\alpha_2}$, holds

$$\Phi^{-1}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2})/(G_1 \times G_2) \cong (G_1 \times G_2) \cdot z/(G_1 \times G_2).$$

This quotient is a point.

□

We observe that using the special properties of proper actions and the moment map we have proved a version of the symplectic leaf correspondence that allows for some singular behaviour, even though starting with conditions closely related to those in Thm. 3.16 which did not show this feature. Furthermore, Thm. 3.16(iii) continues to hold as its proof can easily be redone in our setting.

Beside this, one may explain more explicitly why the reduced spaces are indeed manifolds: Take any $z \in M$ and $\alpha_1 = \Phi_1(z)$. Then the global ineffectivity of the G_{1,α_1}-action on the Φ_1-level is $H_{1,\alpha_1} = \{g_1 \in G_{1,\alpha_1} \mid g_1 \cdot z' = z' \ \forall z' \in \Phi_1^{-1}(\alpha_1)\}$, i.e., the intersection of the stabilizers of all points in the level set. Here, $\Phi_1^{-1}(\alpha_1) = G_2 \cdot z$, hence $G_{1,z'} = G_{1,z} \subseteq H_{1,\alpha_1}$ (see Lemma 3.1), so $H_{1,\alpha_1} = G_{1,z}$. Therefore, the proper action of G_{1,α_1} on $\Phi_1^{-1}(\alpha_1)$ is actually equivalent to a free and proper action of $G_{1,\alpha_1}/H_{1,\alpha_1}$, thus the quotient is a smooth manifold.

As this theorem depends on Prop. 3.20, we can also remove the connectedness condition on the level sets here, see [BW09].

4 Symplectic Dual Pairs Arising from Group Actions seen as Singular Dual Pairs

The orbit correspondence of section 3.3 (Thm. 3.26(i)) includes all orbits, not only generic ones. This indicates that it can be seen in the larger context of singular dual pairs in the sense of [OR04]. Using this framework, we rederive the orbit correspondence in Cor. 4.16.1.

Therefore, we recall quickly the notion of polar pseudogroups and optimal moment maps in a way adapted to our setting. Then we can state the generalized symplectic leaf correspondence and show how to conclude our orbit correspondence from it.

4.1 The Polar Pseudo-Group

The classical definitions are of limited use in singular contexts, which was the motivation for a singular dual pair definition by Ortega and Ratiu. In order to state their definition, the notion of a polar pseudo-group is introduced. Details of this theory are available in Sections 5.5 and 11.3 of [OR04].

Let M be a manifold equipped with a Poisson structure $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$. Assume that a Lie group G is acting on this manifold (by $\Psi : G \times M \to M$) *canonically*, i. e., for all $g \in G$, Ψ_g preserves the Poisson structure on M.

Definition 4.1. The subgroup of the group of Poisson diffeomorphisms corresponding to the action of G is denoted by

$$A_G = \{\Psi_g \mid g \in G\} \subseteq \mathrm{Diff}(M, \{\cdot,\cdot\}),$$

where $\Psi_g : M \to M$ is the diffeomorphism belonging to the action of the element $g \in G$. The orbit of this action through a point $z \in M$ is written $A_G \cdot z$, the tangent space to the orbit at this point is

$$A_G(z) = \left\{\xi_z^M \mid \xi \in \mathfrak{g}\right\} \subseteq T_zM,$$

where ξ^M is the fundamental vector field of the G-action which is generated by an element $\xi \in \mathfrak{g}$.

Having rewritten the action as a subgroup of the diffeomorphism group on M, one may assign to it its polar [OR04, Def. 5.5.2].

Definition 4.2. The pseudogroup of local Poisson diffeomorphisms generated by the flows of the Hamiltonian vector fields of locally defined G-invariant functions, i. e.,

$$A'_G = \left\{F_t^{X_f} \mid f \in C^\infty(U)^G, U \subseteq M \text{ open and } G\text{-invariant}\right\},$$

is called the *polar pseudogroup* of A_G.

Definition 4.3. The *polar distribution* of A_G is the distribution associated to the family

$$\left\{X_f \mid f \in C^\infty(U)^G, U \subseteq M \text{ open and } G\text{-invariant}\right\}.$$

Evaluated at a point $z \in M$, it coincides with the tangent space $A'_G(z)$ to the A'_G-orbit. The pseudogroup A'_G is called *integrable* if the polar distribution to which it corresponds is integrable.

Of course, $\left\{F_t^{X_f} \mid f \in C^\infty(M)^G\right\}$ is contained in A'_G. So, the first natural question that arises in this context is to identify the cases where A'_G can be described using exclusively the globally defined invariant functions. An immediate answer is the following proposition [OR04, Prop. 5.5.3].

Proposition 4.4. *If for all $z \in M$ and any open G-invariant neighbourhood $U \subseteq M$ (of z) and any invariant $f \in C^\infty(U)^G$ there exists an open G-invariant neighbourhood $V \subsetneq U$ (of z) such that $f_{|V}$ can be extended to an invariant $\hat{f} \in C^\infty(M)^G$ on all of M (i. e., $\hat{f}_{|V} = f_{|V}$), then $A'_G(z) = \left\{F_t^{X_f} \mid f \in C^\infty(M)^G\right\}$. In particular, this holds if the G-action on M is proper (in that case, the polar distribution is complete, i. e., it admits a generating family consisting of complete vector fields).*

The following properties are the basis for using the notion of polar pseudogroups in the context of dual pairs, they are proved in [OR04, Prop. 5.5.4].

Proposition 4.5. *Let A_G be as above. Then the following hold:*

(i) The polar pseudogroup A'_G acts canonically and is integrable.

(ii) The group A_G and its polar A'_G commute.

(iii) A'_G acts on M/A_G by requiring the projection $\pi_{A_G} : M \to M/A_G$ to be A'_G-equivariant.

(iv) A_G acts on M/A'_G by requiring the projection $\pi_{A'_G} : M \to M/A'_G$ to be A_G-equivariant.

4.2 Optimal and Standard Moment Map

Another notion that needs to be introduced before defining singular dual pairs is given in the following definition which generalizes the usual notion of a moment map [OR04, Sect. 5.5.5].

Definition 4.6. Let G be a Lie group acting canonically on the Poisson manifold $(M, \{\cdot,\cdot\})$. Let A_G be the associated group of Poisson diffeomorphisms. Then the projection

$$\mathcal{J} : M \to M/A'_G$$

is called the *optimal moment map*. The topology and G-action on M/A'_G are chosen such that $\mathcal{J}$ is continuous, open (quotient topology) and G-equivariant.

The following property of the optimal moment map is called *universality* in [OR04, Thm. 5.5.15] and will allow to compare $\mathcal{J}$ to the usual moment map.

Proposition 4.7. *Let G be a Lie group acting canonically on $(M, \{\cdot,\cdot\})$ and let $K : M \to P$ any continuous map that is invariant under the flows of all Hamiltonian vector fields X_f of G-invariant functions $f \in C^\infty(M)^G$. Then this map factors through $\mathcal{J}$, i. e., there is a unique continuous map $\varphi : M/A'_G \to P$ such that*

$$K = \varphi \circ \mathcal{J},$$

φ being G-equivariant if K is.

Proof. Define the map by $\varphi(\varrho) = K(z)$ for any $\varrho = \mathcal{J}(z) \in M/A'_G$. It is well-defined because if one takes any $z_1, z_2 \in \mathcal{J}^{-1}(\varrho) = A'_G \cdot z_1$, there exists a composition F of flows of invariant functions such that $F(z_1) = z_2$, hence $K(z_2) = K(F(z_1)) = K(z_1)$ by the invariance assumption on K.

If there was another map φ' such that $\varphi' \circ \mathcal{J} = \varphi \circ \mathcal{J} = K$, this would imply $\forall z \in M$: $\varphi'(\mathcal{J}(z)) = \varphi(\mathcal{J}(z))$, so by the surjectivity of $\mathcal{J}$, the uniqueness follows.

Let $U \subseteq P$ be any open subset of P. Then we show that $\varphi^{-1}(U) = \mathcal{J}(K^{-1}(U))$ (the latter being open due to K continuous, $\mathcal{J}$ open):

$$\begin{aligned}\mathcal{J}(K^{-1}(U)) &= \{\mathcal{J}(z) \mid z \in M : K(z) \in U\} \\ &= \{\mathcal{J}(z) \mid z \in M : \varphi(\mathcal{J}(z)) \in U\} = \{\varrho \in M/A'_G \mid \varphi(\varrho) \in U\},\end{aligned}$$

where $K = \varphi \circ \mathcal{J}$ and $\mathcal{J}$ surjective have been used.

Now suppose that K is G-equivariant and take any $g \in G, z \in M$. Then:

$$\varphi(g \cdot \mathcal{J}(z)) = \varphi(\mathcal{J}(g \cdot z)) = K(g \cdot z) = g \cdot K(z) = g \cdot \varphi(\mathcal{J}(z))$$

Again, the surjectivity of $\mathcal{J}$ implies the claim, that is, φ is G-equivariant. □

The definition of the optimal moment map and its universality property are available without requiring M to be symplectic and the G-action on M to be proper. Adding these assumptions permits to give a very precise statement about the orbits of the A'_G-action (which itself does not need to be proper!) [OR04, Thm. 5.5.17].

Theorem 4.8. *Let G be a Lie group acting properly and symplectically on the symplectic manifold (M, ω). Then at any point $z \in M$,*

$$A'_G(z) = (\mathfrak{g} \cdot z)^{\angle} \cap T_z M_{G_z},$$

where M_{G_z} is the submanifold of M containing all points with stabilizer G_z. If the action admits an equivariant moment map $\Phi : M \to \mathfrak{g}^*$, then at any $z \in M$ with $\Phi(z) = \alpha$ and $\mathcal{J}(z) = \varrho$,

$$\mathcal{J}^{-1}(\varrho) = (\Phi^{-1}(\alpha) \cap M^z_{G_z})^z,$$

where the superscript z denotes the connected component of the corresponding object which contains the point z.

In certain cases, this theorem yields an identification of the image of the standard moment map with the quotient M/A'_G.

Proposition 4.9. *Let G be a Lie group acting properly and symplectically on the symplectic manifold (M, ω) with G-equivariant moment map $\Phi : M \to \mathfrak{g}^*$. Suppose further that all level sets of Φ are connected and contained in a single isotropy type of the G-action, i. e., $\forall \alpha \in \Phi(M) : \forall z \in \Phi^{-1}(\alpha) : \Phi^{-1}(\alpha) \subseteq M^z_{G_z}$. Then there is a continuous G-equivariant bijection $\varphi : M/A'_G \to \Phi(M)$. This bijection is a homeomorphism if Φ is closed or open. Further, φ preserves stabilizers, i. e., $G_\varrho = G_{\varphi(\varrho)}$ for any $\varrho \in M/A'_G$.*

Proof. By Thm. 4.8, one concludes that in the present situation, $\Phi^{-1}(\alpha) = \mathcal{J}^{-1}(\varrho) = A'_G \cdot z$ for all appropriate combinations of $z \in M$, $\alpha \in \Phi(M)$ and $\varrho \in M/A'_G$.

The invariance properties of the moment map Φ now allow to apply Prop. 4.7. Hence there is a unique continuous G-equivariant map φ such that $\Phi = \varphi \circ \mathcal{J}$.

Here, φ is a bijection: Let $\varrho_1, \varrho_2 \in M/A'_G$ be such that $\varphi(\varrho_1) = \varphi(\varrho_2)$, hence there exist $z_1, z_2 \in M$ with the same image $\Phi(z_1) = \Phi(z_2)$ in $\Phi(M)$ and $\varrho_i = \mathcal{J}(z_i)\,(i = 1, 2)$. Since the level sets of Φ are A'_G-orbits, there is $F \in A'_G$ mapping z_1 to z_2, therefore $\varrho_2 = \mathcal{J}(z_2) = \mathcal{J}(F(z_1)) = \mathcal{J}(z_1) = \varrho_1$.

Take now any closed set $U \subseteq M/A'_G$. Then, by continuity, $\mathcal{J}^{-1}(U)$ is closed, and by the assumption that Φ is closed, $\Phi(\mathcal{J}^{-1}(U))$ is closed, too. One checks that $\Phi(\mathcal{J}^{-1}(U)) = \varphi(U) = (\varphi^{-1})^{-1}(U)$, hence $(\varphi^{-1})^{-1}(U)$ is closed whenever U is closed and thus φ^{-1} is continuous. An analogous argument applies when Φ is open.

The stabilizers coincide since the equivariance of φ implies $G_\varrho \subseteq G_{\varphi(\varrho)}$ and the equivariance of φ^{-1} yields $G_{\varphi(\varrho)} \subseteq G_\varrho$. □

4.3 Singular Dual Pairs and Orbit Correspondence

The fact that polar pseudogroups play the role of maximal commutants of a group action motivates to compare their action to the action of another (pseudo-)group in the sense of the following definition.

Definition 4.10. Let A and B be subgroups of $\mathrm{Diff}(M, \{\cdot,\cdot\})$. The diagram

$$M/A \xleftarrow{\pi_A} M \xrightarrow{\pi_B} M/B$$

is called a *singular dual pair* if $M/A = M/B'$ and $M/B = M/A'$.

In [OR04, Ex. 11.3.4] it is shown that Lie-Weinstein dual pairs with connected fibres are dual pairs in this new sense. We add the case where such a dual pair is easily obtained from commuting Hamiltonian actions, so singular dual pairs are an appropriate means to study these actions and, more specifically, Howe pairs of symplectic actions. Recall that for G connected, $Z_{C^\infty(M)}(\Phi^* C^\infty(\mathfrak{g}^*)) = C^\infty(M)^G$ holds by Prop. 3.19.

Lemma 4.11. *Given two commuting Hamiltonian actions, both proper, of connected Lie groups G_1, G_2 on the symplectic manifold (M, ω) with equivariant moment maps Φ_1 and Φ_2. If*

$$C^\infty(M)^{G_1} = \Phi_2^* C^\infty(\mathfrak{g}_2^*) \text{ and } C^\infty(M)^{G_2} = \Phi_1^* C^\infty(\mathfrak{g}_1^*),$$

then $M/A_{G_1} \leftarrow M \rightarrow M/A_{G_2}$ forms a singular dual pair.

Proof. By definition (for G_1 acting properly), $A'_{G_1} = \left\langle \{F_t^{X_f} \mid f \in C^\infty(M)^{G_1}\} \right\rangle$. Our assumption on the algebras of invariants allows us to rewrite this as $A'_{G_1} = \left\langle \{F_t^{\xi^{(2)}} \mid \xi \in \mathfrak{g}_2\} \right\rangle = A_{G_2}$. Thus $M/A'_{G_1} = M/A_{G_2}$, and analogously, $M/A'_{G_2} = M/A_{G_1}$ holds, satisfying the conditions for a singular dual pair. □

The classical definition of Lie-Weinstein dual pairs required symplectic complementarity of the kernels of the projections in the legs of a dual pair. In the singular setting, this is weakened to symplectic orthogonality, as the next lemma shows.

Lemma 4.12. *In a singular dual pair* $M/A \xleftarrow{\pi_A} M \xrightarrow{\pi_B} M/B$ *on a symplectic manifold* (M, ω), *the kernels of the projections are symplectically orthogonal, i. e., for any* $z \in M$,

$$\ker T_z\pi_A \subseteq (\ker T_z\pi_B)^{\angle}.$$

Proof. Use $T_z(M/A) = T_z(M/B')$ and note that $\ker T_z\pi_A = B'(z)$. But $\ker T_z\pi_B = B(z)$, hence if we have vector fields X_f and X such that $X_{f|z} \in B'(z)$ and $X_z \in B(z)$, then

$$\omega(X_f, X) = \mathrm{d}f(X) = X(f) = 0,$$

since by definition of B', f is invariant under the action of B. □

Remark 4.13. On sees from the proof that equality holds for $\dim M = \operatorname{def} T_z\pi_A + \operatorname{def} T_z\pi_B = \dim B'(z) + \dim B(z)$.[8] In the setting of commuting Hamiltonian actions as in Lemma 4.11 (put $B = A_{G_2}$), the dimension of $B(z)$ is the dimension of the G_2-orbit through z, i. e., $\dim B(z) = \dim \mathfrak{g}_2 \cdot z$. By Thm. 4.8, the dimension of $B'(z)$ is less or equal the dimension of $(\mathfrak{g}_2 \cdot z)^{\angle}$ – the latter being true if the symplectic complement of $\mathfrak{g}_2 \cdot z$ lies in the tangent space of a single isotropy type. In other words, the orbits satisfying this condition satisfy the Lie-Weinstein condition.

As for the non-singular dual pairs, the aim is to exhibit a correspondence between symplectic leaves. The first step is to generalize the notion of a symplectic leaf to quotients, which can be done as follows [OR04, Def. 11.4.2, Thm. 11.4.3]. Note that these symplectic leaves do no longer carry a symplectic structure, except for special cases. Recall that A'_G acts on M/A_G by requiring the canonical projection to be equivariant (as for the A_G-action on M/A'_G).

Definition 4.14. Let G act canonically on the Poisson manifold $(M, \{\cdot,\cdot\})$, let A_G be the image of G in $\mathrm{Diff}(M, \{\cdot,\cdot\})$. Then a *generalized symplectic leaf* in M/A_G is an orbit of A'_G in M/A_G.

As A'_G is defined through flows of vector fields, its orbits on M are connected; thus the generalized symplectic leaves are connected, too.

Example 4.15. Let $(M, \{\cdot,\cdot\})$ be acted upon by the trivial group $G = \{e\}$. Then $A'_G = \left\langle \{F_t^{X_f} \mid f \in C^\infty(M)\} \right\rangle$. Hence the A'_G-orbits in this case are the integral manifolds of the distribution spanned by all Hamiltonian vector fields, i. e., the symplectic leaves in the usual sense.

In what follows, we will only be concerned with dual pairs whose symplectic leaves are indeed symplectic manifolds; their part will mainly be taken over by coadjoint orbits.

Using the above definition, [OR04, Thm. 11.4.4] shows that the correspondence holds in more general setting of singular dual pairs.

Theorem 4.16 (Symplectic Leaf Correspondence)**.** *Let* $(M, \{\cdot,\cdot\})$ *be a Poisson manifold acted upon canonically by Lie groups* G_1 *and* G_2. *Suppose that* $M/A_{G_1} \leftarrow M \rightarrow M/A_{G_2}$ *is a singular dual pair with* $\pi_i : M \rightarrow M/A_{G_i}$ $(i = 1, 2)$ *the canonical projections. Then the map*

$$(M/A_{G_1})/A_{G_2} \rightarrow (M/A_{G_2})/A_{G_1}, \qquad A_{G_2} \cdot \pi_1(z) \mapsto A_{G_1} \cdot \pi_2(z)$$

[8] Recall that the *defect* of a linear map L is defined to be $\operatorname{def} L = \dim\ker L$.

is a well-defined bijection. Analogously, the map

$$(M/A_{G_1})/A_{G_2} \to M/(A_{G_1} \times A_{G_2}), \qquad A_{G_2} \cdot \pi_1(z) \mapsto \mathcal{K} = (A_{G_1} \times A_{G_2}) \cdot z$$

is a well-defined bijection.

The second bijection is not given by [OR04] but proved in exactly the same way. Unlike the generalized symplectic leaves, $\mathcal{K}$ is not necessarily connected. If the generalized symplectic leaves in a singular dual pair are actually symplectic, then the same relation between the symplectic forms holds that is known from (non-singular) Lie-Weinstein dual pairs; the proof (as in [OR04, Thm. 11.1.9(iii)]) goes through without significant changes.

Lemma 4.16. *Let a symplectic manifold (M,ω) be given and subgroups $A, B \subseteq \mathrm{Diff}(M,\omega)$. Suppose that $M/A \xleftarrow{\pi_1} M \xrightarrow{\pi_2} M/B$ is a singular dual pair. Suppose that the leaves in both M/A and M/B carry symplectic structures, say ω_1 and ω_2 on the leaves $\mathcal{L}_1$ and $\mathcal{L}_2$ in correspondence, respectively. Then the symplectic forms satisfy*

$$(i_{\mathcal{K}})^*\omega = (\pi_{1|\mathcal{K}})^*\omega_1 + (\pi_{2|\mathcal{K}})^*\omega_2,$$

where $i_{\mathcal{K}} : \mathcal{K} \to M$ is the inclusion of the leaf $\mathcal{K}$ in M corresponding to $\mathcal{L}_1$ and $\mathcal{L}_2$.

Proof. Choose any point $z \in M$, let $\mathcal{K}$ be the leaf through z and take $u, v \in T_z\mathcal{K}$. Then write these vectors as sums $u = u_1 + u_2$ and $v = v_1 + v_2$ where $u_1, v_1 \in \ker T_z\pi_1$; $u_2, v_2 \in \ker T_z\pi_2$; such a splitting always exists because the kernels are symplectically orthogonal (it is not unique, though). Then

$$\begin{aligned}
\omega_z(u,v) &= \omega_z(u_1,v_1) + \omega_z(u_2,v_2) \\
&= (\pi_2^*\omega_2)_z(u_1,v_1) + (\pi_1^*\omega_1)_z(u_2,v_2) \\
&= \omega_{2|\pi_2(z)}(T_z\pi_2(u_1), T_z\pi_2(v_1)) + \omega_{1|\pi_1(z)}(T_z\pi_1(u_2), T_z\pi_1(v_2)) \\
&= \omega_{2|\pi_2(z)}(T_z\pi_2(u), T_z\pi_2(v)) + \omega_{1|\pi_1(z)}(T_z\pi_1(u), T_z\pi_1(v)) \\
&= (\pi_2^*\omega_2)_z(u,v) + (\pi_1^*\omega_1)_z(u,v).
\end{aligned}$$

This calculation does clearly not depend on the choice of u_1, u_2, v_1, v_2. ☐

Now, as a corollary to the theorem, one obtains – as in Thm. 3.26(i) – a bijection between coadjoint orbits in the moment images of commuting Hamiltonian actions without any genericity restriction.

Corollary 4.16.1. *Let commuting proper Hamiltonian actions of two connected Lie groups G_1 and G_2 on a symplectic manifold (M,ω) be given. Suppose that the level sets of the equivariant moment maps $\Phi_i : M \to \mathfrak{g}_i^*$ ($i = 1,2$) are connected. Assume these actions satisfy the symplectic Howe condition. Then there is a bijection $\Lambda : \Phi_1(M)/G_1 \to \Phi_2(M)/G_2$.*

Proof. By the symplectic Howe condition and Prop. 3.19, one has $C^\infty(M)^{G_i} = \Phi_j^* C^\infty(\mathfrak{g}_j^*)$ for $i + j = 3$. By Lemma 4.11, we have a singular dual pair in this situation:

$$M/A_{G_1} \leftarrow M \to M/A_{G_2}$$

By the symplectic leaf correspondence, there is a bijection

$$\pi_2 \circ \pi_1^{-1} : (M/A_{G_1})/A_{G_2} \to (M/A_{G_2})/A_{G_1},$$

where $\pi_i : M \to M/A_{G_i}$ are the canonical projections and the symplectic leaves are seen as subsets of the space in which they are lying.

By the symplectic Howe condition and Prop. 3.20, the level sets of the moment maps are $\Phi_i^{-1}(\Phi_i(z)) = G_j \cdot z$ for any point $z \in M$ and $i + j = 3$. By Lemma 3.1, the G_j-orbits and thus the Φ_i-level sets are contained in a single G_i-isotropy type submanifold. Now Prop. 4.9 applies and therefore, $M/A_{G_1} = M/A'_{G_2} = \Phi_2(M)$ and $M/A_{G_2} = M/A'_{G_1} = \Phi_1(M)$ hold.

Explicitly, when $\varphi_i : M/A'_{G_i} = M/A_{G_j} \to \Phi_i(M)$ $(i + j = 3)$ are the bijections coming from Prop. 4.9, then

$$\Lambda = \varphi_2 \circ \pi_1 \circ \pi_2^{-1} \circ \varphi_1^{-1} = \varphi_2 \circ \mathcal{J}_2 \circ (\varphi_1 \circ \mathcal{J}_1)^{-1} = \Phi_2 \circ \Phi_1^{-1}$$

is a well-defined bijection on the level of coadjoint orbits. □

Thus we have placed our orbit correspondence (Thm. 3.26(i)) in the context of singular dual pairs. By Thm. 4.8, we see that for all points $z \in M$, the following generalization of the Lie-Weinstein condition is satisfied $(i + j = 3)$:

$$\mathfrak{g}_i \cdot z = (\mathfrak{g}_j \cdot z)^{\angle} \cap T_z M_{G_{j,z}}.$$

Remark 4.17. Observe that we did not rederive the duality between coadjoint orbits and reduced spaces of Thm. 3.26(iii), although results about reduced spaces are available in the context of singular dual pairs (see, e. g., [OR04, Cor. 11.6.11]).

5 Geometric Quantization of Symplectic Dual Pairs

Before quantizing symplectic dual pairs, we recall some basic properties of representations of Howe dual pairs, followed by a reminder on the procedure of geometric quantization. Thereafter, we describe the behaviour of symplectic dual pairs and the orbit correspondence under quantization.

In particular, we obtain that the orbit correspondence is compatible with the integrality of the orbits, i. e., it maps integral orbits to integral ones. Finally, using results on multiplicities and reduced spaces, we can show that the $(G_1 \times G_2)$-representation on the geometric quantization space admits a decomposition that satisfies the representation-theoretic Howe condition, as recalled below.

5.1 Howe Dual Pairs and Howe Condition

Our final result, Thm. 5.17, will show that the symplectic Howe condition implies a special decomposition of the $(G_1 \times G_2)$-representation obtained via geometric quantization from the commuting Hamiltonian actions of G_1 and G_2.

This decomposition does not come as a surprise – indeed, it is the same one that is known as *Howe's duality* in the representation theory of reductive Lie groups (and in invariant theory), valid for certain representations of Howe (dual) pairs [How89].

Definition 5.1. Let G be a Lie group. A *Howe (dual) pair* in G is a pair (G_1, G_2) of two Lie subgroups of G which are their mutual centralizers in G:

$$Z_G(G_1) = G_2 \quad \text{and} \quad Z_G(G_2) = G_1.$$

All Lie groups are assumed to be real reductive.

We will restrict ourselves to compact groups in the sequel, see a listing of compact examples of Howe dual pairs in App. B. Representations of Howe dual pairs are natural candidates for representations satisfying the following condition.

Definition 5.2. Given a product $G_1 \times G_2$ of two compact Lie groups linearly represented on a finite dimensional complex vector space V ($\varrho : G_1 \times G_2 \to GL(V)$), we say that the representation satisfies the *representation-theoretic Howe condition* if in the unitary dual $\widehat{G_1}$ of G_1, there is a subset $\mathcal{D} \subseteq \widehat{G_1}$, and, defined on it, an injective map $\Lambda : \mathcal{D} \to \widehat{G_2}$ such that

$$V \cong \bigoplus_{[V_\alpha] \in \mathcal{D}} V_\alpha \otimes W_\alpha, \tag{5.1}$$

where V_α represents a class $[V_\alpha]$ in $\mathcal{D}$ and W_α represents the class $\Lambda([V_\alpha])$. The map $[V_\alpha] \mapsto [W_\alpha]$ from $\mathcal{D} \subseteq \widehat{G_1}$ to $\Lambda(\mathcal{D}) \subseteq \widehat{G_2}$ is called a *representation-theoretic Howe duality.*

Condition (5.1) for the representation (ϱ, V) is equivalent to another condition, which we can interpret as quantum counterpart of the symplectic Howe condition. Note that by ϱ, we mean the representation of the Lie group as well as of its Lie algebra.

Lemma 5.3. *Let a product $G_1 \times G_2$ of two compact connected Lie groups be linearly represented on a finite-dimensional vector space V over $\mathbb{C}$ by $\varrho : G_1 \times G_2 \to GL(V)$. The representation-theoretic Howe condition is equivalent to the* double commutant condition

$$Z_{\mathrm{End}(V)}(\varrho_i(\mathcal{U}\mathfrak{g}_i)) = \varrho_j(\mathcal{U}\mathfrak{g}_j) \qquad \text{for } i \neq j, \tag{5.2}$$

where $\mathcal{U}\mathfrak{g}_k$ is the universal enveloping algebra (over $\mathbb{C}$) of a Lie algebra $\mathfrak{g}_k$, $\varrho_k = \varrho_{|G_k}$ is the restriction of the given representation to a member of the product $G_1 \times G_2$ (for $k = 1, 2$), and for $S \subseteq \mathrm{End}(V)$, $Z_{\mathrm{End}(V)}(S)$ is the centralizer of S in the endomorphisms of V.

Proof. This equivalence is a consequence of the double commutant theorem [GW98, Thm. 3.3.7] and the general duality theorem [GW98, Thm. 4.5.12] for the implication (5.2) ⇒ (5.1).

Lemma 3.1.9 in [GW98] and Burnside's theorem show (5.1) ⇒ (5.2). □

Remark 5.4. We see the symplectic Howe condition (Def. 3.17) as the natural classical analog of this double commutant condition for two representations. Here, the endomorphisms in $\mathrm{End}(V)$ play the role of the quantum analog of the classical observables $C^\infty(M)$; and $\varrho_i(\mathcal{U}\mathfrak{g}_i)$ is the quantized counterpart of the collective functions $\Phi_i^* C^\infty(\mathfrak{g}_i^*)$.

Such a situation may be generated from Howe pairs.

5.2 Reminder on Geometric Quatization

Before actually quantizing our symplectic dual pair setting, we give a quick review of geometric quantization.

Given a symplectic manifold (M, ω) with integral symplectic form, i. e., $[\omega] \in H^2_{\mathrm{dR}}(M, \mathbb{Z})$, there exists a complex line bundle $L \to M$ with a connection ∇ and a Hermitean form h on L such that the curvature of ∇ is ω and the connection is metric with respect to the form h. We call such a line bundle a *prequantizing line bundle*. This bundle is unique if there are no non-trivial flat line bundles over M, i. e., in particular, if M is simply connected (this is the case for coadjoint orbits of compact groups: notice that the coadjoint orbits do not change when going to the simply connected cover of the group). In the sequel, we use the following notation.

Notation. If (M, ω) is a symplectic manifold with integral form, we denote by $L(M, \omega)$ a corresponding prequantum line bundle, i. e., a complex line bundle with first de Rham Chern class equal to $[\omega]$. We may omit ω if there is no ambiguity about the symplectic form. For a coadjoint orbit $\mathcal{O}_\alpha$, we will write $L_\alpha = L(\mathcal{O}_\alpha)$, the KKS symplectic form $\omega^{\mathcal{O}_\alpha}$ being understood.

If a group action of a Lie group G is given on the symplectic manifold (M, ω), the question arises whether it can be lifted to L. An answer is given by the following theorem [Kos70, Thm. 4.5.1], which assumes the action to be Hamiltonian with equivariant moment map.

Theorem 5.5. *Let (M, ω) be a symplectic manifold with integral form such that there exist a prequantum line bundle $p : L \to M$, a Hermitean structure on the sections of L, and a connection on L with curvature ω and metric with respect to the Hermitean structure.*

Let a compact, connected and simply connected Lie group G be given with a Hamiltonian action on (M, ω), the moment map $\Phi : M \to \mathfrak{g}^*$ being assumed to be equivariant.

Then there exists a lift of the action of G to L; the fundamental vector fields of this action, for any $\xi \in \mathfrak{g}$, are given by

$$\xi^L = p^* \Phi^\xi \, (2\pi \mathrm{i})^L + \widetilde{\xi^M}, \tag{5.3}$$

where Φ^ξ is the corresponding component of the moment map, $(2\pi\mathrm{i})^L$ the fundamental vector field for the natural fibrewise $U(1)$-action on L, and $\widetilde{\xi^M}$ is the horizontal lift of the fundamental vector field for the action on M.

This theorem does not include torus actions nor of other groups that are not simply connected; the following extension of its scope is possible (compare [Dui96, Prop. 15.4] and [Wur96]).

Proposition 5.6. *If $G = T$ is a torus, then there exists a lift to L of the Hamiltonian action of T on M. If $G = K$ is a compact Lie group, then there exists a finite covering of K whose action admits a lift to L. In both cases, the lift can be chosen to preserve the connection 1-form and the Hermitean structure.*

Assume from now on that a lift of the G-action to L has been chosen. Then the action of a stabilizer G_z of a point $z \in M$ lifts to a linear action on the fibre $L_z = p^{-1}(z)$ in L. As any fibre L_z is isomorpic to $\mathbb{C}$, this action can be seen as a character $\chi : G_z \to \mathbb{C}^\times = GL(1,\mathbb{C})$, even as a unitary character as the lifted action preserves the Hermitean form on L. If G_z is connected, then Kostant's formula (5.3) yields an explicit formula for this character, using that any $h \in G_z$ may be expressed as $h = \exp \xi$:

$$\chi(h) = e^{-2\pi \mathrm{i} \Phi^\xi(z)} \in U(1). \tag{5.4}$$

In order to proceed from prequantization to quantization, one has to take a close look on the space of sections $\Gamma^\infty(M, L)$ of the prequantum line bundle. The G-action on L induces a G-action on the sections. However, this space is often too big to be an irreducible representation and does not satisfy other requirements suggested by canonical quantization in physics (e. g., square integrability). That is why one introduces so-called polarizations, i. e., subbundles of TM with respect to which sections are required to be covariantly constant. Here, we will take the following short-cut: In the case of M being Kähler, the space $\Gamma_{\mathrm{hol}}(M, L)$ of holomorphic sections can be considered as the geometric quantization of (M, ω) – this is what we will do in the sequel. In particular, for G a compact Lie group, its coadjoint orbits are Kähler, and saying that the spaces of polarized sections over these orbits are irreducible as G-representations is a reformulation of the Borel-Weil theorem.

With our results about the duality of orbits and reduced spaces (Thm. 3.26(iii)), we arrived at precise statements about the occurring reduced spaces. This knowledge can be exploited for the quantization, given that we know how reduced spaces behave under geometric quantization. For that reason, we recall the quantization-commutes-with-reduction theorem for the case that the reduced spaces are manifolds [GS82, Thm. 3.2].

Theorem 5.7. *Let (M, ω) be a prequantizable symplectic manifold with Hamiltonian G-action (G compact connected Lie group), equivariant moment map Φ and line bundle $L(M)$ equipped with a connection ∇ on $L(M)$ whose curvature coincides with ω. If the G-action on the level* *set $\Phi^{-1}(0)$ is free, then the reduced space $M_0 = \Phi^{-1}(0)/G$ is a manifold, and there exists a unique line bundle $L(M_0)$ over M_0 such that*

$$\pi^* L(M_0) = i^* L(M) \text{ and } \pi^*\nabla_0 = i^*\nabla,$$

where $\pi : \Phi^{-1}(0) \to \Phi^{-1}(0)/G$ is the quotient map and $i : \Phi^{-1}(0) \to M$ the inclusion.

By the shifting trick, any reduced space may be regarded as a reduced space at 0. More precisely, one has the following folklore result (see, e. g., [OR04, Thm. 6.5.2]).

Theorem 5.8. *The reduced space at $\alpha \in \mathfrak{g}^*$, $\Phi^{-1}(\alpha)/G_\alpha$, is symplectomorphic to the reduction of $M \times \mathcal{O}_\alpha^-$ at 0, where $\mathcal{O}_\alpha^-$ denotes the coadjoint orbit through α with the KKS symplectic form multiplied by -1.*

Let us recall that for G a compact connected semisimple Lie group and $\alpha \in \mathfrak{g}^*$, $\mathcal{O}_\alpha$ is an integral symplectic manifold if and only if there exists a character $\chi_\alpha : G_\alpha \to U(1)$ such that $(\chi_\alpha)_{*e} = 2\pi\mathrm{i}\alpha$. Thus one defines:

Definition 5.9. (i) Let G be a compact connected Lie group and $\alpha \in \mathfrak{g}^*$. We call α *integral* if there exists $\chi_\alpha : G_\alpha \to U(1)$ such that $(\chi_\alpha)_{*e} = 2\pi\mathrm{i}\alpha$.

(ii) If α is integral, we call $\mathcal{O}_\alpha$ an *integral orbit.*

We observe that integral coadjoint orbits are integral symplectic manifolds but the converse statement is not true in general because of the possible presence of a positive-dimensional centre.

5.3 Prequantization: Reduced Spaces and Integral Orbit Correspondence

The aim now is to show that the coadjoint orbit correspondence which we have obtained in Thm. 3.26(i) does preserve the integrality of the orbits. By Thm. 3.26(iii), we know that the preservation of integrality is closely related to the reduced spaces occurring in our setting.

From Thm. 5.8 we conclude that any reduced space M_{α_1} admits a prequantizing line bundle $L(M_{\alpha_1})$ if the coadjoint orbit $\mathcal{O}_{\alpha_1}$ does. Take our symplectomorphism $\tilde\Phi_2 : M_{\alpha_1} \to \mathcal{O}_{\alpha_2}$, then we obtain the bundle $\left(\tilde\Phi_2^{-1}\right)^* L(M_{\alpha_1})$ over $\mathcal{O}_{\alpha_2}$. As coadjoint orbits of compact connected Lie groups are simply connected (which can be seen by going to the simply connected cover of the Lie group which has the same coadjoint orbits), there are no non-trivial flat vector bundles, hence no torsion line bundles. Therefore, for any α_1 such that $(\mathcal{O}_{\alpha_1}, \omega^{\mathcal{O}_{\alpha_1}})$ is an integral symplectic manifold, the bundle $\left(\tilde\Phi_2^{-1}\right)^* L(M_{\alpha_1})$ is the unique prequantum line bundle over $\mathcal{O}_{\alpha_2}$. Thus we have proved:

Proposition 5.10. *Let $\mathcal{O}_{\alpha_1}$ and $\mathcal{O}_{\alpha_2}$ be two coadjoint orbits in correspondence as in Thm. 3.26(i). Then $(\mathcal{O}_{\alpha_1}, \omega^{\mathcal{O}_{\alpha_1}})$ is an integral symplectic manifold if and only if $(\mathcal{O}_{\alpha_2}, \omega^{\mathcal{O}_{\alpha_2}})$ is an integral symplectic manifold, too.*

Let us from now on assume that the $(G_1 \times G_2)$-action on M comes with a fixed "linearization on L", i. e., there is given a fibrewise linear $(G_1 \times G_2)$-action on $L = L(M, \omega)$ covering the $(G_1 \times G_2)$-action on M.

Theorem 5.11. *Let $\mathcal{O}_{\alpha_1}$ and $\mathcal{O}_{\alpha_2}$ be two coadjoint orbits in correspondence as in Thm. 3.26(i). Then $\mathcal{O}_{\alpha_1}$ is integral if and only if $\mathcal{O}_{\alpha_2}$ is integral.*

Proof. Let $\alpha \in \mathfrak{g}_1^*$ be integral. Since the G_2-actions on M and L commute with the G_1-actions, the G_2-action on L descends canonically to a G_2-action on $L(M_{\alpha_1})$ covering the natural G_2-action on M_{α_1}. (See [GS82] for the construction of the bundle $L(M_{\alpha_1})$.)

Denoting the G_2-equivariant symplectomorphism $\tilde{\Phi}_2 : (M_{\alpha_1}, \omega^{M_{\alpha_1}}) \to (\mathcal{O}_{\alpha_2}, \omega^{\mathcal{O}_{\alpha_2}})$ from above by Ψ, we have the bundle $(\Psi^{-1})^* L(M_{\alpha_1}) = L(\mathcal{O}_{\alpha_2})$ together with a G_2-invariant connection ∇ given by pulling back the G_2-invariant connection induced on $L(M_{\alpha_1})$ by a $(G_1 \times G_2)$-invariant connection on $L \to M$. (See again [GS82] for the construction of the connection on $L(M_{\alpha_1})$.) We thus arrive at the following commuting diagram (where p denotes the bundle projection and $L(\mathcal{O}_{\alpha_2})_{\alpha_2} = p^{-1}(\alpha_2)$):

$$\begin{array}{ccc} G_2 \times L(\mathcal{O}_{\alpha_2}) & \longrightarrow & L(\mathcal{O}_{\alpha_2}) \\ \Big\downarrow{\scriptstyle \mathrm{id} \times p} & & \Big\downarrow{\scriptstyle p} \\ G_2 \times \mathcal{O}_{\alpha_2} & \longrightarrow & \mathcal{O}_{\alpha_2}, \end{array}$$

yielding a homomorphism $\chi : G_{2,\alpha_2} \to U(L(\mathcal{O}_{\alpha_2})_{\alpha_2}) = U(1)$.

It remains to show that $\chi = \chi_{\alpha_2}$, i. e., $\chi_{*e} = 2\pi \mathrm{i} \alpha_2$. Since the G_2-action on $L(\mathcal{O}_{\alpha_2})$ and the connection ∇ come from the reduced bundle with connection $L(M_{\alpha_1})$ and this comes in turn from a connection on $L \to M$, we have the usual Kostant formula (5.3) for the fundamental vector fields of the G_2-action on $L(\mathcal{O}_{\alpha_2})$ (compare Thm. 5.5). More precisely, given $\xi \in \mathfrak{g}_2$, $\xi^{L(\mathcal{O}_{\alpha_2})} = \widetilde{\xi^{\mathcal{O}_{\alpha_2}}} + p^*(\Phi_2^{\mathcal{O}_{\alpha_2}})(2\pi \mathrm{i})^{L(\mathcal{O}_{\alpha_2})}$, where ξ^N denotes the fundamental vector field associated to ξ on a G_2-manifold N, $\widetilde{X}$ denotes the ∇-horizontal lift of a vector field X on $\mathcal{O}_{\alpha_2}$ to $L(\mathcal{O}_{\alpha_2})$, $2\pi\mathrm{i} \in \mathrm{i}\mathbb{R} \cong \mathfrak{u}(1)$ also has a fundamental vector field on the bundle by the canonical $U(1)$-action on it, and $\Phi_2^{\mathcal{O}_{\alpha_2}}$ is the G_2-moment map on $\mathcal{O}_{\alpha_2}$.

For $\xi \in \mathfrak{g}_{2,\alpha_2}$, the field $\xi^{L(\mathcal{O}_{\alpha_2})}$ is now tangent to the p-fibre $L(\mathcal{O}_{\alpha_2})_{\alpha_2}$ over α_2 and equals there $(2\pi\mathrm{i})^{L(\mathcal{O}_{\alpha_2})} \langle \alpha_2, \xi \rangle$. Thus $\chi_{*e} : G_{2,\alpha_2} \to U(1)$ is equal to $\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 e^{2\pi\mathrm{i}\langle \alpha_2, \xi \rangle t} = 2\pi\mathrm{i}\langle \alpha_2, \xi \rangle$. □

5.4 Explicit Characters for the Integral Orbit Correspondence

Having checked that integrality is preserved under the orbit correspondence, it is natural to search directly for a relation between the characters $\chi_i : G_{i,\alpha_i} \to U(1)$ and the character $\chi : G_{12,z} \to U(1)$ coming from the $(G_1 \times G_2)$-action on L over a point $z \in M$.

Kostant's result (Thm. 5.5) will now allow us to prove a more explicit version of the integral correspondence, at the price of stronger assumptions on connectedness and simply-connectedness compared to Thm. 5.11.

Theorem 5.12. *Let (M, ω) be an integral symplectic manifold, with commuting Hamiltonian actions of connected and simply connected compact Lie groups G_1 and G_2. Assume the moment maps $\Phi_i : M \to \mathfrak{g}_i^*$ $(i = 1, 2)$ to be equivariant. Abbreviate $G_{12} = G_1 \times G_2$ and $\Phi_{12} = \Phi_1 \oplus \Phi_2$. Require all stabilizers belonging to any occurring action to be connected.*

Assume further that the orbits of G_1 are level sets of Φ_2 and vice versa such that the coadjoint orbits in $\Phi_1(M)$ and $\Phi_2(M)$ are in correspondence via the map Λ of Thm. 3.26.

In this situation, the integrality of an orbit $\mathcal{O}_{\alpha_1} \subseteq \Phi_1(M)$ *implies the integrality of the corresponding orbit* $\mathcal{O}_{\alpha_2} \subseteq \Phi_2(M)$ *and vice versa. The characters* χ, χ_1 *and* χ_2 *of* $G_{12,z}, G_{1,\alpha_1}$ *and* G_{2,α_2}*, resp., satisfy the following relation:*

$$\chi((g_1, g_2)) = \chi_1(g_1)\chi_2(g_2)$$

for any $(g_1, g_2) \in G_{12,z}$*.*

Proof. By Thm. 5.5, the action of the group G_{12} does lift to the prequantization bundle L over M. The action of the stabilizer $G_{12,z}$ of a point $z \in M$ does lift to a linear action on the fibre L_z over z. Hence, it can be expressed as a map $\chi : G_{12,z} \to \mathbb{C}^\times$, as $\mathbb{C}^\times = GL(L_z)$.

If $h \in G_{12,z}$, write it as $h = \exp \xi$ for some $\xi \in \mathfrak{g}_{12,z}$. Then by (5.4), we know that the character is given by $\chi(h) = e^{-2\pi \mathrm{i} \Phi^\xi(z)}$.

Having assumed that the level sets of the moment maps of one action are the orbits of the other action, we may apply Lemma 3.24, which says that the stabilizer G_{1,α_1} of the element $\alpha_1 = \Phi_1(z) \in \mathfrak{g}_1^*$ under the coadjoint action can be expressed as

$$G_{1,\alpha_1} = \{g_1 \in G \mid \exists g_2 \in G_2 : (g_1, g_2) \in G_{12,z}\},$$

and analogously for G_{2,α_2}. Moreover, in this setting, we have the orbit correspondence of Thm. 3.26. Therefore, there is a bijection between the G_{12}-orbits in M, the coadjoint G_1-orbits in $\Phi_1(M)$ and the coadjoint G_2-orbits in $\Phi_2(M)$ given by $G_{12} \cdot z \leftrightarrow \mathrm{Ad}^*(G_1)\alpha_1 \leftrightarrow \mathrm{Ad}^*(G_2)\alpha_2$.

Having this information at hand, we can now verify that the orbit correspondence does send integral coadjoint G_1-orbits to integral coadjoint G_2-orbits. For a stabilizer $G_{12,z}$ of the G_{12}-action on M, we have constructed the character χ. Let $\alpha_i = \Phi_i(z)$ $(i = 1, 2)$ and G_{i,α_i} be its stabilizers. Analogously, a character χ_1 of G_{1,α_1} can be constructed for $\omega^{\mathcal{O}_{\alpha_1}}$; the existence of such a character is the very definition of integrality of the coadjoint orbit. If $h = (h_1, h_2) = (\exp \xi_1, \exp \xi_2) \in G_{12,z}$, then one calculates

$$\begin{aligned} \frac{\chi(h)}{\chi_1(h_1)} &= \frac{e^{-2\pi \mathrm{i} \Phi^\xi(z)}}{e^{-2\pi \mathrm{i} \langle \alpha_1 \,|\, \xi_1 \rangle}} \\ &= \frac{e^{-2\pi \mathrm{i} (\langle \alpha_1 \,|\, \xi_1 \rangle + \langle \alpha_2 \,|\, \xi_2 \rangle)}}{e^{-2\pi \mathrm{i} \langle \alpha_1 \,|\, \xi_1 \rangle}} \\ &= e^{-2\pi \mathrm{i} \langle \alpha_2 \,|\, \xi_2 \rangle}, \end{aligned}$$

thus the quotient does not depend on h_1. Therefore, it is possible to define $\chi_2(h_2) = \chi(h)\chi_1(h_1)^{-1}$; and χ_2 actually is a character. In other words, the integrality of one coadjoint orbit in the correspondence implies the integrality of the other one and the characters are related as claimed. □

5.5 Quantization of Commuting Hamiltonian Actions

In order to make general statements about the geometric quantization of our setting, we will restrict our assumptions further. More precisely, we will assume that M is Kähler, the acting groups are compact and connected and that the moment maps are admissible in the sense of [Sja95, p. 109].

Definition 5.13. Choose a G-invariant inner product on $\mathfrak{g}$ with corresponding norm and define the function $\mu = \|\Phi\|^2$. Let F_t be the gradient flow of $-\mu$. A moment map is called *admissible* if for every $z \in M$, the path of steepest descent $F_t(z)$ through z ($t \geq 0$) is contained in a compact set.

Remark 5.14. Examples of admissible moment maps are all proper moment maps and the moment map of the natural linear $U(n)$-action on $\mathbb{C}^n$.

One then has the following theorem [Sja95, Thm. 2.20].

Theorem 5.15. *Let G be a compact connected Lie group acting by holomorphic transformations on a Kähler manifold M with admissible equivariant moment map Φ. Suppose this action extends to an action of the complexification $G^{\mathbb{C}}$ of G. For every positive weight α of G, the space of holomorphic sections $\Gamma_{\mathrm{hol}}(M_\alpha, L_\alpha)$ of the prequantum line bundle L_α over the symplectic reduced space M_α is naturally isomorphic to $\mathrm{Hom}_G(V_\alpha, \Gamma_{\mathrm{hol}}(M, L))$, the space of intertwining operators from the irreducible representation V_α with highest weight α to the quantization $\Gamma_{\mathrm{hol}}(M, L)$ of M.*

Remark 5.16. The reduced spaces M_α inherit, also in the singular case, "sufficient" structure in order to define holomorphic sections of $L_\alpha \to M_\alpha$ (compare [Sja95]).

Assume now that we are again in the setup of Thm. 3.26 and assume that G_1 and G_2 act by holomorphic transformations, (M, ω) is Kähler, and $[\omega]$ is an integral class. Let a lift of the $(G_1 \times G_2)$-action to the prequantizing line bundle $L \to M$ be fixed. Of course, the lifted transformations are assumed to be holomorphic as well. By the Borel-Weil theorem, the geometric quantizations, $V_{\alpha_i} \cong \Gamma_{\mathrm{hol}}(\mathcal{O}_{\alpha_i}, L_{\alpha_i})$ with α_i integral, realize the irreducible representations of G_i, thus $\Gamma_{\mathrm{hol}}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2}, L_{\alpha_1} \boxtimes L_{\alpha_2})$ the irreducibles for $G_1 \times G_2$. Here $L_{\alpha_1} \boxtimes L_{\alpha_2} \to M_1 \times M_2$ is given as $p_1^*(L_{\alpha_1}) \otimes p_2^*(L_{\alpha_2})$ with $p_j : M_1 \times M_2 \to M_j$ denoting the j-th projection for $j = 1, 2$. Now,

$$\mathrm{Hom}_{G_1 \times G_2}\left(\Gamma_{\mathrm{hol}}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2}, L_{\alpha_1} \boxtimes L_{\alpha_2}), \Gamma_{\mathrm{hol}}(M, L)\right) \cong \Gamma_{\mathrm{hol}}\big(\Phi^{-1}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2})/(G_1 \times G_2), L_{(\alpha_1, \alpha_2)}\big)$$

By Thm. 3.26(iv), the reduced space may be empty (if the coadjoint orbits are not in correspondence), hence the space of sections $\Gamma_{\mathrm{hol}}(\Phi^{-1}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2})/(G_1 \times G_2), L_{(\alpha_1, \alpha_2)})$ is trivial; otherwise the reduced space is a point and the space of sections is simply $\mathbb{C}$. So, one concludes for the multiplicities of $G_1 \times G_2$-representations in the quantization of M:

$$\dim \mathrm{Hom}_{G_1 \times G_2}\left(\Gamma_{\mathrm{hol}}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2}, L_{\alpha_1} \boxtimes L_{\alpha_2}), \Gamma_{\mathrm{hol}}(M, L)\right) \leq 1,$$

the equal sign being true if and only if $\mathcal{O}_{\alpha_2} = \Lambda(\mathcal{O}_{\alpha_1})$.

Interpreting the line bundles $L_{\alpha_1}, L_{\alpha_2}$ and $L_{\alpha_1} \boxtimes L_{\alpha_2}$ as sheaves and their holomorphic sections as their zeroth cohomology group, one concludes from a standard Künneth formula (compare [SW59] and [Kau67]) that

$$\Gamma_{\mathrm{hol}}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2}, L_{\alpha_1} \boxtimes L_{\alpha_2}) \cong \Gamma_{\mathrm{hol}}(\mathcal{O}_{\alpha_1}, L_{\alpha_1}) \otimes \Gamma_{\mathrm{hol}}(\mathcal{O}_{\alpha_2}, L_{\alpha_2}).$$

Recall that by 3.26(iii), the coadjoint orbit corresponding to $\mathcal{O}_{\alpha_1} \in \Phi_1(M)/G_1$ is symplectomorphic to the orbit reduced space of $\mathcal{O}_{\alpha_1}$, i. e.,

$$\Lambda(\mathcal{O}_{\alpha_1}) \cong \Phi_1^{-1}(\mathcal{O}_{\alpha_1})/G_1 = M_{\alpha_1}.$$

This implies that the multiplicity space of one action coincides with an irreducible representation of the other one, i. e., for $\mathcal{O}_{\alpha_2} = \Lambda(\mathcal{O}_{\alpha_1})$,

$$\mathrm{Hom}_{G_1}(V_{\alpha_1}, \Gamma_{\mathrm{hol}}(M, L)) \cong \Gamma_{\mathrm{hol}}(M_{\alpha_1}, L(M_{\alpha_1})) = \Gamma_{\mathrm{hol}}(\mathcal{O}_{\alpha_2}, L_{\alpha_2}).$$

The preceding statements can be summarized as follows.

Theorem 5.17. *Let G_1 and G_2 be compact connected Lie groups acting by holomorphic transformations on a Kähler manifold M such that the actions extend to actions of the respective complexified groups. Suppose that the actions of G_1 and G_2 commute and are Hamiltonian with admissible equivariant moment maps Φ_1 and Φ_2. Denote by L the prequantum line bundle over M, and by $\mathfrak{t}_{\mathbb{Z}}^+$ the integral points in a Weyl chamber $\mathfrak{t}^+$ in the dual $\mathfrak{t}^*$ of a maximal abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{g}_1$.*

Assume further the symplectic Howe condition to be satisfied and that the Φ_1- and Φ_2-level sets are connected so that the orbit correspondence map Λ is available. Then:

$$\Gamma_{\mathrm{hol}}(M, L) \cong \bigoplus_{\alpha_1 \in \Phi_1(M) \cap \mathfrak{t}_{\mathbb{Z}}^+} \Gamma_{\mathrm{hol}}(\mathcal{O}_{\alpha_1}, L_{\alpha_1}) \otimes \Gamma_{\mathrm{hol}}(\mathcal{O}_{\alpha_2}, L_{\alpha_2}),$$

where $\mathcal{O}_{\alpha_2} = \Lambda(\mathcal{O}_{\alpha_1})$.

Remark 5.18. (i) Let (M, ω) be a compact complex manifold together with an integral Kähler form and $L \to M$ a prequantizing holomorphic line bundle. Let furthermore the connected compact Lie group $G_1 \times G_2$ act by holomorphic transformations and in a Hamiltonian fashion on M. If the pair of actions (i. e., the G_1-action and the G_2-action) satisfies the symplectic Howe condition, the preceding theorem applies.

(ii) As in Prop. 3.20 and Thm. 3.26, it is possible to remove the connectedness condition on the level sets of the moment maps, see [BW09].

In the setting of Thm. 5.17 we thus have that the symplectic Howe condition for the actions on M implies the representation-theoretic Howe condition for the linear representation on the (geometric) quantization of (M, ω).

6 The Action of $(U(n), U(m))$ on $\mathrm{Mat}(n, m; \mathbb{C})$

6.1 Definitions

The object of the following considerations will be the action of the Howe pair $(U(n), U(m))$ in $U(nm)$ on the symplectic manifold of $(n \times m)$-matrices $M = \mathrm{Mat}(n, m;\, \mathbb{C})$ $(n \geq m \geq 1)$ with symplectic form given by $\omega_z(A, B) = \mathrm{Im}\,\mathrm{tr}(\bar{A}^{\mathrm{T}} B)$ for $z \in M$ and $A, B \in T_z M$ or, equivalently, in coordinates,

$$\omega = \frac{\mathrm{i}}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathrm{d}z_{ij} \wedge \mathrm{d}\bar{z}_{ij} = -\mathrm{d}\vartheta,$$

i.e., ω is an exact form with potential $\vartheta = \frac{\mathrm{i}}{4} \sum_{i=1}^{m} \sum_{j=1}^{n} (\bar{z}_{ij} \mathrm{d}z_{ij} - z_{ij} \mathrm{d}\bar{z}_{ij})$ or $\vartheta_z(A) = -\frac{1}{2} \mathrm{Im}\,\mathrm{tr}(\bar{z}^{\mathrm{T}} A)$.

On this symplectic manifold, an action is to be defined which preserves the symplectic form. An element $U \in U(n)$ will act on $z \in M$ by matrix multiplication from the left,

$$(U, z) \mapsto U \cdot z;$$

commuting with this action, $V \in U(m)$ will act by matrix multiplication with the inverse of V from the right,

$$(V, z) \mapsto z \cdot V^{-1} = z \cdot \bar{V}^{\mathrm{T}}.$$

Having defined these two obviously commuting actions, the simultaneous action of both groups may be regarded, too.

One notes that the defined action is even Hamiltonian, i.e., there exists a moment map. The fundamental vector fields are as follows (left, right and joint action):

$$\xi_z^{(L)} = \sum_{i,k=1}^{n} \sum_{j=1}^{m} \left(\xi_{ik} z_{kj} \frac{\partial}{\partial z_{ij}} - \xi_{ki} \bar{z}_{kj} \frac{\partial}{\partial \bar{z}_{ij}} \right) = \xi \cdot z \qquad \forall \xi \in \mathfrak{u}(n)$$

$$\eta_z^{(R)} = \sum_{i=1}^{n} \sum_{j,l=1}^{m} \left(-\eta_{lj} z_{il} \frac{\partial}{\partial z_{ij}} + \eta_{jl} \bar{z}_{il} \frac{\partial}{\partial \bar{z}_{ij}} \right) = -z \cdot \eta \qquad \forall \eta \in \mathfrak{u}(m)$$

$$\zeta^M = (\mathrm{pr}_{\mathfrak{u}(n)}(\zeta))^{(L)} + (\mathrm{pr}_{\mathfrak{u}(m)}(\zeta))^{(R)} = \xi \cdot z - z \cdot \eta \qquad \forall \zeta \in \mathfrak{u}(n) \oplus \mathfrak{u}(m),$$

using the projections $\mathrm{pr}_{\mathfrak{u}(n)} : \mathfrak{u}(n) \oplus \mathfrak{u}(m) \to \mathfrak{u}(n)$ and $\mathrm{pr}_{\mathfrak{u}(m)} : \mathfrak{u}(n) \oplus \mathfrak{u}(m) \to \mathfrak{u}(m)$.

By the definition of the moment map, one has $\mathrm{d}\Phi^{\xi} = \xi^M \lrcorner\, \omega = -\xi^M \lrcorner\, \mathrm{d}\vartheta = \mathrm{d}(\xi^M \lrcorner\, \vartheta)$ (by invariance under the group actions) and thus, up to a constant, $\Phi^{\xi} = \xi^M \lrcorner\, \vartheta$:

$$\Phi_{(L)}^{\xi}(z) = \frac{\mathrm{i}}{2} \sum_{i,k=1}^{n} \sum_{j=1}^{m} (\xi_{ik} z_{kj} \bar{z}_{ij}) = -\frac{1}{2} \mathrm{Im}\,\mathrm{tr}(\xi z \bar{z}^{\mathrm{T}})$$

$$\Phi_{(R)}^{\eta}(z) = -\frac{\mathrm{i}}{2} \sum_{i=1}^{n} \sum_{j,l=1}^{m} (\eta_{lj} z_{il} \bar{z}_{ij}) = \frac{1}{2} \mathrm{Im}\,\mathrm{tr}(\eta \bar{z}^{\mathrm{T}} z)$$

The moment map of the joint action is the sum

$$\Phi^{\zeta}(z) = \Phi_{(L)}^{\mathrm{pr}_{\mathfrak{u}(n)}(\zeta)}(z) + \Phi_{(R)}^{\mathrm{pr}_{\mathfrak{u}(m)}(\zeta)}(z) \in \mathfrak{u}(n)^* \oplus \mathfrak{u}(m)^*.$$

For the next calculations, the Hermitian scalar product $\langle\cdot,\cdot\rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}, (v_1, v_2) \mapsto \langle v_1, v_2\rangle = \bar{v}_1^{\mathrm{T}} v_2$ (matrix product between a row and a column) will be needed.

One notes the following property of the moment maps for the action of $U(n)$ and $U(m)$ (which is an adapted version of Witt's theorem):

Lemma 6.1. *Each level set of $\Phi_{(L)}$ is exactly one $U(m)$-orbit and each level set of $\Phi_{(R)}$ is exactly one $U(n)$-orbit.*

Proof. Consider first the case of the map $\Phi_{(R)}$, which is, like $\bar{z}^{\mathrm{T}} z$, obviously invariant under $U(n)$. It has to be shown that for any $z_1, z_2 \in M$ satisfying $\bar{z}_1^{\mathrm{T}} z_1 = \bar{z}_2^{\mathrm{T}} z_2$, there exists $U \in U(n)$ such that $z_2 = U z_1$.

Now interpret $z \in M \cong (\mathbb{C}^n)^m$ as a collection of m column vectors $\{v_1, \ldots, v_m\}$ where $v_i \in \mathbb{C}^n$ for $i = 1, \ldots, m$. Then

$$\bar{z}^{\mathrm{T}} z = \begin{pmatrix} \langle v_1, v_1\rangle & \cdots & \langle v_1, v_m\rangle \\ \vdots & \ddots & \vdots \\ \langle v_m, v_1\rangle & \cdots & \langle v_m, v_m\rangle \end{pmatrix},$$

where $\langle\cdot,\cdot\rangle$ is the Hermitian scalar product. This explicit form of $\bar{z}^{\mathrm{T}} z$ shows that, instead of proving the existence of a $U \in U(n)$ such that $z_2 = U z_1$, it is equivalent to show that for any two ordered sets of m vectors each, $\{v_1, \ldots, v_m\} \subset \mathbb{C}^n$ and $\{w_1, \ldots, w_m\} \subset \mathbb{C}^n$, with $\langle v_i, v_j\rangle = \langle w_i, w_j\rangle \, \forall i, j \in \{1, \ldots, m\}$, there is a $U \in U(n)$ such that $w_i = U v_i \, \forall i \in \{1, \ldots, m\}$.

We start with the case where $n = m$ and both sets consist of linearly independent vectors, hence both sets form a basis for $\mathbb{C}^n$. Then a linear map U defined by $U v_i = w_i$ is automatically unitary: $\langle U v_i, U v_j\rangle = \langle w_i, w_j\rangle = \langle v_i, v_j\rangle$.

Now let $n > m$ and take two sets of m linearly independent vectors. Both sets span a linear subspace of $\mathbb{C}^n$: $E = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_m\}$ and $F = \mathrm{span}_{\mathbb{C}}\{w_1, \ldots, w_m\}$. Then consider the orthogonal complements $E^\perp$ and $F^\perp$, choose orthonormal bases $\{v_i\}_{i=m+1}^n$ and $\{w_i\}_{i=m+1}^n$ on them. Again define U by $U v_i = w_i$. For $1 \le i, j \le m$, one again has $\langle U v_i, U v_j\rangle = \langle w_i, w_j\rangle = \langle v_i, v_j\rangle$ by assumption; for $1 \le i \le m < j \le n$, we have $v_i \in E, v_j \in E^\perp, w_i \in F$ and $w_j \in F^\perp$, therefore $\langle U v_i, U v_j\rangle = \langle w_i, w_j\rangle = 0 = \langle v_i, v_j\rangle$; and for $m < i, j \le n$ the orthonormality of the chosen bases in $E^\perp$ and $F^\perp$ proves that U is unitary.

The next case to be treated is when one has two sets of m vectors $\{v_1, \ldots, v_m\} \subset \mathbb{C}^n$ and $\{w_1, \ldots, w_m\} \subset \mathbb{C}^n$, with $\langle v_i, v_j\rangle = \langle w_i, w_j\rangle \, \forall i, j \in \{1, \ldots, m\}$, where the first k vectors of each set are a maximal linearly independent subset ($k < m \le n$). Note that the assumption on the scalar product implies that k is the same on both sides: Otherwise, we would have k linearly independent vectors in the first set and l in the second. W. l. o. g., assume $l \ge l' > k$, then for any such l' there would be coefficients α_i (not all zero) such that $v_{l'} = \sum_{i=1}^k \alpha_i v_i$. Assume further that $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_l\}$ are orthonormal (always possible with Gram-Schmidt), then ($1 \le j \le k$)

$$0 = \langle w_j, w_{l'}\rangle = \langle v_j, v_{l'}\rangle = \sum_{i=1}^k \alpha_i \langle v_j, v_i\rangle = \alpha_j,$$

thus $k = l$. For these k linearly independent vectors ($k < n$), there exists a map U as in the preceding case (where we had m linearly independent vectors). As all possible scalar products between vectors from the first and the second set, respectively, coincide, Gram-Schmidt yields identical transformations for both sets to obtain orthonormal ones, so assume the first k vectors of both sets are already orthonormal. Then the remaining $m-k$ vectors behave as they are supposed to ($1 \leq j \leq k < l < m$):

$$\langle Uv_j, Uv_l \rangle = \left\langle Uv_j, \sum_{i=1}^{k} \alpha_i U v_i \right\rangle = \left\langle w_j, \sum_{i=1}^{k} \alpha_i w_i \right\rangle = \langle w_j, w_l \rangle = \langle v_j, v_l \rangle,$$

where $\alpha_i = \langle w_i, w_l \rangle = \langle v_i, v_l \rangle$ – so U is unitary.

For $\Phi_{(L)}$, the proof works identically: One has to show that for two sets of n vectors each which are now contained in $\mathbb{C}^m$ and satisfy the analogous condition on the scalar products, there exists a map $V \in U(m)$ mapping the first set to the other one. As $n \geq m \geq k$, these sets will never be linearly independent unless $n = m$. □

6.2 Singular Value Decomposition of $z \in M$

In order to obtain a normal form for the actions of the Howe pair $(U(n), U(m))$, the singular value decomposition of a rectangular matrix $z \in M$ will be recalled.

First note that the products $\bar{z}^{\mathrm{T}} z$ and $\bar{z} z^{\mathrm{T}}$ are Hermitian matrices; consequently, their eigenvalues are real. With respect to the Hermitian scalar product, both are positive-semidefinite (for any $v \in \mathbb{C}^m$, $\bar{v}^{\mathrm{T}}(\bar{z}^{\mathrm{T}} z)v = (\overline{zv}^{\mathrm{T}})(zv) = \|zv\|^2 \geq 0$), therefore, their eigenvalues are non-negative, and one further has:

Lemma 6.2. *The eigenvalues of $\bar{z}^{\mathrm{T}} z$ and $\bar{z} z^{\mathrm{T}}$ are the same; they appear with the same multiplicities (except for the eigenvalue 0).*

Proof. If $v \in \mathbb{C}^m$ is an eigenvector of $\bar{z}^{\mathrm{T}} z$, i. e., $(\bar{z}^{\mathrm{T}} z)v = \lambda v$, then zv is an eigenvector of $z\bar{z}^{\mathrm{T}}$, hence $\overline{zv}$ one of $\bar{z} z^{\mathrm{T}}$, for the same eigenvalue λ – and vice versa. Take $\{v_i\}_{i=1}^k$ to be k linearly independent eigenvectors of $\bar{z}^{\mathrm{T}} z$ for the eigenvalue $\lambda \in \mathbb{R}_+$. Suppose now that $\{zv_i\}_{i=1}^k$ is a linearly dependent set. Then for some constants μ_i which are not all zero,

$$0 = \bar{z}^{\mathrm{T}} z \left(\sum_{i=1}^{k} \mu_i v_i \right) = \lambda \left(\sum_{i=1}^{k} \mu_i v_i \right),$$

which is a contradiction. □

Define a matrix $\Sigma \in M$ by

$$\Sigma = \Sigma(\sigma_1, \dots, \sigma_m) = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_m \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\sigma_i = \sqrt{\lambda_i}$ are the square roots of the eigenvalues of $\bar{z}^{\mathrm{T}} z$, appearing according to their multiplicity and ordered such that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$.

Let k be the highest index for which $\sigma_k \neq 0$. Choose a unitary matrix $V \in U(m)$ where the first k columns V_i are eigenvectors of $\bar{z}^{\mathrm{T}} z$ of length one corresponding to the eigenvalues λ_i, which are orthogonal with respect to the Hermitian form. Complete the remaining $m - k$ columns by choosing $m - k$ arbitrary linearly independent vectors, make them orthonormal using Gram-Schmidt.

Let $U \in U(n)$ be given as follows: For $i = 1, \ldots, k$, give the columns by $U_i = \frac{1}{\sigma_i} z \cdot V_i$. By the definition of V, they are orthogonal of length one (for the Hermitian form). Complete U by adding $n - k$ vectors $\{U_{k+1}, \ldots, U_n\} \subset \mathbb{C}^n$ such that $\{U_i\}_{i=1}^n$ is linearly independent and apply Gram-Schmidt. Then any $z \in M$ may be written as $z = U \Sigma \bar{V}^{\mathrm{T}}$.

The matrix Σ obtained in this decomposition is unique (by the definition of eigenvalues and the imposed ordering), the matrices U and V may differ by an element of the stabilizer, which will be calculated next.

6.3 Slices, Stabilizers and Orbits on M

Slices. From the singular value decomposition one concludes that any orbit of the joint $(U(n) \times U(m))$-action contains exactly one element of the type $\Sigma = \Sigma(\sigma_1, \ldots, \sigma_m) \in M$. Define the set $S = \{\Sigma \in M \mid \Sigma = \Sigma(\sigma_1, \ldots, \sigma_m), \sigma_1 > \ldots > \sigma_m > 0\}$. Its closure $\overline{S} = \{\Sigma \in M \mid \Sigma = \Sigma(\sigma_1, \ldots, \sigma_m), \sigma_1 \geq \ldots \geq \sigma_m > 0\}$ is a global slice for the joint action of $U(n) \times U(m)$, S will turn out to be a slice for the generic orbits.

Orbits of the individual actions of $U(n)$ and $U(m)$ are represented by elements $\Sigma \bar{V}^{\mathrm{T}}$ and $U\Sigma$, respectively.

Stabilizers. In order to describe the orbits corresponding to these actions, the stabilizers of elements of the slice $\overline{S}$ will be calculated. At first, the generic case $\sigma_1 > \ldots > \sigma_m > 0$ is treated. This will give the stabilizer of smallest dimension, hence the generic orbits.

Joint action of $U(n) \times U(m)$ on a generic element. The stabilizer is defined to be the subgroup of $U(n) \times U(m)$ given by

$$\mathrm{Stab}_{U(n) \times U(m)}(\Sigma) = \{(U, V) \in U(n) \times U(m) \mid U \Sigma \bar{V}^{\mathrm{T}} = \Sigma\}.$$

The stabilizer condition is equivalent to $U\Sigma = \Sigma V$, which can be written as

$$\begin{bmatrix} | & & | \\ \sigma_1 U_1, & \cdots, & \sigma_m U_m \\ | & & | \end{bmatrix} = \begin{bmatrix} \text{---} & \sigma_1 V_{\bar{1}} & \text{---} \\ & \vdots & \\ \text{---} & \sigma_m V_{\bar{m}} & \text{---} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \tag{6.1}$$

where U_i is the ith column of U and $V_{\bar{i}}$ the ith row of V.

This equality of matrices yields the following set of conditions:

1. $\sigma_j U_{ij} = \sigma_i V_{ij}$ $(1 \leq i, j \leq m)$

2. $\sigma_j U_{ij} = 0$ $(1 \leq j \leq m < i \leq n)$

3. U_{ij} arbitrary $(1 \leq i \leq n, m+1 \leq j \leq n)$

However, the third condition is restricted, because the second condition, together with unitarity, implies that $U_{ij} = 0$ for $1 \leq i \leq m < j \leq n$. Obviously, the entries U_{ij} for $m+1 \leq i, j \leq n$ form an element of $U(n-m)$.

The interesting part of the stabilizer follows from the first condition. Together with unitarity and the ordering on the σ_i,

$$1 = \sum_{i=1}^{n} \bar{U}_{i1} U_{i1} = \sum_{i=1}^{m} \left(\frac{\sigma_i}{\sigma_1}\right)^2 \bar{V}_{i1} V_{i1} \leq |V_{11}|^2 + \sum_{i=2}^{m} |V_{i1}|^2 = 1.$$

Equality only holds if $\sum_{i=2}^{m} |V_{1i}|^2 = 0$, which yields $V_{11} \in U(1)$ and $V_{i1} = 0$ for $i = 2, \ldots, m$. Repeating this calculation for all rows and columns, one obtains that $V_{ii} \in U(1)$ for all $i = 1, \ldots, m$. Hence $(U, V) \in \mathrm{Stab}_{U(n)\times U(m)}(\Sigma)$ has the form

$$U = \begin{pmatrix} U_{11} & \cdots & 0 & | \\ & \ddots & & 0 \\ 0 & \cdots & U_{mm} & | \\ \text{---} & 0 & \text{---} & \tilde{U} \end{pmatrix}, \qquad V = \begin{pmatrix} U_{11} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & U_{mm} \end{pmatrix},$$

where $U_{ii} \in U(1)$ and $\tilde{U} \in U(n-m)$.

From this calculation follows that, in the generic case, the stabilizer $\mathrm{Stab}_{U(n)\times U(m)}(\Sigma) \cong (U(1))^m \cdot U(n-m)$ has real dimension $m + (n-m)^2$. Noting that $\dim_{\mathbb{R}} M = 2mn$ and $\dim_{\mathbb{R}}(U(n) \times U(m)) = n^2 + m^2$, one concludes that $\dim_{\mathbb{R}}(U(n) \times U(m)) \cdot \Sigma = 2mn - m = \dim_{\mathbb{R}} M - m$. The codimension of the orbit equals the dimension of the slice S, as it is supposed to.

Individual actions of $U(n)$ and $U(m)$ on a generic element. For these stabilizers one simply has $\mathrm{Stab}_{U(n)}(\Sigma) \cong U(n-m)$ and $\mathrm{Stab}_{U(m)}(\Sigma) = \{\mathrm{id}_{\mathbb{C}^m}\}$. The first one is seen from the condition $U\Sigma = \Sigma$, which is equivalent to $U_{ij}\sigma_j = \sigma_j \delta_{ij}$ $(1 \leq i \leq n, 1 \leq j \leq m)$ and no condition on the remaining entries. Therefore, $U_{ij} = \delta_{ij}$ in the upper left block of size $(m \times m)$, the lower right block being an arbitrary element of $U(n-m)$. The second calculation is analogous.

Joint action of $U(n) \times U(m)$ on non-generic elements. Two non-generic cases are going to be considered which represent well the general case. To start with, let $\sigma = \sigma_1 = \ldots = \sigma_k > \sigma_{k+1} > \ldots > \sigma_m > 0$. The second and third condition which follow from (6.1) remain unchanged, only the first one needs to be revisited. By these conditions and unitarity, the following holds for $j \leq k$:

$$1 = \sum_{i=1}^{n} \bar{U}_{ij} U_{ij} = \sum_{i=1}^{m} \left(\frac{\sigma_i}{\sigma}\right)^2 \bar{V}_{ij} V_{ij} = \sum_{i=1}^{k} \bar{V}_{ij} V_{ij} + \sum_{i=k+1}^{m} \left(\frac{\sigma_i}{\sigma}\right)^2 \bar{V}_{ij} V_{ij}$$
$$\leq \sum_{i=1}^{k} \bar{V}_{ij} V_{ij} + \sum_{i=k+1}^{m} \bar{V}_{ij} V_{ij} = 1,$$

where equality holds only for $\sum_{i=k+1}^{m} \bar{V}_{ij} V_{ij} = 0$. Then (U, V) stabilizes $\Sigma = \Sigma(\sigma_1, \ldots, \sigma_m)$ if it is of the form

$$U = \begin{pmatrix} \tilde{U}_1 & & 0 & & 0 \\ & U_{k+1,k+1} & & 0 & \\ 0 & & \ddots & & 0 \\ & 0 & & U_{mm} & \\ 0 & & 0 & & \tilde{U} \end{pmatrix}, \qquad V = \begin{pmatrix} \tilde{U}_1 & & 0 & \\ & U_{k+1,k+1} & & 0 \\ 0 & & \ddots & \\ & 0 & & U_{mm} \end{pmatrix},$$

where $\tilde{U}_1 \in U(k), U_{ii} \in U(1)$ $(i = k+1, \ldots, m), \tilde{U} \in U(n-m)$.

The other non-generic situation under investigation is $\sigma_1 > \ldots > \sigma_k > \sigma_{k+1} = \ldots = \sigma_m = 0$. Here, only the third condition continues to hold, while the first and second condition become less restrictive on the elements in the stabilizer. For $i = 1, \ldots, k$, the diagonal entries $U_{ii} \in U(1)$ appear as before, for the remainder there is no condition. Thus, an element (U, V) of the stabilizer has the form:

$$U = \begin{pmatrix} U_{11} & & & \\ & \ddots & & \\ & & U_{kk} & \\ & & & \tilde{U} \end{pmatrix}, \qquad V = \begin{pmatrix} U_{11} & & & \\ & \ddots & & \\ & & U_{kk} & \\ & & & \tilde{V} \end{pmatrix},$$

where $\tilde{U} \in U(n-k)$ and $\tilde{V} \in U(m-k)$.

The stabilizer of an arbitrary $\Sigma = \Sigma(\sigma_1, \ldots, \sigma_m) \in M$ with $\sigma_1 \geq \ldots \geq \sigma_m \geq 0$ is block-diagonal, each block being a copy of $U(k)$ where k is the multiplicity of the corresponding non-zero eigenvalue. For the zero eigenvalues and the last $n-m$ entries of $U(n)$, there are blocks like $\tilde{U}$ and $\tilde{V}$ in the preceding calculation.

Individual actions of $U(n)$ and $U(m)$ on non-generic elements. For the non-zero entries of $\Sigma = \Sigma(\sigma_1, \ldots, \sigma_m) \in M$ with $\sigma_1 \geq \ldots \geq \sigma_m \geq 0$, the condition from the generic case remains unchanged. For the zero entries (say $\sigma_{k+1} = \ldots \sigma_m = 0$), there is no condition on the corresponding part of the stabilizing unitary matrix, just as in the case of the joint action. Hence, $U \in \mathrm{Stab}_{U(n)}(\Sigma)$ if and only if $U = \mathrm{diag}(1, \ldots, 1, \tilde{U})$ with $\tilde{U} \in U(n-k)$. Analogously, $V \in \mathrm{Stab}_{U(m)}(\Sigma)$ if and only if $V = \mathrm{diag}(1, \ldots, 1, \tilde{V})$ with $\tilde{U} \in U(m-k)$.

Conjugacy of the stabilizer groups. For a smooth group action, the stabilizers of the points of one orbit are all conjugated to each other. In the present setting, there is a further property: The stabilizers $\mathrm{Stab}_{U(n)}(z)$ are all the same for all z which lie in the same $U(m)$-orbit, and vice versa, a special case of Lemma 3.1.

Invariants. One notices that the momentum maps of the left and right action are invariant functions: $\Phi_{(L)}$ is invariant under the action of $U(m)$, and $\Phi_{(R)}$ under $U(n)$. Indeed, the algebra of invariant functions coincides with the collective functions of the other action, i. e., $C^\infty(M)^{U(n)} = \Phi^*_{(R)} C^\infty(\mathfrak{u}(m)^*)$ and $C^\infty(M)^{U(m)} = \Phi^*_{(L)} C^\infty(\mathfrak{u}(n)^*)$, as is seen by the following lemma (keep in mind $M \cong (\mathbb{C}^n)^m$).

Lemma 6.3. *The algebra of $U(n)$-invariant smooth functions on $(\mathbb{C}^n)^m$ (m copies of the defining representation of $U(n)$) is generated by the functions*

$$\langle i \mid j \rangle \text{ for } 1 \leq i \leq j \leq m,$$

where $\langle v_1, v_2 \rangle$ is the Hermitian scalar product between $v_1, v_2 \in \mathbb{C}^n$ and $\langle i \mid j \rangle : (\mathbb{C}^n)^m \to \mathbb{C}, (z_1, \ldots, z_m) \mapsto \langle z_i, z_j \rangle = (\bar{z}^{\mathrm{T}} z)_{ij}$, if the z_i are regarded as columns of $z \in M$.

The proof of this lemma is explained in App. A.

6.4 Symplectic Properties of the Orbits

The first thing to note is that the slice $S = \{\Sigma \in M \mid \Sigma = \Sigma(\sigma_1, \ldots, \sigma_m), \sigma_1 > \ldots > \sigma_m > 0\}$ is isotropic. Two vectors $\Lambda_1, \Lambda_2 \in T_z S = \{\Lambda \in \mathrm{Mat}(n, m; \mathbb{R}) \mid \Lambda = \Sigma(\lambda_1, \ldots, \lambda_m)\}$ have only real matrix entries, so has $\bar{\Lambda}_1^{\mathrm{T}} \Lambda_2$, hence $\omega_z(\Lambda_1, \Lambda_2) = \mathrm{Im}\,\mathrm{tr}(\bar{\Lambda}_1^{\mathrm{T}} \Lambda_2) = 0$, by which $T_z S \subseteq T_z S^{\angle}$.

Coisotropy of the joint $(U(n) \times U(m))$-orbits. We are going to show that the orbits of the joint action are coisotropic. This can be concluded from the orthogonality of the slice S and the orbits.

For an element $(\xi, \eta) \in \mathfrak{u}(n) \times \mathfrak{u}(m)$, the flow of the corresponding vector field on M is $\varphi_t^{(\xi,\eta)}(z) = e^{t\xi} \cdot z \cdot e^{-t\eta}$. Consequently, the tangent space to the orbit, $T_z((U(n) \times U(m)) \cdot z)$, is spanned by vectors $\xi \cdot z - z \cdot \eta$.

Associated to the symplectic form ω, there is a Riemannian metric $g_z(A, B) = \mathrm{Re}\,\mathrm{tr}(\bar{A}^{\mathrm{T}} B)$. Let $\Lambda \in T_z S$ and $\xi \cdot z - z \cdot \eta \in T_z((U(n) \times U(m)) \cdot z)$ for $z = \Sigma$. Then $g_\Sigma(\Lambda, \xi \cdot z - z \cdot \eta) = \mathrm{Re}\,\mathrm{tr}(\bar{\Lambda}^{\mathrm{T}} \xi \Sigma - \bar{\Lambda}^{\mathrm{T}} \Sigma \eta)$. As the trace is purely imaginary, this becomes zero and shows the desired orthogonality. For the trace, one calculates

$$\overline{\mathrm{tr}(\Lambda^{\mathrm{T}} \xi \Sigma)} = \overline{\mathrm{tr}(\Sigma \Lambda^{\mathrm{T}} \xi)} = -\,\mathrm{tr}(\Sigma \Lambda^{\mathrm{T}} \xi^{\mathrm{T}}) = -\,\mathrm{tr}(\xi \Lambda \Sigma^{\mathrm{T}}) = -\,\mathrm{tr}(\Lambda \Sigma^{\mathrm{T}} \xi).$$

But $\Lambda \Sigma^{\mathrm{T}} = \Sigma \Lambda^{\mathrm{T}} = \mathrm{diag}(\lambda_1 \sigma_1, \ldots, \lambda_m \sigma_m, 0, \ldots, 0)$ and thus the claim is true. By the invariance of the symplectic form under the group action, this orthogonality holds everywhere in $\Omega = U(n) \cdot S \cdot U(m)$. Define for now $E = T_z S$ and $F = T_z((U(n) \times U(m)) \cdot z)$. The orthogonality gives, using the associated complex structure J,

$$F^{\angle} = J(F^{\perp}) = JE \subseteq E^{\perp} = F.$$

The inclusion $JE \subseteq E^{\perp}$ follows from $g(e, Je') = \omega(e, e') = 0\, \forall e, e' \in E$ (E isotropic).

Symplectic complementarity of the orbits corresponding to the individual actions. Take $z \in M, \xi \in \mathfrak{u}(n), \eta \in \mathfrak{u}(m)$. Then

$$\begin{aligned}\omega_z(\xi_z^{(L)}, \eta_z^{(R)}) &= \operatorname{Im}\operatorname{tr}(\overline{\xi z}^{\mathrm{T}} \cdot z\eta) \\ &= \frac{1}{2}\Big[\operatorname{tr}(\overline{\xi z}^{\mathrm{T}} \cdot z\eta) - \operatorname{tr}(\bar{\eta}^{\mathrm{T}}\bar{z}^{\mathrm{T}}\xi z)\Big] \\ &= \frac{1}{2}\big[-\operatorname{tr}(\bar{z}^{\mathrm{T}}\xi z\eta) + \operatorname{tr}(\bar{z}^{\mathrm{T}}\xi z\eta)\big] \\ &= 0,\end{aligned}$$

i. e., the orbits of both actions are symplectically orthogonal (which was already clear from the invariance properties of the moment maps we observed earlier). By Lemma 3.4, we know that it suffices to calculate the dimensions of these orbits in order to decide whether they are also complementary.

In section 6.3, the stabilizers of the occuring actions were calculated. From this we know that $\dim_{\mathbb{R}}(U(n) \cdot \Sigma) \le n^2 - (n-m)^2 = 2nm - m^2$ and $\dim_{\mathbb{R}}(\Sigma \cdot U(m)) \le m^2$ for $\Sigma \in \overline{S}$; if $\Sigma \in S$, equality holds. That is, the generic G_1- and G_2-orbits are symplectically complementary because their dimensions add up to $\dim_{\mathbb{R}} M = 2nm$.

6.5 Coadjoint Orbits in the Image of the Moment Map

Now we ask which coadjoint orbits occur in the image of the moment maps. As $\Phi_{(L)}$ and $\Phi_{(R)}$ are both equivariant for one and invariant under the other action, it is sufficient to consider the image (under the moment maps) of matrices $\Sigma = \Sigma(\sigma_1, \dots, \sigma_m), \sigma_1 \ge \dots \ge \sigma_m \ge 0$, i. e., in the closure of the slice S. Attention will be restricted to the action of $U(n)$ from the left, the right action behaves analogously. First note for any $\xi \in \mathfrak{u}(n)$,

$$\Phi_{(L)}^{\xi}(\Sigma) = \frac{\mathrm{i}}{2}\operatorname{tr}(\xi\Sigma\bar{\Sigma}^{\mathrm{T}}) = \frac{\mathrm{i}}{2}\operatorname{tr}(\xi \cdot \operatorname{diag}(\sigma_1^2, \dots, \sigma_m^2, 0, \dots, 0)).$$

Yet the stabilizer of such a point under the coadjoint $U(n)$-action is needed in order to describe the orbit passing through it. An element $U \in U(n)$ stabilizes $\Phi_{(L)}(\Sigma)$ if and only if

$$\operatorname{tr}(\xi\Sigma\bar{\Sigma}^{\mathrm{T}}) = \operatorname{tr}(\xi(U\Sigma)(\overline{U\Sigma})^{\mathrm{T}}) \quad \forall \xi \in \mathfrak{u}(n).$$

This condition is equivalent to $\operatorname{tr}(\xi(\Sigma\bar{\Sigma}^{\mathrm{T}} - (U\Sigma)(\overline{U\Sigma})^{\mathrm{T}})) = 0$, hence to $\Sigma\bar{\Sigma}^{\mathrm{T}} = (U\Sigma)(\overline{U\Sigma})^{\mathrm{T}}$, and eventually to $\sigma_j^2 U_{jk} = \sigma_k^2 U_{jk}$ – which is equivalent to saying

$$U \in \operatorname{Stab}_{U(n)}(\Phi_{(L)}(\Sigma)) \;\Leftrightarrow\; U \in \operatorname{pr}_{U(n)}(\operatorname{Stab}_{U(n)\times U(m)}(\Sigma)),$$

which has been calculated earlier. Analogously,

$$\operatorname{Stab}_{U(m)}(\Phi_{(R)}(\Sigma)) = \operatorname{pr}_{U(m)}(\operatorname{Stab}_{U(n)\times U(m)}(\Sigma)).$$

Taking a generic $\Sigma \in S$, the coadjoint orbits through $\Phi_{(L)}(\Sigma)$ and $\Phi_{(R)}(\Sigma)$ are diffeomorphic to $U(n)/(U(1)^m \cdot U(n-m))$ and $U(m)/U(1)^m$, respectively.

6.6 Duality Properties of the Orbits

From Prop. 3.19 and Lemma 6.3 we conclude that the symplectic Howe condition is satisfied because

$$\Phi^*_{(L)}C^\infty(\mathfrak{u}(n)^*) = C^\infty(M)^{U(m)} = Z_{C^\infty(M)}(\Phi^*_{(R)}C^\infty(\mathfrak{u}(m)^*))$$

and vice versa. Further, the orbits of both group actions as well as the level sets of both moment maps are connected. Therefore, Thm. 3.26 applies in this situation, i. e., we do have a bijective correspondence between the coadjoint orbits in the moment images. Explicitly, the correspondence map is given by $\Phi_{(L)}(\overline{S}) \ni (\lambda_1, \dots, \lambda_m, 0, \dots, 0) \mapsto (-\lambda_1, \dots, -\lambda_m) \in \Phi_{(R)}(\overline{S})$, for $\lambda_1 \geq \dots \geq \lambda_m \geq 0$. Quantizing this setting as in Thm. 5.17 makes us reobtain Thm. 5.2.7 of [GW98] about the $GL(n,\mathbb{C})$-$GL(m,\mathbb{C})$ duality (note that $U(n)$ is maximal compact in $GL(n,\mathbb{C})$).

7 The Action of (G,G) on T^*G

In this section, a natural correspondence between coadjoint orbits will be regarded. This correspondence will be constructed using the moment maps of the simultaneous left and right action of $G \times G$ on the cotangent bundle T^*G of a Lie group G together with its canonical symplectic form. Later, the obtained correspondence can be compared to the decomposition of $L^2(G)$ given by the Peter-Weyl theorem (in the case where G is compact).

The following definitions will be made without further assumptions on the Lie group G, however, they will have to be imposed later on.

7.1 Definitions

Take a Lie group G and its cotangent bundle T^*G, the latter possessing a symplectic structure $\omega = \omega^{T^*G}$ given by the exterior differential of the canonical form $\mathrm{d}\vartheta$,

$$\omega_{\beta_g} = -\mathrm{d}\vartheta_{\beta_g} = -\mathrm{d}(\beta_g \circ T_{\beta_g} \mathrm{pr}),$$

at a point $\beta_g \in T^*_g G$, where $\mathrm{pr} : T^*G \to G$ is the canonical bundle projection ($T_{\beta_g} \mathrm{pr}$ maps a vector from $T_{\beta_g}(T^*G)$ to one in $T_g G$, to which β_g assigns a scalar, hence $\beta_g \circ T_{\beta_g} \mathrm{pr} \in \Omega^1(T^*G)$). This symplectic form is exact.

The bundle T^*G is trivial, hence there exist global trivializations $\mathcal{T} : T^*G \to G \times \mathfrak{g}^*$. For the following calculations, $\mathcal{T}$ is chosen to be the right trivialization (of a $\alpha_g \in T^*_g G$)

$$\mathcal{T} : \alpha_g \mapsto (g, (T^*_e R_g)\alpha_g),$$

using the right action R of G on itself. It is an isomorphism, its inverse being

$$\mathcal{T}^{-1} : (g, \alpha_e) \mapsto (T^*_g R_{g^{-1}})\alpha_e = \alpha_e \circ T_g R_{g^{-1}},$$

where $g \in G$ and $\alpha_e \in T^*_e G \cong \mathfrak{g}^*$. All objects to be defined may be considered both on T^*G and $G \times \mathfrak{g}^*$; the trivialized symplectic form is [OR04, Thm. 6.2.4]

$${}^{\mathcal{T}}\omega_{(g,\alpha)}((u_g,\mu),(v_g,\nu)) = \langle \nu, T_g R_{g^{-1}} u_g \rangle - \langle \mu, T_g R_{g^{-1}} v_g \rangle - \langle \alpha, [T_g R_{g^{-1}} u_g, T_g R_{g^{-1}} v_g] \rangle,$$

where $g \in G$; $\alpha, \mu, \nu \in \mathfrak{g}^*$ and $u_g, v_g \in T_g G$.

On the symplectic manifold (T^*G, ω), a compatible action of $G \times G$ is obtained by lifting the natural left and right actions of G on itself to the bundle. Note that the individual left or right action of G on T^*G is always proper. More properties of these lifts are going to be summarized in the sequel, they can be found in [OR04, pp. 54, 128, and 218].

The left action $\Psi^{(L)}_h = L_h : g \mapsto hg$ $(g, h \in G)$ lifts to

$$\tilde{\Psi}^{(L)}_h = T^*_{hg} L_{h^{-1}} : T^*_g G \to T^*_{hg} G, \quad \beta_g \mapsto \beta_g \circ T_{hg} L_{h^{-1}},$$

and analogously, the action of G from the right, written as a left action, $\Psi^{(R)}_h = R_{h^{-1}} : g \mapsto gh^{-1}$, lifts to

$$\tilde{\Psi}^{(R)}_h = T^*_{gh^{-1}} R_h : T^*_g G \to T^*_{gh^{-1}} G, \quad \beta_g \mapsto \beta_g \circ T_{gh^{-1}} R_h.$$

The action in the trivialization is obtained by requiring $\mathcal{T}$ to be equivariant, this gives ${}^{\mathcal{T}}\tilde{\Psi}^{(L)}_h((g,\alpha)) = (hg, \mathrm{Ad}^*(h)\alpha)$. Note that the orbits of the left G-action on $G \times \mathfrak{g}^* \cong T^*G$ are isomorphic to coadjoint G-orbits. For the right action, one has ${}^{\mathcal{T}}\tilde{\Psi}^{(R)}_h((g,\alpha)) = (gh^{-1},\alpha)$. The orbits of the right G-action are of the form $G \times \{\alpha\}$.

For later calculations, we record the fundamental vector fields for the left and right action of G on T^*G, denoted by $\xi^{(L)}$ and $\eta^{(R)}$ for $\xi,\eta \in \mathfrak{g}$. The infinitesimal generators of these (trivialized) actions are

$$\mathcal{T}_*(\xi^{(L)})_{(g,\alpha)} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 {}^{\mathcal{T}}\tilde{\Psi}^{(L)}_{\exp(t\xi)}((g,\alpha)) = (\exp(t\xi)g, \mathrm{Ad}^*(\exp(t\xi))\alpha) = (\xi^{(L)}_g, \mathrm{ad}^*(\xi)\alpha),$$

where $\xi^{(L)}_g$ is the fundamental vector field for the left G-action on itself, evaluated in $g \in G$, and

$$\mathcal{T}_*(\eta^{(R)})_{(g,\alpha)} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 {}^{\mathcal{T}}\tilde{\Psi}^{(R)}_{\exp(t\eta)}((g,\alpha)) = (g\exp(-t\eta),\alpha) = (\eta^{(R)}_g, 0),$$

where $\eta^{(R)}_g$ is as above, but now for the right action. As both actions commute, the simultaneous action of $G \times G$ is generated by the fundamental vector field

$$\mathcal{T}_*((\xi \oplus \eta)^{T^*G})_{(g,\alpha)} = ((\xi \oplus \eta)^G_g, \mathrm{ad}^*(\xi)\alpha),$$

where $\xi \oplus \eta \in \mathfrak{g} \oplus \mathfrak{g}$. All fundamental vector fields have to be understood as on T^*G or G, depending on the context (i. e., according to the superscript T^*G or G, or whether it is evaluated in a point of T^*G or G).

One easily checks that the action defined above is symplectic; actually, it is Hamiltonian. Using $\mathrm{d}(\xi^{T^*G} \lrcorner\, \vartheta) = \xi^{T^*G} \lrcorner\, \omega$, one has a moment map (determined up to a constant), its components are given by the contraction of a fundamental vector field with the canonical form. The values at the point $\beta_g \in T^*G$ of the moment maps for the left and right action, $\Phi^\xi_{(L)}$ and $\Phi^\eta_{(R)}$, are given by ($\xi,\eta \in \mathfrak{g}$)

$$\Phi^\xi_{(L)|\beta_g} = \beta_g(\xi^{(L)}_g) \text{ and } \Phi^\eta_{(R)|\beta_g} = \beta_g(\eta^{(R)}_g).$$

These moment maps are surjective and equivariant, as one sees immediately from their trivialized forms:

$${}^{\mathcal{T}}\Phi^\xi_{(L)|(g,\alpha)} = \alpha(\xi) \text{ and } {}^{\mathcal{T}}\Phi^\eta_{(R)|(g,\alpha)} = -[\mathrm{Ad}^*(g^{-1})\alpha](\eta).$$

The trivialized forms of the moment maps also permit to read off their level sets. The moment map for the right G-action is constant precisely on the left G-orbits; the right G-orbits are the level sets of the left moment map.

7.2 Properties of the Orbits on T^*G

Now the orbit types will be determined. Therefore, the stabilizers corresponding to the different actions are calculated. In the trivialization $\mathcal{T}$, a point $(g,\alpha) \in G \times \mathfrak{g}^*$ is fixed by the following subgroup of $G \times G$ under the simultaneous action:

$$\begin{aligned}
\mathrm{Stab}^{\mathcal{T}}_{G\times G}((g,\alpha)) &= \{(g_1,g_2) \in G \times G \mid (g_1,g_2)\cdot(g,\alpha) = (g,\alpha)\} \\
&= \{(g_1,g_2) \in G \times G \mid (g_1 g g_2^{-1}, \mathrm{Ad}^*(g_1)\alpha) = (g,\alpha)\} \\
&= \{(g_1, g^{-1}g_1 g) \in G \times G \mid g \in \mathrm{Stab}_G(\alpha)\} \\
&\cong \mathrm{Stab}_G(\alpha) = \{g_1 \in G \mid \mathrm{Ad}^*(g_1)\alpha = \alpha\}
\end{aligned}$$

By specializing to $g_1 = e$ and $g_2 = e$, resp., the stabilizers of the left and right action of G turn out to be trivial, thus the individual actions to be free:

$$\mathrm{Stab}^{\mathcal{T}}_{\{e\}\times G}((g,\alpha)) = \mathrm{Stab}^{\mathcal{T}}_{G\times\{e\}}((g,\alpha)) = \{e\}$$

Note that the stabilizer of (g,α) under the simultaneous action is isomorphic to the stabilizer of α under the coadjoint action, hence it does not need to be compact if G is not – the simultaneous left and right action of G on T^*G is, in general, not proper. However, for a compact group G, the generic stabilizer is isomorphic to a maximal torus in G. Note further that triviality of the left and right stabilizer implies that the moment maps of the individual actions are submersions because by Prop. 2.11, the image of the tangent map of the moment map is the annihilator of the stabilizer Lie algebra.

Knowing now the structure of the $(G\times G)$-orbits, we continue by showing that these orbits are coisotropic. Let V be the tangent space to the $(G\times G)$-orbit at (g,α):

$$V = T_{(g,\alpha)}(G\times G)\cdot(g,\alpha) = \{(T_eR_g(\xi) - T_eL_g(\eta), \mathrm{ad}^*(\xi)\alpha) \mid \xi,\eta\in\mathfrak{g}\},$$

which is seen from the explicit form of the fundamental vector fields. As the action of G on itself is transitive, T_eR_g and $T_eL_g : \mathfrak{g}\to T_gG$ are surjective and we may write:

$$V = \{((T_eL_g)\xi, \alpha\circ\mathrm{ad}(\eta)) \mid \xi,\eta\in\mathfrak{g}\}$$

It can be split into the direct sum $V = E\oplus F$, where $E = \{((T_eL_g)\xi, 0) | \xi\in\mathfrak{g}\} = T_gG\times\{0\}$ and $F = \{(0,\alpha\circ\mathrm{ad}(\eta)) \,|\, \eta\in\mathfrak{g}\}$. The symplectic complement of V is now given by $V^\angle = E^\angle\cap F^\angle$. Therefore, we calculate the complements of E and F separately.

$$\begin{aligned}
(u_g,\mu)\in E^\angle \Leftrightarrow\\
0 &= {}^{\mathcal{T}}\omega_{(g,\alpha)}((u_g,\mu),((T_eL_g)\xi,0))\\
&= -\langle\mu,(T_gR_{g^{-1}})(T_eL_g)\xi\rangle - \langle\alpha,[T_gR_{g^{-1}}u_g,\mathrm{Ad}(g)\xi]\rangle\\
&= -\langle\mu,\mathrm{Ad}(g)\xi\rangle - \langle\alpha,\mathrm{ad}(T_gR_{g^{-1}}u_g)(\mathrm{Ad}(g)\xi)\rangle\\
&= -\langle\mu,\mathrm{Ad}(g)\xi\rangle + \langle\mathrm{ad}^*(T_gR_{g^{-1}}u_g)\alpha,\mathrm{Ad}(g)\xi\rangle\\
&= \langle\mathrm{ad}^*(T_gR_{g^{-1}}u_g)\alpha-\mu,\mathrm{Ad}(g)\xi\rangle \qquad \forall\xi\in\mathfrak{g}\\
&\Leftrightarrow \mu = \mathrm{ad}^*(T_gR_{g^{-1}}u_g)\alpha
\end{aligned}$$

$$\begin{aligned}
(u_g,\mu)\in F^\angle \Leftrightarrow\\
0 &= {}^{\mathcal{T}}\omega_{(g,\alpha)}((u_g,\mu),(0,\alpha\circ\mathrm{ad}(\eta)))\\
&= \langle\alpha\circ\mathrm{ad}(\eta),T_gR_{g^{-1}}u_g\rangle\\
&= -\langle\alpha,\mathrm{ad}(T_gR_{g^{-1}}u_g)\eta\rangle\\
&= \langle\mathrm{ad}^*(T_gR_{g^{-1}}u_g)\alpha,\eta\rangle \qquad \forall\eta\in\mathfrak{g}\\
&\Leftrightarrow \mathrm{ad}^*(T_gR_{g^{-1}}u_g)\alpha = 0
\end{aligned}$$

From this follows:

$$V^\angle = E^\angle\cap F^\angle = \{(u_g,0) \mid \mathrm{ad}^*(T_gR_{g^{-1}}u_g)\alpha = 0\} \subseteq T_gG\times\{0\} = E\subseteq V$$

This means that the $(G \times G)$-orbits on T^*G are coisotropic.

The orbits of the individual left or right action do not have a particular symplectic property. If they were isotropic or coisotropic, they would be Lagrangian (by dimension) and the symplectic form would vanish when restricted to the tangent space of the orbit. Consequently, the moment map would be constant on the connected components of the orbits, its image being zero-dimensional (for G having a countable number of connected components). But, in general, this is not the case. It it were symplectic, the dimension of the orbit would not change under the moment map, which is not true, either.

For the individual left and right action, one easily finds a global slice: $T_e^*G \subset T^*G$ meets every $G \times \{e\}$- and $\{e\} \times G$-orbit exactly once. As the actions on T^*G are lifts from transitive actions on G, the tangent spaces of the orbits are complementary to those of T_e^*G. As this slice is common for both actions, there is a slice for the simultaneous $G \times G$-action which lies in T_e^*G: Note that $\mathrm{diag}(G) \subset G \times G$ acts on $T_e^*G = \mathfrak{g}^*$ by the coadjoint action, where slices exist under the conditions of Thm. 2.30. As the moment maps, when restricted to T_e^*G, are simply the identity, the slice for the $G \times G$-action on T^*G is again mapped to a slice under $\Phi_{(L)}$ and $\Phi_{(R)}$. Here, the preservation of slices which was shown in Lemma 3.11 can be observed for global slices (as the orbit structure in this example is very simple).

7.3 Duality Properties

The computation which establishes the coisotropy of the orbits allows for another conclusion: Note that the vector space E is at the same time the tangent space at $g \in G$ for the orbit of the right action. $E^{\angle}$ turns out to be the tangent space for the left action. Hence the tangent spaces of both orbits are symplectically complementary and thus satisfy the Lie-Weinstein condition for a symplectic dual pair.

If G is connected, one easily deduces the following identities, which show that the symplectic Howe condition is satisfied, i. e., [9]

$$Z_{C^\infty(M)}(\Phi_{(L)}^* C^\infty(\mathfrak{g}^*)) = {}^G C^\infty(M) = \Phi_{(R)}^* C^\infty(\mathfrak{g}^*)$$

and

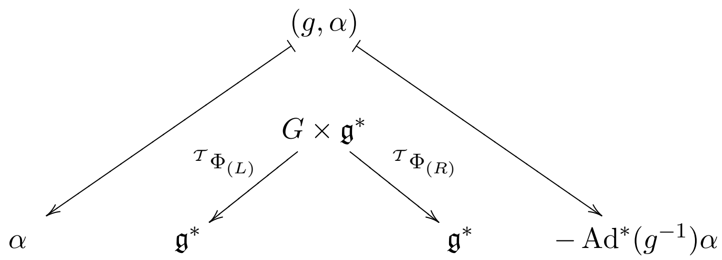
$$Z_{C^\infty(M)}(\Phi_{(R)}^* C^\infty(\mathfrak{g}^*)) = C^\infty(M)^G = \Phi_{(L)}^* C^\infty(\mathfrak{g}^*).$$

We have used Prop. 3.19 and the fact that the components of the moment maps separate the orbits (because the level sets of each moment map are orbits of the other action).

As both moment maps are surjective submersions, we are in a completely generic situtation and thus we may apply Thm. 3.16 directly (of course, the general orbit correspondence of Thm. 3.26 also holds here). Therefore, we have a bijection between the symplectic leaves in $\mathfrak{g}^*$ (i. e., the coadjoint orbits) for the left and the right action. The maps are represented

[9] The superscript G denotes invariance under the left and right G-action on M, resp.

by a diagram:



Both moment maps being equivariant, there is a well-defined map

$$\Lambda : \mathfrak{g}^*/G \to \mathfrak{g}^*/G, \quad \mathcal{O}_\alpha \mapsto \mathcal{O}_{-\alpha},$$

providing an explicit orbit correspondence between the coadjoint orbits $\mathcal{O}_\alpha$ (containing α) and $\mathcal{O}_{-\alpha}$ (containing $-\alpha$). Further, by Thm. 3.26(iii), we reobserve the known fact that the reduced spaces for the moment map of the right G-action are the coadjoint orbits in the image of the moment map of the left G-action (see also [OR04, Thm. 6.1.4]).

The orbit correspondence we have found matches the decomposition of the space $L^2(G)$ (which is identified with the geometric quantization of T^*G with respect to the standard vertical polarization) given by the Peter-Weyl theorem.

Theorem 7.1 (Peter-Weyl)**.** *Let G be a compact connected Lie group. Let G act on $L^2(G)$ via the left-regular representation. Then*

$$L^2(G) \cong \bigoplus_{\alpha \in \widehat{G}} V_\alpha \otimes V_\alpha^*,$$

where V_α denotes the representation space corresponding to the highest weight α and V_α^* its dual, which corresponds to the lowest weight $-\alpha$.

A Invariant functions on $\mathrm{Mat}(n,m;\mathbb{C})$ under the action of $U(n)$

Let an element $U \in U(n)$ act on $z \in M = \mathrm{Mat}(n,m;\mathbb{C})$ by matrix multiplication from the left: $(U,z) \to Uz$, i. e., each column of z is acted on independently, columns are not mixed. We interpret M as a real $2nm$-dimensional vector space with complex structure given by multiplication with the imaginary unit i.

The aim is to find all smooth (but not necessarily holomorphic as we work over $\mathbb{R}$) functions on M which are invariant under this action. By the following theorem of G. Schwarz ([Sch75], see also [OR04, Thm. 2.5.3]), this problem actually reduces to finding the invariant real polynomials.

Theorem A.1. *Let K be a compact Lie group acting linearly on the real vector space V and suppose $\{p_1,\dots,p_k\}$ generates the algebra of K-invariant real polynomials on V. Then the map*

$$p : C^\infty(\mathbb{R}^k) \to C^\infty(V)^K, \qquad f \mapsto f \circ (p_1,\dots,p_k)$$

is surjective.

Therefore, we will find a Hilbert basis for the $U(n)$-invariants in $\mathbb{R}[M]$ in order to describe $C^\infty(M)^{U(n)}$. We first record the following identity:

Lemma A.2.

$$\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Hom}_{\mathbb{R}}(M,\mathbb{R}) = \mathrm{Hom}_{\mathbb{C}}(M,\mathbb{C}) \oplus \mathrm{Hom}_{\overline{\mathbb{C}}}(M,\mathbb{C})$$

Here, $\mathrm{Hom}_{\overline{\mathbb{C}}}$ denotes the $\mathbb{C}$-antilinear homomorphisms.

Proof. This becomes obvious when one writes out bases for these vector spaces. Observe first

$$\mathrm{Hom}_{\mathbb{R}}(M,\mathbb{R}) = \mathrm{span}_{\mathbb{R}}\{z \mapsto \mathrm{Re} z_{ij}, z \mapsto \mathrm{Im} z_{ij} \mid 1 \le i \le n, 1 \le j \le m\},$$

i. e., the real dual space to M is spanned by the real and imaginary parts of all coordinate functions. Its complexification can be written as follows:

$$\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Hom}_{\mathbb{R}}(M,\mathbb{R}) = \mathrm{span}_{\mathbb{C}}\{z \mapsto z_{ij}, z \mapsto \bar{z}_{ij} \mid 1 \le i \le n, 1 \le j \le m\}.$$

These are the $\mathbb{C}$-linear and the $\mathbb{C}$-antilinear complex coordinate functions on M. Both parts can be considered separately,

$$\mathrm{Hom}_{\mathbb{C}}(M,\mathbb{C}) = \mathrm{span}_{\mathbb{C}}\{z \mapsto z_{ij} \mid 1 \le i \le n, 1 \le j \le m\}$$

and

$$\mathrm{Hom}_{\overline{\mathbb{C}}}(M,\mathbb{C}) = \mathrm{span}_{\mathbb{C}}\{z \mapsto \bar{z}_{ij} \mid 1 \le i \le n, 1 \le j \le m\}.$$

By comparison, the claim follows. □

Remark A.3. Note that, if $S(V)$ denotes the algebra of symmetric tensors over a vector space V, we have $\mathbb{C}[M] = S(\mathrm{Hom}_{\mathbb{C}}(M,\mathbb{C}))$, $\mathbb{C}[\overline{M}] = S(\mathrm{Hom}_{\overline{\mathbb{C}}}(M,\mathbb{C}))$ and $\mathbb{R}[M] = S(\mathrm{Hom}_{\mathbb{R}}(M,\mathbb{R}))$. From the preceding lemma,

$$S(\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Hom}_{\mathbb{R}}(M,\mathbb{R})) = \mathbb{C}[M \oplus \overline{M}].$$

It still needs to be shown that

$$S(\mathbb{C}\otimes_{\mathbb{R}}\mathrm{Hom}_{\mathbb{R}}(M,\mathbb{R}))^{U(n)}=\mathbb{C}\otimes_{\mathbb{R}}(\mathbb{R}[M]^{U(n)}),$$

in order to link the preceding identity to the determination of the invariants on $\mathbb{R}[M]$. The fact that symmetrization and complexification commute is clear as symmetrization is purely combinatorial and does not depend on the underlying field.

Lemma A.4.

$$(\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}[M])^{U(n)}=\mathbb{C}\otimes_{\mathbb{R}}(\mathbb{R}[M]^{U(n)})$$

Proof. In fact, $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}[M]=\mathbb{C}[M]$, but if $q\in\mathbb{C}[M]^{U(n)}$, then $\mathrm{Re}q$ and $\mathrm{Im}q$ lie in $\mathbb{R}[M]^{U(n)}$, hence $q\in\mathbb{C}\otimes_{\mathbb{R}}(\mathbb{R}[M]^{U(n)})$. The opposite inclusion is clear because any complex-linear combination $\sum_i\lambda_iq_i\in\mathbb{C}[M]$, with $\lambda_i\in\mathbb{C}$ and $q_i\in\mathbb{R}[M]^{U(n)}$, is invariant under $U(n)$, thus lies in $(\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}[M])^{U(n)}$. □

At this point, we know that determining the invariants in $\mathbb{C}[M\oplus\overline{M}]$ is equivalent to describing $\mathbb{C}\otimes_{\mathbb{R}}(\mathbb{R}[M]^{U(n)})$.

By the definition of the unitary group, for any $U\in U(n)$ we have

$$\overline{U}=(U^{-1})^{\mathrm{T}}.$$

This can be interpreted as equivalence between the complex conjugate representation of $U(n)$ on $\mathbb{C}^n$ and the contragredient representation of $U(n)$ on $(\mathbb{C}^n)^*$ [Sep07, Cor. 2.20(1)]. Using this and $M=(\mathbb{C}^n)^m$, one sees

$$\mathbb{C}[M\oplus\overline{M}]^{U(n)}=\mathbb{C}[(\mathbb{C}^n)^m\oplus(\mathbb{C}^n)^{*m}]^{U(n)}.$$

Using the fact that $U(n)$ is, at the same time, a maximal compact subgroup and a real form of $GL(n,\mathbb{C})$, we may apply the unitary trick.

Suppose $K=U(n)$ acts on a vector space W by the representation $\varrho:K\to GL(W)$. Let $w\in W$ be such that $\varrho(k)w=w\ \forall k\in K$, i. e., $\varrho(k)_{|\mathbb{C}w}=\mathrm{id}_{\mathbb{C}w}$ and $\mathbb{C}w$ is a one-dimensional invariant subspace on which K acts trivially. Consequently, one has for the derivative $\varrho_*(X)w=0$ and $\varrho_*(X)_{|\mathbb{C}w}=0$ on the whole Lie algebra $\mathfrak{k}$ of K. If ϱ_* is extended $\mathbb{C}$-linearly to the complexification $\mathfrak{k}\oplus\mathrm{i}\mathfrak{k}$, this writes $\varrho_*(Y+\mathrm{i}Z)_{|\mathbb{C}w}=0$ for any $Y,Z\in\mathfrak{k}$. By the exponential map, this maps to $\varrho(\exp(Y+\mathrm{i}Z))_{|\mathbb{C}w}=\exp(\varrho_*(Y+\mathrm{i}Z)_{|\mathbb{C}w})=\mathrm{id}_{\mathbb{C}w}$. As $K=U(n)$ is a real form of $GL(n,\mathbb{C})$, the complexification of $\mathfrak{k}$ equals $\mathfrak{k}\oplus\mathrm{i}\mathfrak{k}=\mathfrak{gl}(n,\mathbb{C})$. Therefore, the the invariant subspaces for the actions of $U(n)$ and $GL(n,\mathbb{C})$ are the same, hence

$$\mathbb{C}[(\mathbb{C}^n)^m\oplus(\mathbb{C}^n)^{*m}]^{U(n)}=\mathbb{C}[(\mathbb{C}^n)^m\oplus(\mathbb{C}^n)^{*m}]^{GL(n,\mathbb{C})}.$$

The latter algebra of invariants is known by the First Fundamental Theorem for the general linear group [Pro07, p. 245].

Theorem A.5 (FFT for $GL(n,\mathbb{C})$)**.** *The ring of polynomial functions on $(\mathbb{C}^n)^m\oplus(\mathbb{C}^n)^{*m}$ that are invariant under the action of $GL(n,\mathbb{C})$ is generated by the functions $f_i(z_j)$ where $(z_1,\dots,z_m,f_1,\dots,f_m)\in(\mathbb{C}^n)^m\oplus(\mathbb{C}^n)^{*m}$.*

Translating this back to the initial problem by $f_i(z_j)=\langle w_i,z_j\rangle$, the algebra of invariants $\mathbb{R}[M]^{U(n)}$ is generated by the real and imaginary parts of the Hermitian scalar products on M.

B Structure and Classification of Howe Dual Pairs

Howe dual pairs are the natural candidates for creating examples of commuting Hamiltonian actions. Of course, their classification and structure theory are well-studied. For a complete classification, we refer to e. g., [Rub94] and [Sch99]. Mainly based on the former, we will give a very brief summary of some of their structural properties, in terms of the notion of a Howe dual pair of Lie algebras, i. e., a pair of real reductive Lie subalgebras $(\mathfrak{g}_1, \mathfrak{g}_2)$ in a real reductive Lie algebra $\mathfrak{g}$ satisfying $Z_{\mathfrak{g}}(\mathfrak{g}_1) = \mathfrak{g}_2$ and $Z_{\mathfrak{g}}(\mathfrak{g}_2) = \mathfrak{g}_1$. One easily sees [Rub94, Lemma 5.1 and 5.2]:

Lemma B.1. *Write $\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. If $(\mathfrak{g}_1, \mathfrak{g}_2)$ is a Howe dual pair in $\mathfrak{g}$ then $\mathfrak{g}_1 = Z(\mathfrak{g}) \oplus \mathfrak{h}_1$ and $\mathfrak{g}_2 = Z(\mathfrak{g}) \oplus \mathfrak{h}_2$ where $(\mathfrak{h}_1, \mathfrak{h}_2)$ is a Howe dual pair in $[\mathfrak{g}, \mathfrak{g}]$. Conversely, any Howe dual pair in $[\mathfrak{g}, \mathfrak{g}]$ gives a Howe dual pair in $\mathfrak{g}$ by this construction. Further, $\mathfrak{g}_1 \cap \mathfrak{g}_2 = Z(\mathfrak{g}_1) = Z(\mathfrak{g}_2) \supseteq Z(\mathfrak{g})$.*

Lemma B.2. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $(\mathfrak{g}_1, \mathfrak{g}_2)$ a Howe dual pair in $\mathfrak{g}$ with $\mathfrak{g}_i$ complex ($i = 1, 2$), and define $\mathfrak{z} = Z(\mathfrak{g}_1) = Z(\mathfrak{g}_2)$. Further, define $\mathfrak{l}_{\mathfrak{z}} = Z_{\mathfrak{g}}(\mathfrak{z})$. Then $\mathfrak{l}_{\mathfrak{z}}$ is a Levi subalgebra[10] of $\mathfrak{g}$, the centre of $\mathfrak{l}_{\mathfrak{z}}$ is $\mathfrak{z}$. Write $\mathfrak{l}_{\mathfrak{z}} = \mathfrak{z} \oplus [\mathfrak{l}_{\mathfrak{z}}, \mathfrak{l}_{\mathfrak{z}}]$, $\mathfrak{g}_1 = \mathfrak{z} \oplus [\mathfrak{g}_1, \mathfrak{g}_1]$ and $\mathfrak{g}_2 = \mathfrak{z} \oplus [\mathfrak{g}_2, \mathfrak{g}_2]$. Then $([\mathfrak{g}_1, \mathfrak{g}_1], [\mathfrak{g}_2, \mathfrak{g}_2])$ is a Howe dual pair in $[\mathfrak{l}_{\mathfrak{z}}, \mathfrak{l}_{\mathfrak{z}}]$. Conversely, any Howe dual pair in a Levi subalgebra of $\mathfrak{g}$ is a Howe dual pair in $\mathfrak{g}$.*

If one has a complex Howe dual pair in a complex semisimple Lie algebra $\mathfrak{g}$, then the decompositions into simple ideals of $\mathfrak{g}$ and of the Lie subalgebras in the pair are compatible. Further, one knows the following statement about Cartan subalgebras in a complex Howe dual pair [Rub94, Thm. 5.4].

Theorem B.3. *Let $\mathfrak{g}_1$ be a complex Lie subalgebra of the complex Lie algebra $\mathfrak{g}$ belonging to a Howe dual pair in $\mathfrak{g}$, and let $\mathfrak{h}_1$ be a Cartan subalgebra of $\mathfrak{g}_1$. Then $\mathfrak{h}_1$ is the centre of a Levi subalgebra in $\mathfrak{g}$.*

Further, we provide a table of the Howe dual pairs in a compact classical Lie group, extracted from the complete classification of irreducible reductive Howe dual pairs (see, e. g., [Sch99, Table 4]).

G	G_1	G_2
$O(n_1n_2)$	$O(n_1)$	$O(n_2)$
$O(2n_1n_2)$	$U(n_1)$	$U(n_2)$
$O(4n_1n_2)$	$Sp(n_1)$	$Sp(n_2)$
$U(n_1n_2)$	$U(n_1)$	$U(n_2)$
$Sp(n_1n_2)$	$O(n_1)$	$Sp(n_2)$
$Sp(n_1n_2)$	$U(n_1)$	$U(n_2)$

[10] By a Levi subalgebra, we mean a subalgebra of $\mathfrak{g}$ which is constructed as follows: Take a Cartan subalgebra in $\mathfrak{g}$, choose a $\pm$-symmetric subset of the root system which is closed under addition, then the Levi subalgebra is the direct sum of the Cartan subalgebra and the root spaces corresponding to the chosen symmetric subset.

References

[AMR88] Ralph Abraham, Jerrold E. Marsden, and Tudor S. Ratiu. *Manifolds, tensor analysis, and applications*, volume 75 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1988.

[Ati82] Michael F. Atiyah. Convexity and commuting Hamiltonians. *Bull. London Math. Soc.*, 14(1):1–15, 1982.

[Bil04] Harald Biller. Characterizations of proper actions. *Math. Proc. Cambridge Philos. Soc.*, 136(2):429–439, 2004.

[Bla01] Anthony D. Blaom. A geometric setting for Hamiltonian perturbation theory. *Mem. Amer. Math. Soc.*, 153(727):xviii+112, 2001.

[Bla02] David E. Blair. *Riemannian geometry of contact and symplectic manifolds*, volume 203 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2002.

[BW09] Carsten Balleier and Tilmann Wurzbacher. On the geometric quantization of symplectic Howe pairs. *preprint*, 2009.

[DK00] Johannes J. Duistermaat and Johan A. C. Kolk. *Lie groups*. Universitext. Springer-Verlag, Berlin, 2000.

[Dui96] Johannes J. Duistermaat. *The heat kernel Lefschetz fixed point formula for the spin^c Dirac operator*. Progress in Nonlinear Differential Equations and their Applications, 18. Birkhäuser Boston Inc., Boston, MA, 1996.

[GS82] Victor Guillemin and Shlomo Sternberg. Geometric quantization and multiplicities of group representations. *Invent. Math.*, 67:515–538, 1982.

[GS90] Victor Guillemin and Shlomo Sternberg. *Symplectic techniques in physics*. Cambridge University Press, Cambridge, second edition, 1990.

[GW98] Roe Goodman and Nolan R. Wallach. *Representations and invariants of the classical groups*, volume 68 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1998.

[Hel78] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[HNP94] Joachim Hilgert, Karl-Hermann Neeb, and Werner Plank. Symplectic convexity theorems and coadjoint orbits. *Compositio Math.*, 94(2):129–180, 1994.

[How85] Roger Howe. Dual pairs in physics: harmonic oscillators, photons, electrons, and singletons. In *Applications of group theory in physics and mathematical physics (Chicago, 1982)*, volume 21 of *Lectures in Appl. Math.*, pages 179–207. Amer. Math. Soc., Providence, RI, 1985.

[How89] Roger Howe. Remarks on classical invariant theory. *Trans. Amer. Math. Soc.*, 313(2):539–570, 1989.

[HW90] Alan T. Huckleberry and Tilmann Wurzbacher. Multiplicity-free complex manifolds. *Math. Ann.*, 286(1-3):261–280, 1990.

[Kau67] Ludger Kaup. Eine Künnethformel für Fréchetgarben. *Math. Z.*, 97:158–168, 1967.

[KL97] Yael Karshon and Eugene Lerman. The centralizer of invariant functions and division properties of the moment map. *Illinois J. Math.*, 41(3):462–487, 1997.

[Kos70] Bertram Kostant. Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*, pages 87–208. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.

[OR04] Juan-Pablo Ortega and Tudor S. Ratiu. *Momentum maps and Hamiltonian reduction*, volume 222 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2004.

[Pal61] Richard S. Palais. On the existence of slices for actions of non-compact Lie groups. *Ann. of Math. (2)*, 73:295–323, 1961.

[Pfl01] Markus J. Pflaum. *Analytic and geometric study of stratified spaces*, volume 1768 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[Pro07] Claudio Procesi. *Lie groups*. Universitext. Springer, New York, 2007. An approach through invariants and representations.

[Rub94] Hubert Rubenthaler. Les paires duales dans les algèbres de Lie réductives. *Astérisque*, (219):121, 1994.

[Sch75] Gerald W. Schwarz. Smooth functions invariant under the action of a compact Lie group. *Topology*, 14:63–68, 1975.

[Sch99] Matthias Schmidt. Classification and partial ordering of reductive Howe dual pairs of classical Lie groups. *J. Geom. Phys.*, 29(4):283–318, 1999.

[Sep07] Mark R. Sepanski. *Compact Lie groups*, volume 235 of *Graduate Texts in Mathematics*. Springer, New York, 2007.

[Sja95] Reyer Sjamaar. Holomorphic slices, symplectic reduction and multiplicities of representations. *Ann. of Math. (2)*, 141(1):87–129, 1995.

[Sja98] Reyer Sjamaar. Convexity properties of the moment mapping re-examined. *Adv. Math.*, 138(1):46–91, 1998.

[SW59] Joseph H. Sampson and Gerard Washnitzer. A Künneth formula for coherent algebraic sheaves. *Illinois J. Math.*, 3:389–402, 1959.

[Var77] Veeravalli S. Varadarajan. *Harmonic analysis on real reductive groups*. Lecture Notes in Mathematics, Vol. 576. Springer-Verlag, Berlin, 1977.

[Wur96] Tilmann Wurzbacher. unpublished notes. 1996.

Index