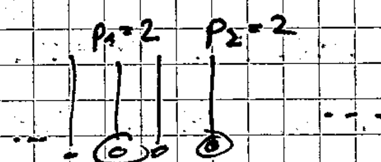
0. <u>Introduction</u> Marcel

<u>Aim</u> - explore + understand some of the techniques in the study of coh. sheaves on weighted proj. lines (Geigle / Lenzing)
- use the above to prove an analog of Kac's theorem for w.p.l. ($\mathbb{X}_{p,\lambda}$) (following CB).

I <u>What is $\mathbb{X}_{p,\lambda}$</u>?
Start with $\mathbb{P}^1$: loc. free sheaves
ind : $\mathcal{O}(n)$, $n \in \mathbb{Z}$ (grothendieck
torsion sheaves
ind : $S_p(\ell)$ length

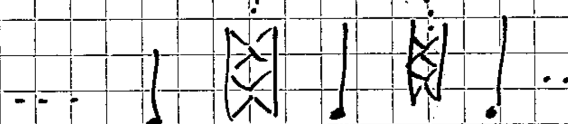
<u>AR</u>

$$\cdots \searrow \mathcal{O} \nearrow \mathcal{O}(1) \searrow \mathcal{O}(2) \nearrow \cdots$$



Choose certain points in $\mathbb{P}^1$ $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ and associate to them certain weights $\underline{p} = (p_1, \dots, p_n)$
Morally, to obtain coh $\mathbb{X}_{\underline{p},\underline{\lambda}}$, one blows up to chosen tubes

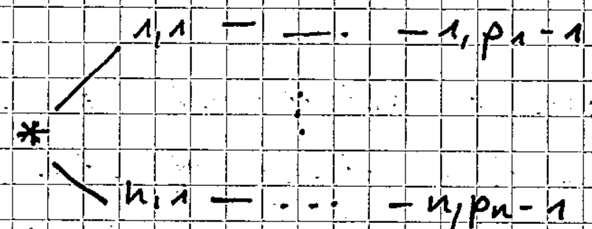
changes accordingly



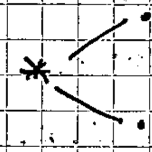
<u>Aim</u> Prove an analog of Kac's Thm in this setting, that is describe the "dimension types" of indec. coh. sheaves on $\mathbb{X}_{p,\lambda}$ in terms of "<u>roots</u>"

What are these?

One can associate a star shaped diagram with $\mathbb{X}_{\underline{p},\underline{\lambda}}$, namely



In our example we get



The desired root system will be the root system of the "loop algebra" assoc. with this diagram.

Main Thm (CB) $\mathbb{X}_{p,\lambda}$, $k=\bar{k}$. There ex. an indec. sheaf of "dim type" ϕ iff ϕ is a positive root. There ex. a unique indec. for a real root and infinitely many for an imaginary root.

II Classical situation Kac's Thm for quivers

Q $\rightsquigarrow$ dim. vectors of indec. rep.

$\mathfrak{g}$ (assoc. KM algebra) $\rightsquigarrow$ combinatorially $\rightsquigarrow$ $\Delta^+(\mathfrak{g})$ pos. roots

dim. vectors of indec. rep. $\xleftrightarrow{\text{Kac's Thm}}$ $\Delta^+(\mathfrak{g})$ pos. roots

Special case of this: Gabriel's Thm

Dynkin quivers $\downarrow$ dim. vectors of indec. $\xleftrightarrow{\text{Gabriel's Thm}}$ pos. roots

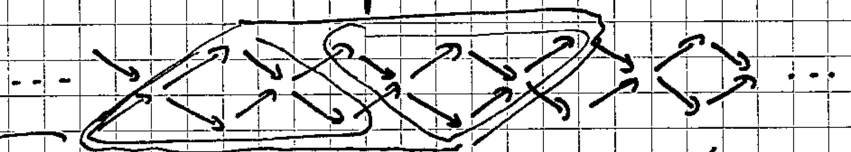
f.d. simple Lie algebras A, D, E $\downarrow$ pos. roots

We see that there exists a strong connection between rep. of quivers and KM algebras.

<u>Question</u>. Can this Lie algebra be recovered by the data given by $\mathrm{rep}_k Q$?

<u>Observation</u> (Happel). Ex. $Q \quad 1 \to 2 \to 3$

Consider $D^b(\mathrm{rep}_k Q)$ | $\mathfrak{g} = sl_4$, # pos. roots $= 6$



Now, consider $D^b(\mathrm{rep}_k Q)/_{T^2}$ = root category
$\quad T$: shift

indec $= 2$ # pos. roots of sl_4

We have found a suitable candidate for the pos. and neg. part of a Lie algebra. Over finite fields Peng + Xiao showed that the root category can indeed be equipped with the structure of a Lie algebra structure (using Hall algebra methods) = "counting of triangles"

$$D^b(\mathrm{rep}_k Q) \xrightarrow[\text{Peng Xiao}]{} L(Q)$$

and moreover $L(Q) \cong \mathfrak{g} \leftarrow$ KM algebra assoc. to Q

III <u>Analogue of Kac's Thm for w.p.l. (CB)</u>

k - finite field.

The construction of Peng and Xiao can be applied to the setting of $D^b(\mathrm{coh}\, \mathbb{X}_{p,\lambda})/T^2$. One obtains a Lie algebra $L(\mathbb{X}_{p,\lambda})$. The main observation of CB is that $L(\mathbb{X}_{p,\lambda})$

is related to the loop algebra $\mathcal{L}\mathfrak{g}$ assoc. with the star shaped diagram given earlier. Namely, there is a map of Lie alg

$$\mathcal{L}\mathfrak{g} \longrightarrow L(X_{p,\lambda}).$$

Coherent sheaves on w.p.l. I

Jue Le

1.1 graded alg.

1.2 graded module cat

1.3 Gorenstein ring

1.1 k - any field, $n \geq 2$

$p := (p_1, p_2, \ldots, p_n)$ weight sequence, $p_i \in \mathbb{Z}_{\geq 1}$

$$L = L(p) \cong \bigoplus_{i=1}^{n} \mathbb{Z} x_i \Big/ p_1\vec{x_1} = p_2\vec{x_2} = \cdots = p_n\vec{x_n} = \vec{c}$$

<u>Lemma 1.</u> (1) $L \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$, rank $L = 1$

(2) $L/\mathbb{Z}\vec{c} \cong \prod_{i=1}^{n} \mathbb{Z}/\mathbb{Z}p_i$

(3) $\forall\, \vec{l} \in L$ is uniquely written in its normal form

$$\vec{l} = \sum_{i=1}^{n} l_i \vec{x_i} + l\vec{c} \qquad 0 \leq l_i \leq p_i - 1, \quad l \in \mathbb{Z}$$

$L_+ := \sum_{i=1}^{n} \mathbb{Z}_{\geq 0} \vec{x_i}$, $\vec{l}$ positive $\overset{\text{def}}{\iff} \vec{l} \in L_+$ $\iff l \geq 0$

$\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ (normalized) parameter sequence, $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$, $\lambda_i \neq \lambda_j \in \mathbb{P}^1(k) = k \cup \{\infty\}$

$$S = S(p, \underline{\lambda}) := k[x_1, x_2, \ldots, x_n] \Big/ (x_i^{p_i} - x_2^{p_2} + \lambda_i x_1^{p_1},\ i = 1, \ldots, n)$$

the string singularity of type (p, λ)

$x_i = X_i + (\ldots)$

S is L-graded by $\deg x_i = \vec{x_i}$

$\mathfrak{p} \subseteq S$ is homog. prime ideal $\iff \begin{cases} 1) \ \mathfrak{p} \text{ homog. ideal} \\ 2) \ \forall\, a, b \in S \text{ homog.} \\ \quad ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}. \end{cases}$

Lemma 1.2. (1) S is graded noetherian, graded local.

~~Defn~~ Proof. S noeth $\Rightarrow$ S graded noeth. $m = (x_1, x_2, \dots, x_n)$ is the unique homog. max ideal of S

(2) Fix $L \ni \vec{l} = \sum_{i=1}^{n} l_i \vec{x_i} + l\vec{c}$, every element in $S_{\vec{l}}$ is uniquely written as

$$x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n} f(x_1^{p_1}, x_2^{p_2})$$

where $f = f(t_1, t_2) \in k[t_1, t_2]$ is a homog. polynom. of degree l.

In particular, S is graded domain.

Pf. $R := k[t_1, t_2]$, $\deg t_i = \vec{c}$.

Define $\theta : R \longrightarrow S(p, \lambda)$

$$t_i \longmapsto x_i^{p_i} \qquad i = 1, 2$$

$$t_i \longmapsto x_i^{p_i}$$

$$\begin{tikzcd} R' = k[t_1, \dots, t_n] \arrow[r, hook, "\theta'"] \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowleft"] & S' = k[x_1, x_2, \dots, x_n] \arrow[d, two heads] \\ R \arrow[r] & S \end{tikzcd}$$

$t_1 \mapsto t_1$, $t_2 \mapsto t_2$, $t_i \mapsto t_2 - \lambda_i t_1$

- $R \cong R' / (t_i - t_2 + \lambda_i t_1,\ i \geq 3)$
- $S \cong S' \otimes_{R'} R$
- $S' = \bigoplus_{0 \leq l_i < p_i} R' x_1^{l_1} \cdots x_n^{l_n}$

$\Big\} \Rightarrow S \cong \bigoplus_{0 \leq l_i < p_i} R x_1^{l_1} \cdots x_n^{l_n}$

∎

Let $0 \neq a \in S$ homog. element. Call a prime element if $(a) \subseteq S$ is a homo prime ideal.

Let $f = f(t) \in k[t]$ a non-constant monic irred. polynomial other than $t - \lambda_i$ ($i = 2, \dots, n$).

$f^h(t_1, t_2) = t_1^{\deg(f)} f\left(\frac{t_2}{t_1}\right)$ homogenization of f

<u>Lemma</u> 1.2. (3) Up to scalar, all the homog. prime elements in S are $x_1, \dots, x_n$, $f^h(x_1^{p_1}, x_2^{p_2})$.

Hence, every homog. element in S is a unique product of prime ones.

<u>Pf.</u> x_i is homog. prime.

$$S/(x_1) = k[x_2, x_3, \dots, x_n] / (x_i^{p_i} - x_2^{p_2})$$

$$S/(x_2) = k[x_1, \dots, x_n] / (x_i^{p_i} + \lambda_i x_1^{p_1})$$

$$S/(x_i) = k[x_1, x_2, \dots, x_n] / \begin{pmatrix} x_2^{p_2} + \lambda_i x_1^{p_1} \\ x_j^{p_j} - x_2^{p_2} + \lambda_i x_1^{p_1} \end{pmatrix}$$

$$= k[x_1, \dots, x_n] / (x_j^{p_j} - (\lambda_i - \lambda_j) x_1^{p_1})$$

$$S/(f^h(x_1^{p_1}, x_2^{p_2})) \longleftarrow k[t_1, t_2] / (f^h(t_1, t_2))$$ $\square$

<u>graded spectrum of S</u>

$$\mathrm{Spec}(S) = \{ p \triangleleft S \mid p \text{ homog prime} \}$$
$$= \{ \underset{\text{generic point}}{0}, \underbrace{(x_1), \dots, (x_n)}_{\text{exceptional}}, \underset{\text{ordinary}}{(f^h(x_1^{p_1}, x_2^{p_2}))}, \underset{\text{max}}{m} \}$$

<u>Note</u> $\forall\, p \neq q$ of height 1, the smallest prime ideal containing p and q is m.

<u>Zariski topology</u>.

- Open subset : $\bigcup_a D(a)$ $a \in S$ homo

  $D(a) = \{ p \in \mathrm{Spec}^{\mathbb{Z}}(S) \mid a \notin p \}$

- Closed subset : $V(T)$, T subset of S consisting of homog. elements

$V(T) = \bigcap_{x \in T} V(x)$, $V(x) = \{ p \mid x \in p \}$

$\forall x$

$x \in p \iff \exists$ prime factor of $x \in p$

$p = (y)$, $p = m$

ϕ $\mathrm{Spec}^L(R)$

finite subsets containing m.

1.2 <u>Graded module cat.</u>

$\mathrm{Mod}^L S :=$ the cat. of L-graded S-modules

degree shift autom. $(\vec{l}) : \mathrm{Mod}^L S \longrightarrow \mathrm{Mod}^L S$

$M \longmapsto M(\vec{l})$

$M(\vec{l})_{\vec{l}'} = M_{\vec{l}+\vec{l}'}$

$\mathrm{mod}^L S \subseteq \mathrm{Mod}^L S$ f.g.

$\mathrm{mod}^L_0 S \subseteq \mathrm{mod}^L S$ f. ~~fl.~~ l. modules

$\forall\, p \in \mathrm{Spec}^L(S)$ $S_p \cong$ homog. localization

$= \bigoplus_{\vec{l} \in L} \left\{ \frac{a}{s} \;\middle|\; \begin{array}{l} a, s \in p \text{ homog elements} \\ \deg a - \deg s = \vec{l} \end{array} \right\}$

- S_p L-graded
- $pS_p \subseteq S_p$ unique homog. maximal ideal
- S_p graded local
- $k(p) = S_p / pS_p$ (L-graded ring
  $\nexists$ non-trivial homog. ideal)
- Let $p \in \mathrm{Spec}^L R$ with $\mathrm{ht}\, p = 1$. Then
  S_p is graded DVR, i.e. S_p graded local,
  graded noetherian, graded domain

pS_p for some homo $t \neq 0$.

$$\begin{tikzcd} 0 \arrow[r] & pS_p \arrow[r] & S_p \arrow[r] & k(p) \arrow[r] & 0 \\ 0 \arrow[r] & S_p(\vec{l}) \arrow[u, "\cong\, t"] \arrow[ur, "t"] & & & \end{tikzcd}$$

$\deg(t) = -\vec{l}$.

proj. dim $k(p) = 1$

$\Rightarrow gl.\dim^L(S_p) = 1$.

Consider simple obj. in $\mathrm{mod}_0^L S_p$.

1) M simple $\Rightarrow \exists\ 0 \neq m \in M_{\vec{l}}$.

$S_p(\vec{l}) \longrightarrow M$

$1 \longmapsto m$ $\quad \Rightarrow \quad M \cong k(p)(\vec{l})$

2) $L\text{-supp}(M) \equiv \{\vec{l} \in L \mid M_{\vec{l}} \neq 0\}$ L-support of M

$L\text{-supp}(k(p))$ subgroup of L.

$k(p)(\vec{l}) \cong k(p)(\vec{l'}) \iff k(p) \cong k(p)(\vec{l'} - \vec{l})$

$\iff \vec{l'} - \vec{l} \in L\text{-supp}(k(p))$

$\vec{l_1}, \dots, \vec{l_r}$ rep. of coset of $L / L\text{-supp}(k(p))$

$k(p)(\vec{l_1}), \dots, k(p)(\vec{l_r})$ are all simple objects in $\mathrm{mod}_0^L S_p$.

$\forall\, p \in \mathrm{Spec}^L(S) \Rightarrow S/p$ graded domain

$Q(S/p) \cong \bigoplus_{\vec{l'} \in L} \{\frac{a}{s} \mid a, s \text{ homo } s \neq 0\}$

fractional field $\quad$ ($\Rightarrow$ graded field)

$Q(S/p) \cong k(p)$

If $ht\, p = 1$.

$k(q) \cong Q(S/p) \Rightarrow L\text{-supp}(k(p)) =$ subgroup of L gen. by $L\text{-supp}(S/p)$

<u>Case 1</u>. $p = (x_i)$ $\quad S/p = S/(x_i)$

$L\text{-supp}(S/p) = \sum_{i=2}^{n} \mathbb{Z}_{\geq 0} \vec{x}_i$

$L\text{-supp}(k(p)) = \langle \vec{x}_2, \dots, \vec{x}_n \rangle$

$L / L\text{-supp}(k(p)) \cong \mathbb{Z}/\mathbb{Z}p_1$

rep: $\vec{0}, \vec{x}_1, \dots, (p_1 - 1)\vec{x}_1$

$\rightsquigarrow k(p), k(p)(\vec{x}_1), \dots, k(p)((p_1-1)\vec{x}_1)$

$\rightsquigarrow S_p / p^l S_p (j\vec{x}_1) \quad j = 0, \dots, p_1 - 1, \; l \geq 1$

all indec. objects in $\mathrm{mod}_0^L S_p$

<u>Case 2</u>. $p = (f^u(x_1^{p_1}, x_2^{p_2}))$

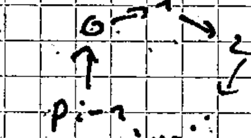
$L\text{-Supp}(k(p)) = L, \quad L / L\text{-supp}(k(p)) = 0$

$\rightsquigarrow$ unique simple module $k(p)$.

$\rightsquigarrow S_p / p^l S_p \quad l \geq 1$

<u>Proposition 1.3</u>. (1) Let $p = (x_i)$

Then, $\mathrm{mod}_0^L S_p \simeq$ f.d. nilp. rep. of C_{p_i}

C_{p_i} 

(2) Let $p = (f^u(x_1^{p_1}, x_2^{p_2}))$

Then, $\mathrm{mod}_0^L S_p \simeq$ f.d. nilp. rep. of $k[t]$.

<u>Remark</u>. mod $k[t]$ = f.d. rep. of $\cdot\circlearrowleft$

If k is perfect, then f.d. nilpotent rep. of $k[t] \simeq$ nilp. rep. of $\cdot\circlearrowleft$ over $k' = k[t]/(f(t))$

1.3 <u>Gorenstein property</u>

$\vec{\omega} := \sum_{i=1}^{n} (p_i - 1)\vec{x}_i - 2\vec{c} = -\sum_{i=1}^{n} \vec{x}_i + (n-2)\vec{c}$

dualizing element in L

$$\mathrm{EXT}_S^i(M,N) \cong \bigoplus_{\vec{\ell}\in L} \mathrm{Ext}^i_{\mathrm{Mod}^L S}(M, N(\vec{\ell})) \quad (i \geq 0)$$

<u>Prop 1.4</u>. $\mathrm{EXT}_S^i(k, S) \cong \delta_{i,2}\, k(-\vec{\omega}) \quad (i \geq 0)$

i.e. S is <u>graded Gorenstein</u> of dim. 2.

In particular, inj. dim. ${}^L S = 2$.

Let $M \in \mathrm{Mod}^L S$.

$\Gamma_m(M) \cong \{ a \in M \mid m^i a = 0 \text{ for some } i \}$

graded submodule

$H^i_m(M) := R^i \Gamma_m(M)$

<u>Theorem</u> 1.5. (local duality). $\exists$ graded S-module isom.

$$H^i_m(M) \cong D\,\mathrm{EXT}_S^{2-i}(M, S(\vec{\omega})) \quad \forall i$$

D - graded duality.

$$\begin{aligned} MCM^L(S) &:= \{ M \in \mathrm{mod}^L(S) \mid \mathrm{EXT}_S^i(k, M) = 0 \text{ for } i = 0, \dots, n \} \\ &= \{ M \in \mathrm{mod}^L(S) \mid \mathrm{EXT}_S^i(M, S) = 0 \text{ for } i = 1, 2 \} \end{aligned}$$

<u>maximal Cohen-Macaulay-module</u>

<u>Fact</u>. $M \in \mathrm{mod}^L S$

$M \in MCM^L(S) \iff {}_R M$ is graded free

$R = k[t_1, t_2]$.

§ 2.1 Graded alg. geometry
§ 2.2 $\mathbb{X} = \mathbb{X}(p;\underline{\lambda})$, $\mathrm{coh}^L(\mathbb{X})$ and tilting
§ 2.3 Grothendieck group and CB's
Kac's Thm

$L = L(p)$
k - a field

§ 2.1, 2.2 Baer, Dowbor, Geigle, Lenzing [98]
§ 2.3 CB 2005

Remark. Many ways to define $\mathbb{X}$
- axiomatic
- sheaf of noncomm. alg.
- stack / orbifold
- tilting procedure
- $\boxed{\text{graded sheaf}}$

§ 2.1 Graded alg. geometry
- rings – L-graded, comm. k-alg
- (L-graded) ringed space $(X, \mathcal{O}_X)$
  - X top. space
  - $\mathcal{O}_X$ structure sheaf
    $U \subseteq X \rightsquigarrow \mathcal{O}_X(U)$
    $U \longrightarrow V \rightsquigarrow \mathrm{res}_{V,U} : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$
- $\forall x \in X$, stalk $\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U)$
  is L-graded

Morphism of ringed space
$(f, f^\#) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$

$f : X \to Y$ continuous map
$V \underset{\text{open}}{\subseteq} Y$, $\mathcal{O}_Y(V) \xrightarrow{f_V^\#} \mathcal{O}_X(f^{-1}(V))$
morphism of L-graded rings
compatible with "res"

$\rightsquigarrow \forall x \in X$, $f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$

$\mathrm{Mod}^L \mathcal{O}_X$ = the cat of L-graded $\mathcal{O}_X$-modules
$\mathcal{M} : U \underset{\text{open}}{\subseteq} X \rightsquigarrow \mathcal{M}(U) \in \mathrm{Mod}^L \mathcal{O}_X(U)$
$\rightsquigarrow \mathcal{M}_x \in \mathrm{Mod}^L \mathcal{O}_x$
$\rightsquigarrow \mathrm{Mod}^L \mathcal{O}_X$ is abelian with $\oplus$.

<u>Support</u> $\mathrm{supp}(\mathcal{M}) = \{x \in X \mid \mathcal{M}_x \neq 0\}$

e.g. $\mathcal{O}_X \in \mathrm{Mod}^L \mathcal{O}_X$

<u>degree-shift</u> $(\vec{\ell}) : \mathrm{Mod}^L \mathcal{O}_X \to \mathrm{Mod}^L \mathcal{O}_X$
$\mathcal{M} \longmapsto \mathcal{M}(\vec{\ell}) : U \mapsto \mathcal{M}(U)(\vec{\ell})$

e.g. $\mathcal{O}_X(\vec{\ell}) \in \mathrm{Mod}^L \mathcal{O}_X$ <u>twisted structure sheaf</u> of X

$\Gamma(\mathcal{M}) = \mathcal{M}(X) \in \mathrm{Mod}^L \Gamma(\mathcal{O}_X)$ $\quad (\Gamma(\mathcal{O}_X) = \mathcal{O}_X(X))$
$\cong \bigoplus_{\vec{\ell} \in L} \mathrm{Hom}(\mathcal{O}_X, \mathcal{M}(\vec{\ell}))$

elements = <u>global sections</u> of $\mathcal{M}$

$\Gamma(-) : \mathrm{Mod}^L \mathcal{O}_X \to \mathrm{Mod}^L \Gamma(\mathcal{O}_X)$ <u>global section functor</u>

$f : X \to Y$
$f_* : \mathrm{Mod}^L \mathcal{O}_X \to \mathrm{Mod}^L \mathcal{O}_Y$
$\mathcal{M} \longmapsto f_* \mathcal{M}(V) = \mathcal{M}(f^{-1}(V))$
<u>direct image functor</u> of f

(e.g. $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ in $\mathrm{Mod}^L \mathcal{O}_Y$)

$$f^* : \mathrm{Mod}^L \mathcal{O}_Y \longrightarrow \mathrm{Mod}^L \mathcal{O}_X$$
$$W \longmapsto \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1} W$$

<u>inverse image functor</u> of f.

<u>Lemma 1</u>. (f^*, f_*) adjoint!

<u>Example 2</u>. R ring, $X = \mathrm{Spec}^L(R) = \{ p \subseteq R \mid p \text{ homog prime} \}$

$f \in R$ homog.

$D(f) = \{ p \in X \mid f \notin p \}$

$\mathcal{O}_X$ on X : $\mathcal{O}_X(D(f)) = R_f$

$$= \bigoplus_{\vec{\ell} \in L} \left\{ \frac{r}{f^i} \;\middle|\; \begin{array}{l} r \in R \text{ hom.} \\ \deg r - i \cdot \deg f = \vec{\ell} \end{array} \right\}$$

$\mathrm{Spec}^L(R) = (X, \mathcal{O}_X)$

Note : $\mathcal{O}_{X,p} = R_p$, $\Gamma(\mathcal{O}_X) = R$

(L-graded) <u>affine scheme</u> $(X, \mathcal{O}_X) \cong \mathrm{Spec}^L R$ for some R

<u>scheme</u> $(X, \mathcal{O}_X)$ s.t. · $X = \bigcup_{i \in \Lambda} U_i$ open

· $(U_i, \mathcal{O}_X|_{U_i})$ affine scheme

(U_i <u>affine open subset</u> of X)

$\rightsquigarrow$ · $\forall x \in X$, $\mathcal{O}_{X,x}$ is graded local, $m_{X,x}$

· $k(x) = \mathcal{O}_{X,x} / m_{X,x}$ <u>residue field</u> of x.

$f : X \to Y$ <u>morphism of schemes</u> s.t.

$$f_x : \mathcal{O}_{Y, f(x)} \longrightarrow \mathcal{O}_{X,x}$$
$$m_{Y, f(x)} \longrightarrow m_{X,x}$$

e.g. X scheme

$U \subseteq X$, $(U, \mathcal{O}_X|_U)$ is a scheme, open subscheme

$\mathrm{inc}: U \hookrightarrow X$ is a morphism, open immersion.

Example 3. $X = \mathrm{Spec}^L(R)$, $M \in \mathrm{Mod}^L R$.

$\tilde{M} \in \mathrm{Mod}^L \mathcal{O}_X$: $\tilde{M}(D(f)) = M_f \in \mathrm{Mod}^L \mathcal{O}_X(D(f))$
$= \mathrm{Mod}^L R_f$

In general, $\mathcal{M} \in \mathrm{Mod}^L \mathcal{O}_X$ is quasi-coherent if $X = \bigcup_{i \in \Lambda} U_i$ affine open s.t. $\mathcal{M}|_{U_i} \simeq \tilde{M}_i$
$\in \mathrm{Mod}^L \mathcal{O}_X|_{U_i}$

$\mathrm{QCoh}^L(X) \subseteq \mathrm{Mod}^L \mathcal{O}_X$ abelian, $\oplus$.

Fact 4

$$\begin{tikzcd} \mathrm{Mod}^L R \arrow[r] & \mathrm{Qcoh}^L(\mathrm{Spec}^L(R)) \arrow[l, bend left, "\Gamma"] \end{tikzcd}$$
$$M \longmapsto \tilde{M}$$

an equivalence.

A scheme X is noetherian if $X = \bigcup_{i=1}^{n} U_i$ affine open, $\mathcal{O}_X(U_i)$ graded noetherian.

$\Rightarrow x \in X$, $\mathcal{O}_{X,x}$ is noetherian.

$\mathcal{M} \in \mathrm{Qcoh}^L(X)$ is coherent if $\mathcal{M}|_{U_i} \simeq \tilde{M}_i$ with M_i is f.g. over $\mathcal{O}_X(U_i)$.

$\mathrm{coh}^L(X) \subseteq \mathrm{Qcoh}^L(X)$ abelian subcat.

$\boxed{\text{From now on, all schemes are noetherian.}}$

Fact 5. (1) $\mathrm{coh}^L(X)$ is small, noetherian, abelian category

(2) $\mathrm{coh}^L(\mathrm{Spec}^L(R)) \underset{\Gamma}{\overset{(\tilde{-})}{\simeq}} \mathrm{mod}^L R$

(3) $\mathrm{gl.dim}\ \mathrm{coh}^L(X) = \sup_{x \in X} \mathrm{gl.dim}^L \mathcal{O}_{X,x}$

(4) $\mathcal{M} \in \mathrm{coh}^L(X)$, $\mathrm{supp}(\mathcal{M}) \subseteq X$ is closed.

$$\boxed{\begin{array}{ccc} \mathrm{coh}^L(X) & \supseteq & \mathrm{coh}^L_0(X) \\ \cup\!\!\mid & & \\ \mathrm{vect}^L(X) & & \end{array}}$$

$\mathrm{coh}_0^L(X)$ = the cat of <u>finite length</u> coherent sheaves on X

(abelian subcat. of $\mathrm{coh}^L(X)$)

Fact 6. $\mathcal{M} \in \mathrm{coh}_0^L(X) \iff \mathrm{supp}(\mathcal{M})$ = finite set of closed points.

$x \in X$ closed point

$$\mathrm{coh}^L(X)_x = \{ \mathcal{M} \in \mathrm{coh}^L(X) \mid \mathrm{supp}\, \mathcal{M} \subseteq \{x\} \} \subseteq \mathrm{coh}^L_0(X)$$

skyscraper sheaves at x.

Fact 7. $f : X \to Y$. $\mathrm{Qcoh}^L(Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Qcoh}^L(X)$

$x \in X$ closed point

$x \subseteq U = \mathrm{Spec}\, R \subseteq X$

$x = \mathfrak{m}$

$i_x : \operatorname{Spec} \mathcal{O}_{X,x} \hookrightarrow \operatorname{Spec}^L(R) \subseteq X$

$M \in \operatorname{mod}_0^L \mathcal{O}_{X,x} \longmapsto (i_x)_*(\tilde{M}) \in \operatorname{coh}^L(X)_x$

$k(x) \longmapsto S_x$

<u>Prop 8</u>. (1) $\operatorname{mod}_0^L \mathcal{O}_{X,x} \xrightarrow{\sim} \operatorname{coh}^L(X)_x$

(2) $\operatorname{coh}_0^L(X) \xrightarrow{\sim} \coprod_{\substack{x \in X \\ \text{closed}}} \operatorname{coh}^L(X)_x$

$\Rightarrow S_x(\vec{l}),\ \vec{l} \in L$ are all the simple sheaves on X.

$\mathcal{M} \in \operatorname{coh}^L(X)$ <u>locally free of rank r</u> (<u>vector bundle</u>)
if $X = \bigcup_{i=1}^n U_i$, $\mathcal{M}|_{U_i} = \bigoplus_{j=1}^r \mathcal{O}_X|_{U_i}(\vec{l}_{ij})$

v. b. of rank 1 = <u>line bundles</u> &
$\operatorname{vect}^L(X) \subseteq \operatorname{coh}^L(X)$

<u>Fact 9</u>. Assume X has a generic point ξ ($\overline{\{\xi\}} = X$), then

(1) $\mathcal{M} \in \operatorname{vect}^L(X)$ iff $\forall x \in X$, $\mathcal{M}_x$ is graded free $\mathcal{O}_{X,x}$-module

(2) $\operatorname{vect}^L(\mathbb{X}) \subseteq \operatorname{coh}^L(\mathbb{X})$ closed under ext. and direct summands

(3) Line bundles are indec.

§2.2 $\mathbb{X}$ $\operatorname{coh}^L(\mathbb{X}) \simeq \mathcal{H}$, tilting.

<u>Recall</u>. $S = S(p,\lambda) = k[X_1, X_2, \ldots, X_n] \Big/ \left(X_i^{p_i} = X_2^{p_2} - \lambda_i X_1^{p_1},\ i \geq 3\right)$

$\mathfrak{m} = (x_1, x_2, \ldots, x_n)$

Def (G, L 1989). <u>w. p. l.</u> of type $(p, \underline{\lambda})$
15
$$\mathbb{X} = \mathbb{X}(p, \underline{\lambda}) = \mathrm{Spec}^L(S) \setminus \{m\} = \bigcup_{i=1} D(x_i)$$

<u>Remark 11</u>. (1) $\mathbb{X}$ is a noetherian scheme

(2) $\mathbb{X} = \{ \underset{\text{generic}}{0}, \underbrace{(x_1), \ldots, (x_n)}_{\text{exceptional}}, \underset{\text{ordinary}}{(f^u(x_1^{p_1}, x_2^{p_2}))} \}$

closed subsets : $\mathbb{X}$, ϕ finite sets of closed points.

$\mathbb{X} \simeq \mathbb{P}^1_k$ homeomorphism

(3) $\mathcal{O}_{\mathbb{X}, \xi}$ = fractional field of S, <u>function field of $\mathbb{X}$</u>

$\mathcal{O}_{\mathbb{X}, p} = S_p$ graded DVR ($\mathrm{gl.dim}^L = 1$)

$\rightsquigarrow$ (H1) $\mathcal{H}$ small, noetherian, hereditary

$$\widetilde{(-)} : \mathrm{Mod}^L S \xrightarrow{\sim} \mathrm{Qcoh}^L(\mathrm{Spec}^L(S)) \xrightarrow{\text{res}} \mathrm{Qcoh}^L(\mathbb{X})$$
$$M \longmapsto \tilde{M} \longmapsto \tilde{M}|_{\mathbb{X}}$$

<u>sheafification</u>

$$S(\vec{l}) \longmapsto \mathcal{O}_{\mathbb{X}}(\vec{l})$$

<u>Theorem 12</u>. $\widetilde{(-)}$ induces
$$\mathrm{mod}^L S / \mathrm{mod}_0^L S \xrightarrow{\sim} \mathcal{H}.$$

<u>Corollary 13</u>. $\mathcal{M} \in \mathcal{H}$.
$$\bigoplus \mathcal{O}_{\mathbb{X}}(\vec{l}_i) \twoheadrightarrow \mathcal{M}.$$

$\mathcal{M} \in \mathcal{H}$: $\mathrm{rk}\,\mathcal{M} = \dim_{k(x)} \mathcal{M}_\xi$ <u>rank</u>.

<u>Prop 14.</u> (1) $\mathcal{M} \in \mathcal{H}_0 = \mathrm{coh}_0^L(X) \iff \mathrm{rk}\,\mathcal{M} = 0$

$\iff \mathrm{supp}\,\mathcal{M} = \text{finite set}$

(2) $\mathcal{H}_0 = \coprod_{\substack{x \in X \\ \text{closed}}} \mathrm{mod}_0^L\, \mathcal{O}_{X,x}$

Fundamental
s.e.s. in $\mathcal{H}$

(1) $x \in X$ ordinary $x = p = (f^n(x_1^{p_1}, x_2^{p_2}))$

$\mathcal{S}_x$ only simple in x

$$0 \longrightarrow \mathcal{O}_X(-\deg(f)\vec{c}) \xrightarrow{f^n} \mathcal{O}_X \longrightarrow \mathcal{S}_x \longrightarrow 0$$

(2) $x \in X$ exceptional

$x = (x_i) = p$. $\quad k(x)(j\vec{x}_i) \quad j = 0, 1, \ldots, p_i - 1$

$\in \mathrm{mod}_0^L\, \mathcal{O}_{X,x}$

S_{ij} simple at x

$j = 0, 1, \ldots, p_i - 1$

$$0 \longrightarrow \mathcal{O}_X((j-1)\vec{x}_i) \xrightarrow{x_i} \mathcal{O}_X(j\vec{x}_i) \longrightarrow S_{i,j} \longrightarrow 0$$

<u>Prop 15</u>. (1) $\forall$ indec. object in $\mathcal{H}$ lies in $\mathcal{H}_0$ or $\mathcal{H}_+ = \mathrm{vect}^L(X)$

(2) all the line bundles are $\mathcal{O}_X(\vec{\ell})$

(3) $M \subset M^L(S) \underset{\Gamma(-)}{\overset{\widetilde{(-)}}{\rightleftarrows}} \mathrm{vect}^L(X)$

<u>Thm 18</u> (Serre duality). $\mathcal{M}, \mathcal{N} \in \mathcal{H}$

$$D\,\mathrm{Ext}^1(\mathcal{M}, \mathcal{N}) \simeq \mathrm{Hom}(\mathcal{N}, \mathcal{M}(\vec{\omega}))$$

<u>Pf</u> follows from local duality.

(H2) $\mathcal{H}$ is Hom, Ext - finite and Serre - duality.

<u>Recall</u> $\mathcal{T} \in \mathcal{H}$ is <u>tilting</u> if

(1) proj. dim $\mathcal{T} \leq 1$

(2) $\mathrm{Ext}^1(\mathcal{T}, \mathcal{T}) = 0$

(3) $\mathcal{T}$ generates $\mathcal{D}^b(\mathcal{H})$

$\rightsquigarrow \mathcal{D}^b(\mathcal{H}) \simeq \mathcal{D}^b(\mathrm{mod}\ \mathrm{End}(\mathcal{T}))$

<u>Thm 16</u> (G, L, M)

(1) $\mathcal{T} = \bigoplus_{0 \leq \vec{\ell} \leq \vec{c}} \mathcal{O}_{\mathbb{X}}(\vec{\ell})$ is tilting

(canonical tilting)

$$\mathrm{End}(\mathcal{T}) = \begin{tikzcd}[column sep=small] & \vec{x}_1 \arrow[r, "x_1"] & 2\vec{x}_1 \arrow[r] & \cdots \arrow[r] & (p_1-1)\vec{x}_1 \arrow[dr] & \\ \vec{0} \arrow[ur, "x_1"] \arrow[r] \arrow[dr, "x_n"'] & {} & & & & \vec{c} \\ & \vec{x}_n \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & (p_n-1)\vec{x}_n \arrow[ur] & \end{tikzcd}$$

$$x_i^{p_i} = x_2^{p_2} - \lambda_i \cdot x_1^{p_1} \qquad i \geq 3$$

<u>canonical algebra</u> of Ringel

(2) $\mathcal{T}' = \mathcal{O}_{\mathbb{X}} \oplus \mathcal{O}_{\mathbb{X}}(\vec{c}) \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{p_i-1} \mathcal{S}_{i,0}^{[j]}$ ← sheaf at (x_i) of length j

is a tilting sheaf.

$$\mathrm{End}(\mathcal{T}') = \begin{tikzcd}[column sep=small] & & (1,1) \arrow[r] & (1,2) \arrow[r] & \cdots \arrow[r] & (1,p_1-1) \\ 0 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \vec{c} \arrow[ur, "y_1"] \arrow[r] \arrow[dr, "y_n"'] & \cdots & & \vdots & \\ & & (n,1) \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & (n,p_n-1) \end{tikzcd}$$

$$y_1 a = 0 = y_2 b; \quad y_i(b - \lambda_i a) = 0$$

<u>squid algebra</u> of Ringel.

(H3) $\mathcal{H}$ has a tilting object.

Remark. $\mathrm{coh}^L(\mathbb{X})$ is characterized by (H1) – (H3) + ~~(H4)~~ (H4)

§2.3 Grothendieck group and CB's Thm

Recall. $\mathcal{A}$ abelian cat.

$$K_0(\mathcal{A}) = \bigoplus \mathbb{Z}\langle [M] \mid M \in \mathcal{A} \rangle \Big/ \begin{matrix}[M]+[N]-[L] \\ \text{for } 0 \to M \to L \to N \to 0\end{matrix}$$

Grothendieck group of $\mathcal{A}$

Fact (1) $K_0(\mathcal{A}) \xrightarrow{\sim} K_0(D^b(\mathcal{A}))$

(2) Λ f.d. alg. of gl. dim. $< \infty$

$$K_0(\mathrm{mod}\,\Lambda) = \bigoplus_{\substack{P \text{ ind. proj.} \\ \text{over } \Lambda}} \mathbb{Z}[P]$$

Prop. $K_0(\mathcal{H}) = \mathbb{Z}[\mathcal{O}_{\mathbb{X}}] \oplus \mathbb{Z}[\mathcal{O}_{\mathbb{X}}(\vec{c})]$
$$\bigoplus_{i=1}^{t} \bigoplus_{j=1}^{p_i-1} \mathbb{Z}[S_{ij}]$$

Pf. tilting + fund s.e.s. $\square$

Notation. $\alpha_* = [\mathcal{O}_*]$

$\delta = [\mathcal{O}_*(\vec{c})] - \alpha_*$

$\alpha_{ij} = [S_{ij}] \qquad 1 \leq j \leq p_i - 1$

Fund. s.e.s. $\Rightarrow [\mathcal{O}_*(r\vec{c})] = \alpha_* + r\delta$

$[S_x] = d(x)\cdot\delta$, $x = (f^u)$ ordinary

$d(x) = d(f)$

$$[S_{i,0}] = \delta - \sum_{j=1}^{p_i-1} \alpha_{ij}$$

$$\Rightarrow K_0(\mathcal{H}) = \mathbb{Z}\alpha_* \oplus \mathbb{Z}\delta \oplus \bigoplus \mathbb{Z}\alpha_{ij} = \hat{\Gamma}$$

$$\hat{\Gamma}_+ = \mathbb{N}(\alpha_* + \mathbb{Z}\delta) + \mathbb{N}\delta + \mathbb{N}\Big(\delta - \sum_{j=1}^{p_i-1} \alpha_{ij}\Big) + \mathbb{N}\alpha_{ij}$$

pos. cone $\qquad 1 \le i \le n, \quad 1 \le j \le p_i - 1$

<u>Fact</u> (1) $\alpha \in \hat{\Gamma}_+ \iff \alpha = [M]$, $M \in \mathcal{H}$

(2) $0 \ne \alpha = c_*\alpha_* + c_\delta\delta + \sum_{i,j} c_{ij}\alpha_{ij}$

$$\alpha \in \hat{\Gamma}_+ \iff \begin{cases} c_* > 0 \\ \text{or } c_* = 0 \quad c_\delta > 0 \\ \text{or } c_* = 0 = c_\delta \quad c_{ij} > 0 \end{cases}$$

<u>Star-shaped diagram</u>

$$\begin{array}{lllll} & (1,1) & - \cdots - & (1, p_1 - 1) & \\ * & \vdots & & & I = \{*, (i,j)\} \\ & (n,1) & - \cdots - & (n, p_n - 1) & \end{array}$$

$\Pi = \{\alpha_\nu \mid \nu \in I\}$ ~~fundamental~~ simple roots

$$\Gamma = \bigoplus_{\nu \in \Pi} \mathbb{Z}\alpha_\nu, \quad (\alpha_\nu, \alpha_\mu) = \begin{cases} 2 & \nu = \mu \\ -\#\,\mathrm{edg}\ \nu - \mu, & \nu \ne \mu \end{cases}$$

$$r_\alpha(\beta) = \beta - \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha, \quad \alpha \in \Pi, \ \beta \in \Gamma.$$

$r_\alpha \in \mathrm{Aut}(\Gamma)$

$W = \langle r_\alpha \mid \alpha \in \Pi \rangle \le \mathrm{Aut}(\Gamma)$ <u>Weyl group</u>

$\Delta^{re}(\Gamma) = \bigcup_{w \in W} w(\Pi)$ real roots of Γ.

$\alpha \in \Gamma$

$\alpha = \sum_{\nu \in I} k_\nu \alpha_\nu$ supp (α) = full diagram containing $\nu \in I$ s.t. $k_\nu \neq 0$.

Fundamental set

$$M = \{ \alpha \in \Gamma_+ \setminus \{0\} \mid (\alpha, \alpha_\nu) \leq 0, \ \text{supp}(\alpha) \text{ connected} \}$$

$$\Delta^{im}(\Gamma) = \bigcup_{w \in W} w(\pm M) \quad \text{imaginary roots}$$

$$\Delta = \Delta(\Gamma) = \Delta^{re}(\Gamma) \,\dot{\cup}\, \Delta^{im}(\Gamma)$$

$$\Delta_+ = \Delta \cap \Gamma_+$$

$$\hat{\Gamma} = \Gamma \oplus \mathbb{Z}\delta \quad (= K_0(\mathcal{H})).$$

$$\hat{\Delta} = \{ \alpha + r\delta \mid \alpha \in \Delta, \ r \in \mathbb{Z} \} \cup \{ r\delta \mid r \neq 0 \}$$

$$\hat{\Delta}^{re} = \{ \alpha + r\delta \mid \alpha \in \Delta^{re}, \ r \in \mathbb{Z} \}$$

$$\hat{\Delta}_+ = \hat{\Delta} \cap \hat{\Gamma}_+, \quad \hat{\Delta}^{im} = \hat{\Delta} \setminus \hat{\Delta}^{re}$$

<u>Kac Thm</u> (CB 2005) $\alpha \in \hat{\Gamma}$ $k = \overline{k}$

(1) $\alpha \in \hat{\Delta}_+ \iff \alpha = [M]$, $M \in \text{ind}\, \mathcal{H}$

(2) $\alpha \in \hat{\Delta}_+^{re} \iff$ such M is unique

$\alpha \in \hat{\Delta}_+^{im} \iff$ such M is infinite $\square$

Kac-Moody-Lie-Algebras I

Martin

General construction $\to \tilde{\mathfrak{g}}(A)$ factor out by r: $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/r$

$A \in \mathrm{Mat}_n(\mathbb{C}) \xrightarrow[\text{ass.}]{} \{\mathfrak{g}(A)$ complex Lie-Algebra$\}$

(a_{ij})

$\cup\!\!\mid$

$\{$Kac-Moody LA, A is a gen. Cartan Mat (GCM) i.e. $a_{ii} = 2$, $a_{ij} \leq 0$, $a_{ij} = 0 \Rightarrow a_{ji} = 0$ and $A \in \mathrm{Mat}_n(\mathbb{Z})$ $\}$

$\cup\!\!\mid$ $\qquad$ $\cup\!\!\mid$

Nico: classification $\to$ $\{$fin. dim. semisimple LA$\}$ $\qquad$ $\{$affine type Lie Algebra$\}$

Thilo will give an explicit construction in terms of a loop algebra construction.

Def. $A \in \mathrm{Mat}_n(\mathbb{C})$. A realization of A is a tuple $(\mathfrak{h}, \Pi, \Pi^\vee)$ s.t. $\mathfrak{h}$ is a compl. VSp of dim $2n - \mathrm{rk}(A)$ and lin. independent subsets $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$, $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subseteq \mathfrak{h}$

simple roots $\qquad$ simple coroots

s.t. $\alpha_i(\alpha_j^\vee) = a_{ij}$

Prop. There ex. a unique (up to isom) realization for every $A \in \mathrm{Mat}_n(\mathbb{C})$

Def. $Q_{(+)} = \sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\alpha_i$ pos. root lattice

If $\alpha = \sum_i k_i \alpha_i \in Q$ the height $\mathrm{ht}\,\alpha := \sum_i k_i$

Define $\geq$ on $\mathfrak{h}^*$: $\lambda \geq \mu \Leftrightarrow \lambda - \mu \in Q_+$.

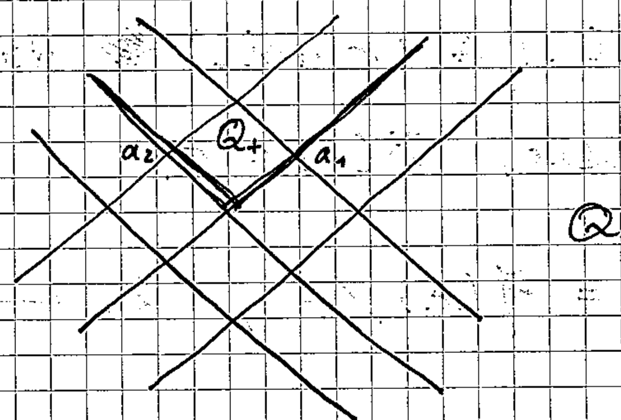
Ex. $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \rightsquigarrow sl_3$

Realization: h := space of 3×3 trace 0 diag. matrices
$\cong \mathbb{C}^2 = \mathbb{C}^{2 \cdot 2 - 2}$

$\alpha_1^\vee = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \quad \alpha_2^\vee = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad \varepsilon_i \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} := a_i$

$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3$

Root lattice



Def. $A \in \mathrm{Mat}_n(\mathbb{C})$

$$\tilde{g}(A) := \frac{\text{"free" Lie Alg } \langle e_i, f_i, h \rangle}{\text{ideal } \left\langle \begin{array}{ll} [e_i, f_j] = \delta_{ij} \alpha_i^\vee & i,j = 1,\dots,n \\ [h, h'] = 0 & \forall\, h, h' \in h \\ \left.\begin{array}{l} [h, e_i] = \alpha_i(h) e_i \\ [h, f_i] = -\alpha_i(h) f_i \end{array}\right\} & \forall i = 1,\dots,n \end{array} \right\rangle}$$

Set $\tilde{n}_+$ = Subalg. gen. by $e_i \subseteq \tilde{g}(A)$
$\tilde{n}_-$ = —— " —— $f_i \subseteq \tilde{g}(A)$

Thm. (a) $\tilde{g}(A) = \tilde{n}_- \oplus h \oplus \tilde{n}_+$
(b) $\tilde{n}_+$ is freely gen. by e_i's
$\tilde{n}_+$ —— " —— f_i's
(d) $\tilde{g}(A) = \bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \tilde{g}(A)_{-\alpha} \oplus h \oplus \bigoplus_{\substack{\alpha \in Q_+ \\ \alpha \neq 0}} \tilde{g}(A)_\alpha$

where $\tilde{\mathfrak{g}}(A)_\alpha = \{ x \in \tilde{\mathfrak{g}}(A) \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h} \}$

$[\tilde{\mathfrak{g}}(A)_\alpha, \tilde{\mathfrak{g}}(A)_\beta] \subseteq \tilde{\mathfrak{g}}(A)_{\alpha+\beta}$

and $\dim \tilde{\mathfrak{g}}(A)_\alpha \leq n^{ht\,\alpha} < \infty$

c) $\tilde{\omega} : \tilde{\mathfrak{g}}(A) \longrightarrow \tilde{\mathfrak{g}}(A)$, $\tilde{\omega}^2 = id$.

$e_i \longmapsto -f_i, \ f_i \longmapsto -e_i, \ h \longmapsto -h$

d) There ex. a unique max. ideal $\mathfrak{r}$ among those ideals J which intersect $\mathfrak{h}$ trivially.

$\mathfrak{r} = (\mathfrak{r} \cap \tilde{\mathfrak{n}}_-) \oplus (\mathfrak{r} \cap \tilde{\mathfrak{n}}_+)$ dir. sum of ideals

in particular, $\tilde{\omega}(\mathfrak{r}) = \mathfrak{r}$.

<u>Def</u>. $\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\mathfrak{r}$, $\mathfrak{h} \hookrightarrow \mathfrak{g}(A)$

<u>Consequence</u>. Every $J \subset \mathfrak{g}(A)$ which intersects $\mathfrak{h}$ trivially is 0.

In this context:

- A is Cartan-Matrix of $\mathfrak{g}(A)$
- $\operatorname{rk} \mathfrak{g}(A) := \operatorname{rk} A$
- $\mathfrak{h} \subset \mathfrak{g}(A)$ is called Cartan subalgebra (CSA)
- e_i, f_i are called Chevalley-Generators (they gen $\mathfrak{g}'(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]$ derived subalg ($= \mathfrak{g}(A) \Leftrightarrow \det A \neq 0$)).

<u>Rem./Prop</u>. • $\mathfrak{g}(A) = \mathfrak{g}'(A) + \mathfrak{h}$

• $\mathfrak{g}'(A) \cap \mathfrak{h} = \sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee = \mathfrak{h}'$

Set $\mathfrak{g}_\alpha = \{ x \in \mathfrak{g}(A) \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h} \}$

$\mathfrak{g}'(A) \cap \mathfrak{g}_\alpha = \mathfrak{g}_\alpha \quad \alpha \neq 0$

Define $\text{mult}\,\alpha := \dim \mathfrak{g}_\alpha$ multiplicity of α

$\rightsquigarrow$ $\text{mult}\,\alpha \leq \dim \tilde{\mathfrak{g}}(A)_\alpha < \infty$

Def $\alpha \in Q$ root iff $\text{mult}\,\alpha \neq 0$ and $\alpha \neq 0$.

$$\Delta = \{\text{roots}\} = \underbrace{(\Delta \cap (-Q_+))}_{\substack{=\Delta_- \\ \text{neg roots}}} \cup \underbrace{(\Delta \cap Q_+)}_{=\Delta_+}$$

Cor of Thm $\mathfrak{n}_- =$ subalg. gen. by e_i's $\in \mathfrak{g}(A)$

$\mathfrak{n}_+ =$ —— " —— f_i's $\in \mathfrak{g}(A)$

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

$$\mathfrak{n}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha \qquad \mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$$

Note $[e_{i_1}[e_{i_2} \dots [e_{i_{n-1}}, e_{i_n}] \dots]$ with $\alpha_{i_1} + \dots + \alpha_{i_n} = \alpha$

gen. $\mathfrak{g}_\alpha$ for $\alpha \in \Delta_+$. $\alpha \in \Delta_-$ take f_i - - -

therefore, $\mathfrak{g}_{\alpha_i} = \mathbb{C} e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C} f_i$, $\mathfrak{g}_{s\alpha_i} = 0$ if $|s| > 1$

Recall $\tilde{\omega}(\mathfrak{r}) = \mathfrak{r}$

$\rightsquigarrow$ $\tilde{\omega}$ induces $\omega : \mathfrak{g}(A) \to \mathfrak{g}(A)$

Chevalley involution.

So $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ $\rightsquigarrow$ $\text{mult}\,\alpha = \text{mult}\,{-\alpha}$

$\rightsquigarrow$ $\Delta_- = -\Delta_+$

Prop 1.6 (Kac's book)

The center $C = \{x \in \mathfrak{g}(A) \mid [x,y] = 0 \;\forall y \in \mathfrak{g}(A)\}$ of $\mathfrak{g}(A)$ equals $\{h \in \mathfrak{h} \mid \alpha_i(h) = 0 \;\forall i = 1, \dots, n\}$ and $\dim C = n - \text{rk}(A)$

Prop 1.7 a (Kac).

$\mathfrak{g}(A)$ is simple $\iff$ $\det A \neq 0$ and A is indecomp.

$\cancel{\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}}$

Def. $A \in Mat_n(\mathbb{C})$ is called symmetrizable if there ex. $D = diag(\varepsilon_1, .., \varepsilon_n)$ (non-degenerate) s.t. $A = DB$ with B symmetric.

Thm 9.11 (Kac)

Let A be a symmetrizable gen. Cartan matrix

"Serre relation" (i) $(ad\, e_i)^{1-a_{ij}} e_j$
(ii) $(ad\, f_i)^{1-a_{ij}} f_j$ $\Big\}$ $i \neq j$ $1, .., n$

they gen. $(r \cap \tilde{n}_+)$ and $(r \cap \tilde{n}_-)$ resp.

In part., r is gen. by i) and ii).

Kac-Moody-Lie Alg II

Nico

1. Bilinear form
2. Classification

Let A GCM $\quad a_{ii} = 2$

$a_{ij} \leq 0$

$a_{ij} = 0 \iff a_{ji} = 0$

<u>Def</u>. A bilinear form $(-,-) : \mathfrak{g}(A) \times \mathfrak{g}(A) \longrightarrow \mathbb{C}$ is called <u>invariant</u> if

$$([x,y],z) = (x,[y,z]) \quad \forall\, x,y,z$$

<u>Ex</u>. If $\mathfrak{g}(A)$ f.d. Then, the Killing form $(\mathrm{Tr}(\mathrm{ad}\,x\;\mathrm{ad}\,y))$ is invariant, sym. and non-degenerate bilinear form.

<u>Thm</u>. A GCM. There ex. an invariant sym. non-degenerate bilinear form iff A is symmetrizable

Then: 1) $(\mathfrak{g}(A)_\alpha, \mathfrak{g}(A)_\beta) = 0$ if $\alpha + \beta \neq 0$

2) $[x,y] = (x,y)\,\nu^{-1}(\alpha) \quad x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$

$\nu : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*, \; h \longmapsto (h,-)$

<u>Pf</u>. "$\Leftarrow$": A symmetrizable, $D = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_n)$, $B = (b_{ij})_{ij}$, $b_{ij} \in \mathbb{Z}$ s.t. $A = DB$.

By induction, $\quad \mathfrak{g}(A) = \sum_{N \in \mathbb{N}} \mathfrak{g}(N)$

$$\mathfrak{g}(N) = \bigoplus_{\substack{\alpha \in \Delta \cup \{0\} \\ |ht\,\alpha| \leq N}} \mathfrak{g}_\alpha$$

$\mathfrak{h}' = \sum \mathbb{C}\alpha_i^\vee$

Choose $\mathfrak{h}''$ as $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$

$(\alpha_i^\vee, h) = \alpha_i(h)\,\varepsilon_i \qquad$ Show: symmetric

$(\mathfrak{h}'', \mathfrak{h}'') = 0 \qquad$ non-degenerate

$N = 1 \quad (e_i, f_i) = \varepsilon_i$

Show : invariant.

Let $1 < ht\, \alpha =: N$, $x \in \mathfrak{g}(A)_\alpha$, $y \in \mathfrak{g}(A)_{-\alpha}$

$y = \sum_i [u_i, v_i]$, $u_i, v_i \in \mathfrak{g}(N-1) \cap n$

$$(x, y) := \sum_i ([x, u_i], v_i)$$

Show : well defined, symm, inv.

$\mathrm{Ker}(-,-)$ is an ideal of $\mathfrak{g}(A)$,

$\mathfrak{h} \cap \mathrm{Ker}(-,-) = 0 \implies \mathrm{Ker}(-,-) = 0$.

<u>Lemma</u> A symble GCM, $(-,-)$ such a bilinear form on $\mathfrak{g}(A)$

1) $\mathfrak{h}^*$: $\quad (\alpha, \beta) := (\nu^{-1}(\alpha), \nu^{-1}(\beta))$

bilinear form.

2) $\nu(\alpha_i^\vee) = \varepsilon_i \alpha_i$

3) $(\alpha_i, \alpha_j) = \dfrac{a_{ij}}{\varepsilon_i}$

4) $\alpha_i^\vee = \dfrac{2}{(\alpha_i, \alpha_j)} \nu^{-1}(\alpha_i)$

5) $a_{ij} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$

2. Classification

<u>Def</u> $v = (v_i)_i \in \mathbb{Z}^n$

$v \geq 0 \iff v_i \geq 0 \quad \forall i$

$v > 0 \iff v_i > 0 \quad \forall i$

<u>Thm</u> A GCM, indec.

Then, one and only one of the following properties holds

(Fin) 1. $\det A \neq 0$ $\qquad$ of finite type

2. $\exists u > 0 : Au > 0$

3. $\forall\, Av \geq 0 : v = 0 \vee v > 0$

(AFF) 1. $\operatorname{rk} A = n-1$ — of affine type

2. $\exists\, u > 0 : Au = 0$

3. $\forall\, Av \geq 0 : Av = 0$

(Ind) 1. $\exists\, u > 0 : Au < 0$ — of indefinite type

2. $\forall\, Av \geq 0,\ v \geq 0 : v = 0$

EX

$$\tilde{A}_{n-1} = \begin{pmatrix} 2 & -1 & & 0 & -1 \\ -1 & \ddots & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & -1 \\ -1 & 0 & & -1 & 2 \end{pmatrix}$$

$$\tilde{A}_{n-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0 \implies \tilde{A}_{n-1} \text{ of affine type}$$

Lemma. An ind. symmetric GCM A is of finite type iff positive definite, is of affine type iff pos. semidefinite.

Def. Let $S \subsetneq \{1, \dots, n\}$, $A = (a_{ij})_{ij}$ GCM. Then, the proper principal submatrix

$$A_S = (a_{ij})_{i,j \in S} \quad \text{is again a GCM}$$

Lemma. A indec GCM of finite or affine type. Then, any p.p.s.m. A_S is of finite type.

Pf. $A = \begin{bmatrix} A_S & Q \\ R & T \end{bmatrix}$ (w.l.o.g.)

$\exists\, u > 0$ s.t. $Au \geq 0$. So $u = (u', u'')$

$0 \leq A_S u' + Q u'' \leq A_S u'$ $\Rightarrow$ not of ind. type

$0 \neq Q$ $\quad A_S u' \neq 0 \Rightarrow$ not of aff. type

$\Rightarrow A_S$ of finite type.

Dynkin diagrams

<u>Def</u>. Let A GCM $\rightsquigarrow S(A)$ Dynkin diagram

vertices $\{1, \dots, n\}$

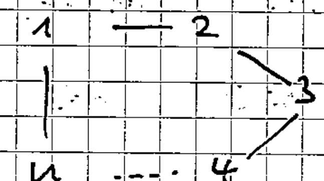
$i \quad\quad j \qquad a_{ij} = a_{j0} = 0$

$i - j \quad : \quad a_{ij} = a_{ji} = -1$

$i \xrightarrow{|a_{ji}|} j \qquad a_{ij} = -1 \quad a_{ji} < -1$

$i \Leftarrow\Rightarrow j \qquad a_{ij} = -2 = a_{ji}$

$i \xrightarrow{\;|a_{ij}||a_{ji}|\;} j \qquad$ otherwise

e.g. $S(\tilde{A}_{n-1}) =$ 

<u>Lemma</u>. Let A GCM of affine type s.t. $S(A)$ contains a cycle. Then, $A \cong \tilde{A}_{n-1}$.

<u>Cor</u>. Let A GCM of fin. or affine type. Then, A symmetrizable.

<u>Cor</u>. Let A be an ind. GCM. Then, A is of finite type $\iff$ $\det(A_S) > 0 \quad \forall S$

$\det(A) > 0$

A of affine type $\iff$ $\det(A_S) > 0$ and

$\det(A) = 0$.

<u>Thm</u>. Let A be an ind. GCM.

a) A is of finite type iff $S(A)$ is listed in (Fin) (see Appendix)

b) A is of affine type iff $S(A)$ is listed in (Aff 1 – 3)

c) The numerical labels in Tab. Aff 1 – 3 are the coordinates of a vector $u > 0$ s.t. $Au = 0$.

Pf. c) Calc. $\Rightarrow$ ($S(A)$ listed in Aff 1 – 3 ~~are listed~~ $\Rightarrow A$ of affine type.)

$\Rightarrow$ ($S(A)$ listed in (Fin) $\Rightarrow$ A of fin. type).

Appendix

TABLE Fin

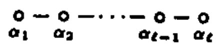
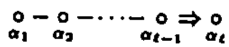
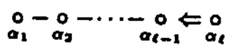
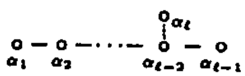
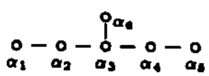
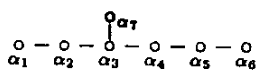
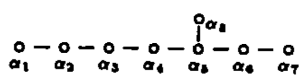
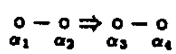
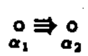
A_ℓ		$(\ell+1)$
B_ℓ		(2)
C_ℓ		(2)
D_ℓ		(4)
E_6		(3)
E_7		(2)
E_8		(1)
F_4		(1)
G_2		(1)

TABLE Aff 1

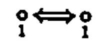
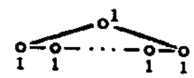
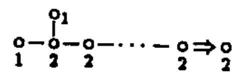
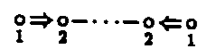
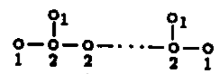
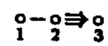
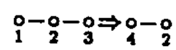
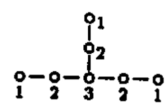
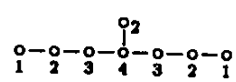
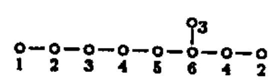
$A_1^{(1)}$	
$A_\ell^{(1)}(\ell \geq 2)$	
$B_\ell^{(1)}(\ell \geq 3)$	
$C_\ell^{(1)}(\ell \geq 2)$	
$D_\ell^{(1)}(\ell \geq 4)$	
$G_2^{(1)}$	
$F_4^{(1)}$	
$E_6^{(1)}$	
$E_7^{(1)}$	
$E_8^{(1)}$	

TABLE Aff 2

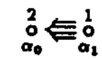
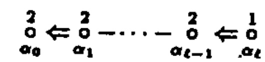
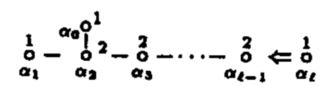
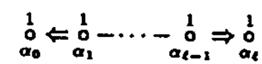
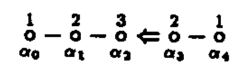
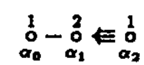
$A_2^{(2)}$	
$A_{2\ell}^{(2)}(\ell \geq 2)$	
$A_{2\ell-1}^{(2)}(\ell \geq 3)$	
$D_{\ell+1}^{(2)}(\ell \geq 2)$	
$E_6^{(2)}$	

TABLE Aff 3

$D_4^{(3)}$	

Kac-Moody and loop algebras

Thilo

<u>Plan</u>. 1) For $\mathfrak{g}$ simple, fin. dim. construct $\hat{\mathfrak{g}}$ corresponding affine as loop algebra $\mathcal{L}\mathfrak{g}$
2) Generalize the loop algebra construction for $\mathfrak{g}$ Kac-Moody.

1) $\mathcal{L}\mathfrak{g}$ is constructed in 3 steps

<u>1. Step</u>. $L := \mathbb{C}[t, t^{-1}]$. Set $L\mathfrak{g} := \mathfrak{g} \otimes_{\mathbb{C}} L$

induced Lie structure:

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m}$$

<u>2. Step</u> Construct $\tilde{L}\mathfrak{g}$ as the universal central extension of $L\mathfrak{g}$

$$0 \longrightarrow \mathbb{C}c \longrightarrow \tilde{L}\mathfrak{g} \longrightarrow L\mathfrak{g} \longrightarrow 0$$

In detail: Recall:

<u>Thm</u> $\left\{ \begin{array}{l} 1 \text{ dim central} \\ \text{extensions of } \mathfrak{g}' \end{array} \right\} \xleftrightarrow{1:1} \overline{2\text{-cocycles}}$

$\mathfrak{g}'$ Lie Alg

where • a central extension of $\mathfrak{g}'$ (through ξ) is a s.e.s. of Lie Algebras

$$0 \longrightarrow \xi \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{g}' \longrightarrow 0$$

s.t. $\xi \subseteq$ Center of $\mathfrak{a}$

This is called 1-dim if $\dim \xi = 1$

• a 2-cocycle is a bilinear $\mathbb{C}$-form ψ on $\mathfrak{g}'$

$$\begin{cases} \psi(a,b) = -\psi(b,a) \\ \psi([a,b],c) + \psi([b,c],a) + \psi([c,a],b) = 0 \end{cases}$$

$H^2(\mathfrak{g}', \xi)$

Hence, in order to construct some central extension, we need only give a 2-cocycle.

Fix $(\cdot|\cdot)$ the non-degenerate bilinear $\mathbb{C}$-form on $\mathfrak{g}$.

Extend this to $L\mathfrak{g}$

$$(a\otimes P, b\otimes Q) = (a|b)\, P\cdot Q$$

This gives a $\mathcal{L}$-form on $L\mathfrak{g}$

Set $\psi(a,b) := \cancel{\mathrm{Res}(\ldots)}\ \mathrm{Res}\left(\frac{d}{dt} a|b\right)$, $a,b \in L\mathfrak{g}$

e.g. $\psi(x\otimes t^n, y\otimes t^m) = \mathrm{Res}\left((x|y)\, n t^{n-1+m}\right)$

$$= \delta_{n,-m}\ (x|y)\, n$$

<u>Fact</u>. ψ is a 2-cocycle corresponding to

$$\eta: 0 \longrightarrow \mathbb{C}c \longrightarrow \tilde{L}\mathfrak{g} \longrightarrow L\mathfrak{g} \longrightarrow 0$$

Moreover, any other choice of a 2-cocycle yields $\lambda\psi$, $\lambda \in \mathbb{C}$.

This translates to the universal property

$$\begin{array}{ccccccccc}
\eta: & 0 & \longrightarrow & \mathbb{C}c & \longrightarrow & \tilde{L}\mathfrak{g} & \longrightarrow & L\mathfrak{g} & \longrightarrow 0 \\
 & & & \exists\downarrow & & \exists\downarrow & & \| & \\
 & 0 & \longrightarrow & \xi & \longrightarrow & \mathfrak{n} & \longrightarrow & L\mathfrak{g} & \longrightarrow 0
\end{array}$$

$$\tilde{L}\mathfrak{g} = L\mathfrak{g} \oplus \mathbb{C}c$$

Lie bracket
$$\begin{cases} c \text{ central} \\ [x\otimes t^n, y\otimes t^m] \\ \quad = [x,y]\otimes t^{n+m} \\ \quad\quad + \delta_{n,-m}\, n(x|y)\cdot c \end{cases}$$

<u>3. step</u>. Fix ∂ the derivation which acts on
$$\begin{cases} c & \text{via } 0 \\ x\otimes t^n & \text{by } t\cdot\frac{d}{dt} \end{cases}$$

The loop algebra $\mathcal{L}\mathfrak{g}$ is defined as

$$\mathcal{L}\mathfrak{g} := \tilde{L}\mathfrak{g} \oplus \mathbb{C}\partial$$

Lie bracket

$$[a+\lambda\partial, b+\mu\partial] = [a,b] + \lambda\partial(b) - \mu\partial(a)$$

Want $\mathcal{L}\mathfrak{g} \simeq \hat{\mathfrak{g}}$.
Start with identifying $\mathfrak{h} \subseteq \mathfrak{g}$ with $\mathfrak{h}\otimes 1 \subseteq \mathcal{L}\mathfrak{g}$. Then,

$\hat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}\partial$ is Cartan $\subseteq \mathcal{L}\mathfrak{g}$.

Extend $\mathfrak{h}^*$ to $\hat{\mathfrak{h}}$ via $\mathfrak{h}^*|_{\mathbb{C}c\oplus\mathbb{C}\partial} = 0$.

Then, $\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\varphi \oplus \mathbb{C}\delta$

where φ is the dual of c
δ —— " —— of ∂.

Fix e_i, f_i Chevalley generators of $\mathfrak{g}$, $1\le i\le n$.
Denote $E_i := e_i \otimes 1$, $F_i := f_i \otimes 1$.
Fix θ the ~~highest weight~~ longest root of $\mathfrak{g}$ and elements $e_\theta \in \mathfrak{g}_\theta$, $f_\theta \in \mathfrak{g}_{-\theta}$ s.t. $(e_\theta | f_\theta) = 1$.

Denote $E_0 := f_\theta \otimes t$

$F_0 := e_\theta \otimes t^{-1}$

<u>Fact</u>. The Es, Fs, $\hat{\mathfrak{h}}$ generate $\mathcal{L}\mathfrak{g}$.
Indeed, $\mathfrak{g}\otimes 1 \subseteq$ the subalg generated, denoted $\tilde{\mathcal{L}}\mathfrak{g}$.

Moreover, $\mathfrak{g}\otimes t \subseteq \tilde{\mathcal{L}}\mathfrak{g}$: $I = \{x\in\mathfrak{g} \mid x\otimes t \in \tilde{\mathcal{L}}\mathfrak{g}\}$
Then $0 \underset{\neq}{\subseteq} I \subset \mathfrak{g}$. I is an ideal:

$f_\theta \in I$

$$[\underset{I}{x}, y]\otimes t = [x\otimes t, y\otimes 1]$$
both in $\tilde{\mathcal{L}}\mathfrak{g}$.

But: $\mathfrak{g}$ simple $\Rightarrow I = \mathfrak{g}$

<u>Thm</u>. $\mathcal{L}\mathfrak{g} \simeq \hat{\mathfrak{g}}$

<u>Benefits of the construction</u>:

<u>Lemma</u>. Δ the root system of $\mathfrak{g}$
$\hat{\Delta}$ —— " —— $\hat{\mathfrak{g}}$.

Then, $\hat{\Delta} = (\Delta \oplus \mathbb{Z}\delta) \cup \mathbb{Z}^{*}\delta$

<u>Indeed</u>, $\hat{\mathfrak{g}} = \Big(\bigoplus_{\alpha \neq 0} \hat{\mathfrak{g}}_\alpha\Big) \oplus \hat{h}$

Any element in here ↑ is of the form

$x \otimes t^n$, $x \in \mathfrak{g}_\alpha$, $n \in \mathbb{Z}$, $(\alpha, n) \neq (0,0)$

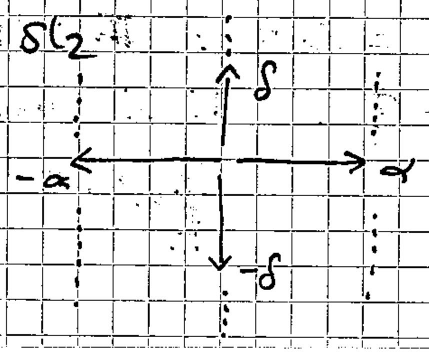
$$[\underbrace{h \otimes 1 + \lambda c + \mu d}_{y}, x \otimes t^n] = \underbrace{[h, x]}_{\alpha(h)x} \otimes t^n + \mu n (x \otimes t^n)$$

$$= (\alpha + n\delta)(y) \; x \otimes t^n$$

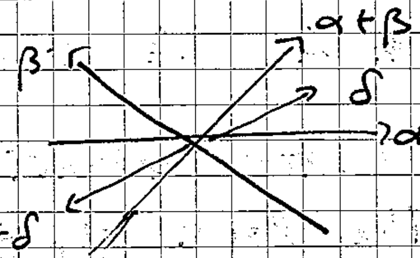
$$\Rightarrow \hat{\mathfrak{g}}_{\alpha + n\delta} = \begin{cases} \mathfrak{g}_\alpha \otimes t^n & \alpha \neq 0 \\ h \otimes t^n & \text{else and } n \neq 0 \end{cases}$$

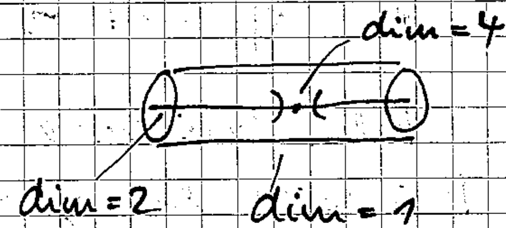
<u>Corollary</u>. $\dim \hat{\mathfrak{g}}_{\alpha + n\delta} = \begin{cases} \dim \mathfrak{g}_\alpha = 1, & \alpha \neq 0 \\ \dim h & \cancel{\text{else}}\ \alpha = 0,\ n \neq 0 \\ \dim \hat{h} & \text{else} \end{cases}$

<u>Example</u>. $\mathfrak{g} = sl_2$



$\mathfrak{g} = sl_3$





2) <u>Motivation</u>. $\tilde{\mathcal{L}}\mathfrak{g}$ is natural.

∂ accounts for $\dim \tilde{\mathfrak{h}} = 2n - \mathrm{rk}\, A$

$$\tilde{\mathcal{L}}\mathfrak{g} = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$$

Proposition. [Garland] *Let $\mathfrak{g} = \mathfrak{g}(A)$ be a simple fin. dim. Lie algebra, I the index set of the simple roots of $\mathfrak{g}$. Then $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$ is isomorphic to the Lie algebra with generators:*

$c,\ e_{ik},\ f_{ik},\ h_{ik}\quad i \in I, k \in \mathbb{Z}$

relations:

- c central
- $[e_{vr}, e_{vs}] = 0 = [f_{vr}, f_{vs}]$
- $[e_{ur}, f_{vs}] = \delta_{u,v}(h_{v,r+s} + r\delta_{r+s,0}\, c)$
- $[h_{ur}, e_{vs}] = a_{uv} e_{v,r+s}$
- $[h_{ur}, f_{vs}] = -a_{uv} f_{v,r+s}$
- $[h_{ur}, h_{vs}] = r a_{uv} \delta_{r+s,0}\, c$
- $(\mathrm{ad} e_{u0})^{1-a_{uv}}(e_{v,s}) = 0 = (\mathrm{ad} f_{u0})^{1-a_{uv}}(f_{v,s})$ if $u \neq v$

Definition. For a general Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$, we take this as the definition of the loop algebra $\mathcal{L}\mathfrak{g}$.

What is $\mathcal{L}\mathfrak{g}$ in general? $L\mathfrak{g} \oplus \mathbb{C}c$

Always have $\mathcal{L}\mathfrak{g} \longrightarrow \cancel{L\mathfrak{g} \otimes \mathbb{C}A} \longrightarrow 0$

$$c \longmapsto c$$

What if $\mathfrak{g}$ is simple and look at $\mathcal{L}\mathcal{L}\mathfrak{g}$

Moody, Rao, Yokonuma: its again a universal central extension. Explicitely,

$$\mathcal{L}\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[s^{\pm}, t^{\pm}] \oplus K$$

$$K = \Omega^1 \mathbb{C}[s^{\pm}, t^{\pm}] / d\,\mathbb{C}[s^{\pm}, t^{\pm}]$$

Different approach to generalize $\mathcal{L}\mathfrak{g}$: Instead of

$$\mathfrak{g} \rightsquigarrow \cancel{\mathfrak{g}} \quad \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$$

do $\mathfrak{g} \rightsquigarrow \mathfrak{g} \otimes A$, A comm. $\mathbb{C}$ alg

take universal central extension.

This will be of the form $\mathfrak{g}_A \oplus \Omega^1 A/dA$

Triangulated and Derived categories. Thorsten

1. Triangulated categories

Let $\mathcal{C}$ ~~abelian~~ additive category with a translation functor T.

<u>Definition</u>. A sextuple in $\mathcal{C}$ is given by $X, Y, Z \in \mathcal{C}$ and morphisms

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

$$\begin{tikzcd} & Z \arrow[dl, "w"'] & \\ X \arrow[rr, "u"'] & & Y \arrow[ul, "v"'] \end{tikzcd}$$

A morphism is given by a triple of morphisms (f, g, h) such that

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "v"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r, "w"] \arrow[d, "h"] \arrow[dr, phantom, "\circlearrowright"] & TX \arrow[d, "Tf"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & TX' \end{tikzcd}$$

A set of sextuples is called a triangulation $\mathcal{T}$ of $\mathcal{C}$ (elements are called triangles) if

(TR 1) • Every sextuple isomorphic to a triangle is a triangle

• Every morphism $u: X \to Y$ can be embedded into (X, Y, Z, u, v, w)

• $(X, X, 0, 1_X, 0, 0) \in \mathcal{T}$

(TR2) $(X, Y, Z, u, v, w) \in \mathcal{T}$
$\Rightarrow (Y, Z, TX, v, w, -Tu) \in \mathcal{T}$

(TR3) Let $(X_i, Y_i, Z_i, u_i, v_i, w_i) =: T_i \in \mathcal{T}$ $i = 1,$
and $f: X_1 \to X_2$, $g: Y_1 \to Y_2$
such that $u_2 \circ f = g \circ u_1$. Then, there exists $(f, g, h): T_1 \to T_2$.

$$
\begin{tikzcd}
(TR4) & T^{-1}Y' \arrow[r] \arrow[d, "T^{-1}g"'] & X \arrow[r, equal] \arrow[d, "u"] & X \arrow[d, "v\circ u"] & & \\
& T^{-1}X' \arrow[r] & Y \arrow[r, "v"] \arrow[d, "i"] & Z \arrow[r, "j"] \arrow[d, "k"] & X' \arrow[r, "j'"] \arrow[d, equal] & TY \arrow[d, "Ti"] \\
& & Z' \arrow[r, dashed, "f"] \arrow[d, "i'"] & Y' \arrow[r, dashed, "g"] \arrow[d, "k'"] & X' \arrow[r, "Ti\circ j'"] & TZ' \rightsquigarrow \text{triangle} \\
& & TX \arrow[r, equal] & TX & &
\end{tikzcd}
$$

The triple $(\mathcal{C}, T, \mathcal{T})$ is called a triangulated category.

An additive functor $H : \mathcal{C} \to \mathcal{A}$, $\mathcal{A}$ abelian is called cohomological if $\forall (X, Y, Z, u, v, w) \in \mathcal{T}$

$$\cdots \to H(T^i X) \xrightarrow{H(T^i u)} H(T^i Y) \to H(T^i Z) \to H(T^{i+1} X) \to \cdots$$

is exact.

2) <u>Derived categories</u>

$\mathcal{A}$ ~~additive~~ abelian category. A complex over $\mathcal{A}$ is given by

$$M = \cdots M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} M^{i+2} \to \cdots \quad i \in \mathbb{Z}$$

s.t. $d^{i+1} \circ d^i = 0$. A complex is bounded if $X^i = X^{-i} = 0$ for all $i \geq n \in \mathbb{N}$.

Define the ith homology of M by

$$H^i(M) = \ker d^i / \operatorname{Im} d^{i-1}.$$

A morphism $f^\bullet : M^\bullet \to N^\bullet$ is called a quasi-isomorphism if it induces isomorphisms in homology.

Two morphisms $f, g : M \to N$ are called homotopic if $\exists\ k^i : M^i \to N^{i-1}$ such that $f^i - g^i = k^{i+1} \circ d_M^i + d_N^{i-1} \circ k^i$.

Then, define $\mathcal{K}(\mathcal{A}) = C(\mathcal{A})/\sim$

The derived category is the localization of the category of complexes with respect to the class of quasi-isomorphisms.

Take objects to be chain complexes and morphisms $\mathrm{Mor}(L^\bullet, M^\bullet)$ are given by

$$\begin{tikzcd} & M'^\bullet & \\ L^\bullet \arrow[ur, "f"] & & M^\bullet \arrow[ul, "s"'] \end{tikzcd}$$

where s is an quasi-isom.

and $(f, s) \sim (f', s')$ if

$$\begin{tikzcd} & M' \arrow[d] & \\ L \arrow[ur, "f"] \arrow[r, "f''"] \arrow[dr, "f'"'] & M''' & M \arrow[ul, "s"'] \arrow[l, "s''"'] \arrow[dl, "s'"] \\ & M'' \arrow[u] & \end{tikzcd}$$

$\rightsquigarrow$ derived category $\mathcal{D}(\mathcal{A})$

<u>Thm</u>. $\mathcal{D}^b(\mathcal{A})$ is a triangulated category with translation functor $T : X \to X[1]$ ($X[1]_n = X_{n+1}$ with differential $-d$).

<u>Lemma</u>. Let $M \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}$ hereditary, i.e. $\mathrm{Ext}^2(A, B) = 0\ \forall\ A, B \in \mathcal{A}$.

Then, we have $M \cong \bigoplus_{n \in \mathbb{Z}} H^n(M)[-n]$

<u>Pf</u>. Let $Z^n = \ker d^n$ and consider

$$0 \to Z^{n-1} \longrightarrow M^{n-1} \xrightarrow{\delta} Z^n \longrightarrow H^n \longrightarrow 0$$

(δ induced by d). Since $\mathcal{A}$ is hereditary, we have $M^{n-1} \xrightarrow{\varepsilon} E^n \xrightarrow{\xi} Z^n$ where ε is mono, ξ epi s.t. $Z^{n-1} = \ker \xi$ and $H^n = \operatorname{coker} \varepsilon \Rightarrow H^n[-n] \cong S_n$ where $S^n = \ldots \to 0 \longrightarrow M^{n-1} \xrightarrow{\varepsilon} E^n \longrightarrow 0$

$$\Rightarrow M \xleftarrow{\sim} \bigoplus_{n \in \mathbb{Z}} S_n \xrightarrow{\sim} \bigoplus_{n \in \mathbb{Z}} H^n(M)[-n]$$

$\square$

3) Applications.

$\mathcal{C}$ a triangulated category such that $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ are f.d. vector spaces. A triangle
$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is called
AR triangle if

(AR1) X, Z are indec.

(AR2) $w \neq 0$

(AR3) $f : W \to Z$ no retraction, then there exists $f' : W \to Y$ with $v \circ f' = f$

Thm i) Let $\mathcal{A}$ be a f.d. k-alg with finite gl. dim $\Rightarrow D^b(\mathcal{A})$ has AR-triangles.

ii) ~~Iff~~ Let $\mathcal{A}$ be a Krull-Schmidt-category. If $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is a AR-sequence, then t.f.a.e.

1) $X \to Y \to Z \to TX$ is a AR triangle

2) inj dim, proj dim ≤ 1

Recall that the AR quiver of a k-linear category is defined by the vertices $[X]$, X indecomposable, and

$$\#\{\alpha : [X] \longrightarrow [Y]\} = \dim \operatorname{Irr}(X,Y) = \dim \operatorname{rad}(X,Y)/\operatorname{rad}^2(X,Y)$$

<u>Fact</u>. If Γ is the AR quiver of $k\vec{\Delta}$ ($k\vec{\Delta}$ f.d. alg (hereditary)), we get the AR-quiver of $\mathcal{D}^b(k\vec{\Delta})$ as follows

$$\mathcal{D}^b(k\vec{\Delta})_0 = \coprod_{i\in\mathbb{Z}} (\Gamma_i)_0 \qquad \Gamma_i = \Gamma$$

$$\mathcal{D}^b(k\vec{\Delta})_1 = \coprod_{i\in\mathbb{Z}} (\Gamma_i)_1 \cup \{\tilde{\alpha} : I(a) \to P(b)[1] \mid \alpha : a \to b\}$$

Let $\mathbb{C}$ be a weighted proj. line.
Recall that $T = \bigoplus_{0\le\vec{x}\le\vec{c}} \mathcal{O}_{\mathbb{C}}(\vec{x})$ is a tilting sheaf ($\mathbb{C} = \mathbb{C}(p,\lambda)$).
We get functors $\overset{F}{\overbrace{\operatorname{Hom}(T,-)}} : \operatorname{Qcoh}(\mathbb{C}) \to \operatorname{Mod}(\Lambda)$
$\Lambda = \operatorname{End} T$, and $\underset{G}{\underbrace{-\otimes_\Lambda T}} : \operatorname{Mod}\Lambda \to \operatorname{Qcoh}(\mathbb{C})$

<u>Thm</u> (Lenzing, Geigle). F and G induce equivalences of triangulated categories

$$R^*F : \mathcal{D}^b(\operatorname{coh}(\mathbb{C})) \longrightarrow \mathcal{D}^b(\operatorname{mod}\Lambda)$$

$$L^*G : \mathcal{D}^b(\operatorname{mod}\Lambda) \longrightarrow \mathcal{D}^b(\operatorname{coh}(\mathbb{C}))$$

Let $\mathbb{C} = \mathbb{P}^1$, then $T = \mathcal{O}(-1) \oplus \mathcal{O}$. We have

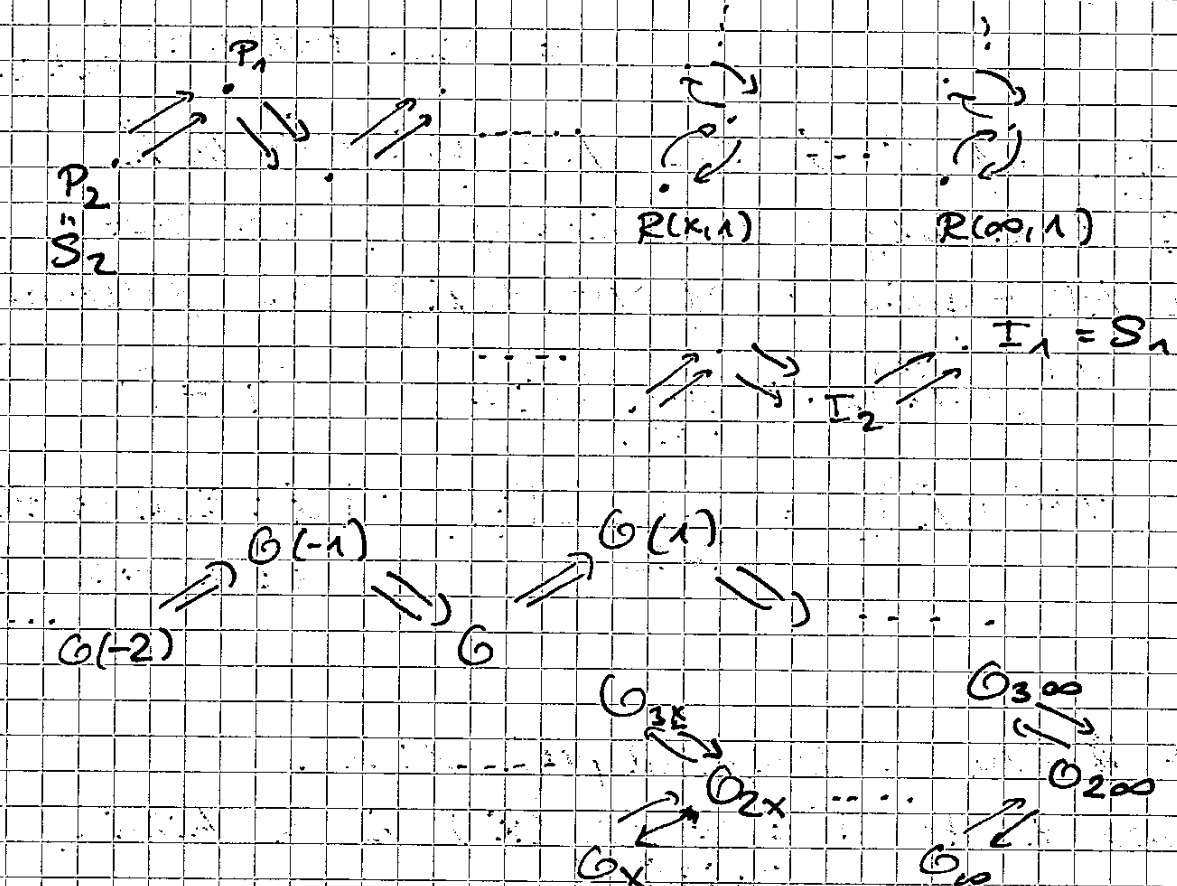
$$\operatorname{Hom}(\mathcal{O}(-1), \mathcal{O}) = S^1(V)$$

$$\operatorname{Hom}(\mathcal{O}, \mathcal{O}) = \operatorname{Hom}(\mathcal{O}(-1), \mathcal{O}(-1)) = S^0(V)$$

 $\operatorname{Hom}(\mathcal{O}, \mathcal{O}(-1)) = 0$

where $V = \Gamma(\mathbb{P}^1, \mathcal{O}(1))$ and $S^\circ(V)$ symm. alg.

$\Rightarrow \; \mathrm{End}(T) = \bullet \rightrightarrows \bullet$



Julia 10

Kac's theorem (for quivers)

"correspondence" between dimension vectors of indecomposable rep's of quivers and the positive roots of Kac-Moody algebras with symmetric Cartan matrices

(5 Min)

Kac-Moody algebra (of $A = {}^tA$)

$\Gamma := \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$

$\mathfrak{g} := \mathfrak{g}(A) = \bigoplus_{\alpha\in\Gamma} \mathfrak{g}_\alpha$ complex Γ-graded Lie algebra defined by

1. $I \subset \mathfrak{g}$ Γ-graded ideal, $I \cap \mathfrak{g}_0 = 0 \Rightarrow I = 0$
2. $\mathfrak{g}$ generated by e_i, f_i, h_i, $1 \le i \le n$:
   $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$, $\mathfrak{g}_0 = \bigoplus_{i=1}^{n} \mathbb{C}h_i$
   $[h_i, h_j] = 0$, $[e_i, f_j] = \delta_{ij} h_i$
   $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$

$\longleftarrow$

symmetric generalized Cartan matrices

$A = {}^tA \in M(n; \mathbb{Z})$ $\quad (n \in \mathbb{N})$

with $a_{ii} = 2$, $a_{ij} \in -\mathbb{N}_0$ $\forall i \neq j$

$\longrightarrow$

$\longleftarrow$ - - - (*)

finite graphs without edge loops

$S := S(A) = (S_0, S_1)$ "Dynkin diagram of A"

$S_0 = \{p_1, \ldots, p_n\}$

S_1: $p_i \underbrace{\overset{\vdots}{\text{———}}}_{-a_{ij} \text{ edges}} p_j$ $\quad \forall i \neq j$

(*) rem: Up to the following equivalence relation it is a bijection:

$A \sim B :\Longleftrightarrow \exists$ permutation matrix $P \in GL_n(\mathbb{Z})$: $A = PBP^{-1}$

---

root data of S (or A or $\mathfrak{g}$):

$\Gamma = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i \supset \Gamma_+ := \bigoplus_{i=1}^{n} \mathbb{N}_0 \alpha_i$

$\cup$

$\Delta := \{\alpha \in \Gamma \mid \mathfrak{g}_\alpha \neq 0\}$ — roots

$\cup$

$\Delta_+ := \Delta \cap \Gamma_+$ — positive roots

$\cup$

$\Pi := \{\alpha_1, \ldots, \alpha_n\}$ — simple roots

$r_i \in \mathrm{Aut}(\Gamma)$ via $r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$

$W := \langle r_i \mid 1 \le i \le n \rangle$ — Weyl group

$\Delta^{re} := W\Pi = \bigcup_{i=1}^{n} W\alpha_i$ — real roots

$\Delta^{im} := \Delta \setminus \Delta^{re}$ — imaginary roots

---

Fix: $\left\{\begin{array}{l} Q \text{ finite quiver with underlying graph } S = S(A) \text{ with } A = {}^tA \text{ gen. Cartan matrix} \\ \mathbb{F} = \overline{\mathbb{F}} \text{ field (algebraically closed!)} \\ \alpha = \sum_{i=1}^{n} k_i\alpha_i \in \Gamma \end{array}\right.$

Set: $\mathcal{M}^\alpha(Q) := \bigoplus_{\substack{\alpha: i \to j \\ \text{arrow in } Q}} \mathrm{Hom}(\mathbb{F}^{k_i}, \mathbb{F}^{k_j}) \supset \mathcal{M}^\alpha_{ind}(Q) = \{M \in \mathcal{M}^\alpha(Q) \mid M \text{ indec}\}$

$\circlearrowleft$

$G^\alpha := \prod_{i=1}^{n} Gl_{k_i}(\mathbb{F})$

$\mu_\alpha(Q) := \max\{\dim Z \mid \exists\, Y \subset \mathcal{M}^\alpha_{ind}(Q)$ G^α-invariant with Z is the geom. quotient $Y /\!\!/ G^\alpha\}$

$\mathbb{M}_{ind}(Q) := \bigcup_{\alpha\in\Gamma} \{G^\alpha\text{-orbits in } \mathcal{M}^\alpha_{ind}(Q)\} = \{$isomclasses of indec. rep's of $Q\}$

Kac's thm: Q, $\mathbb{F} = \overline{\mathbb{F}}$, α (see left)

Consider $\underline{\dim}: \mathbb{M}_{ind}(Q) \longrightarrow \Gamma$

$\Longrightarrow$

1. $\underline{\dim}(\mathbb{M}_{ind}(Q)) = \Delta_+(S)$
2. $\underline{\dim}\big|_{\underline{\dim}^{-1}(\Delta^{re} \cap \Delta_+)}$ injective
3. $\alpha \in \Delta^{im} \cap \Delta_+ \Longrightarrow \mu_\alpha(Q) = 1 - (\alpha, \alpha) > 0$
   $[\Rightarrow \# \underline{\dim}^{-1}(\alpha) = \infty]$

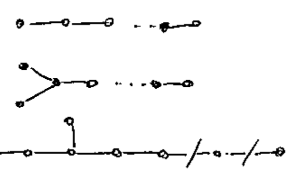
$S=(S_0,S_1)$ connected graph, $\#S_0, \#S_1 < \infty$

a) S finite

$\cancel{\Leftrightarrow}$ $\#W < \infty$ $\cancel{\Leftrightarrow}$ $\#\Delta < \infty$ $\cancel{\Leftrightarrow}$ $\Delta^{im} = \emptyset$

$\Longleftrightarrow$ S is one of the following list (subscript $= n°$ of vertices)

$A_n (n \geq 1)$, $D_n (n \geq 4)$, $E_{6/7/8}$



rem: S without edge-loops $\Rightarrow$ $S = S(A)$ with $A = {}^tA$ Cartan matrix and:

S finite $\Longleftrightarrow$ $\dim_{\mathbb{C}} \mathfrak{g}(A) < \infty$

$$\boxed{\Delta_+ = \Delta_+^{re} = \{\alpha \in \mathbb{N}_0^n = \Gamma_+ \mid (\alpha,\alpha) = 1\}}$$

$\boxed{S = A_n}$ $1 \; 2 \; \dots \; n$ $\quad (\alpha_i, \alpha_j)_{ij} = \begin{pmatrix} 1 & -\frac12 & & \\ -\frac12 & \ddots & \ddots & \\ & \ddots & \ddots & -\frac12 \\ & & -\frac12 & 1 \end{pmatrix}$

$\Delta_+ = \{\alpha_i + \alpha_{i+1} + \dots + \alpha_j \mid 1 \leq i \leq j \leq n\}$, $\# = \frac{n(n+1)}{2}$

$W \cong S_{n+1}$ $(r_{\alpha_i} \mapsto (i, i+1))$

check for example: $W(A_2) = \langle \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \rangle \cong S_3$

$\boxed{S = D_n}$ $n-1, n, n-2, \dots, 2, 1$ $\quad (\alpha_i, \alpha_j)_{ij} = \begin{pmatrix} 1 & -\frac12 & & & & \\ -\frac12 & \ddots & \ddots & & & \\ & \ddots & 1 & -\frac12 & -\frac12 \\ & & -\frac12 & 1 & 0 \\ & & -\frac12 & 0 & 1 \end{pmatrix}$

$\Delta_+ = \{\alpha = \sum k_i \alpha_i \mid k_i \in \{0,1\},\ \mathrm{supp}\,\alpha \text{ connected}\}$

$\cup \left\{ {}^{1}_{1} 2 2 \dots \underset{i}{2} 1 \dots \underset{j}{1} 0 \dots 0 \;\middle|\; 1 \leq j < i \leq n-2 \right\}$

$\# = n^2 - n$

$W \cong$ semidirect product of S_n and $\langle (\text{diag}(1,\dots,-1,-1,\dots,1)) \mid 1 \leq i \leq n-1 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^{n-1}$

$\boxed{S = E_{6/7/8}}$ $\quad \#\Delta_+(E_6) = 36$

$\#\Delta_+(E_7) = 64$

for a complete list see: Bourbaki: Groupes et algè-

b) S tame

$\cancel{\Leftrightarrow}$ $\exists \delta \in \Gamma \setminus \{0\}: \Delta^{im} = \mathbb{Z}\setminus\{0\} \cdot \delta$

$= \ker(-,-) \setminus \{0\}$

$\Longleftrightarrow$ S is one of the following list (subscript $+1 = n°$ of vertices)

Vertices are labelled by $a_i \in \mathbb{N}_0$ s.t. $\delta = \sum_{i=1}^{n+1} a_i \alpha_i$; $a_{n+1} = 1$

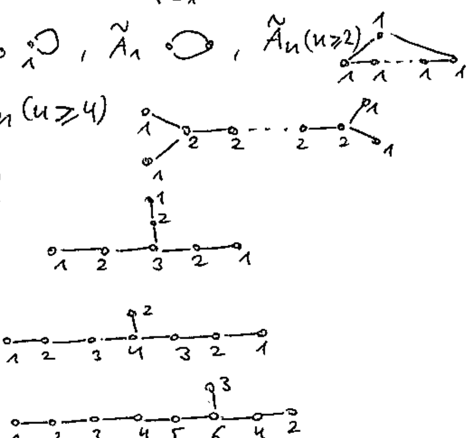
$\tilde{A}_0$, $\tilde{A}_1$, $\tilde{A}_n (n \geq 2)$

$\tilde{D}_n (n \geq 4)$

$\tilde{E}_6$

$\tilde{E}_7$

$\tilde{E}_8$



$$\boxed{\begin{aligned} \Delta_+^{re} &= \{\alpha \in \Gamma_+ \mid (\alpha,\alpha) = 1\} \\ \Delta_+^{im} &= \{\alpha \in \Gamma_+ \mid (\alpha,\alpha) \leq 0\} \end{aligned}}$$

Let $\tilde{X}_n \in \{\tilde{A}_n, \tilde{D}_n, \tilde{E}_{6/7/8}\}$, fix $X_n \subset \tilde{X}_n$

$\Longrightarrow \Delta_+(X_n) \subset \Delta_+^{re}(\tilde{X}_n)$ and

$\Delta_+^{re}(\tilde{X}_n) = \{\alpha + m\delta \mid \alpha \in \pm\Delta_+(X_n), m \geq 1\}$

$\cup \; \Delta_+(X_n)$

$W(\tilde{X}_n) \cong \mathbb{Z}^n \rtimes W(X_n)$ is an infinite group

c) S wild

$\Longleftrightarrow \exists \alpha \in \Delta_+, \mathrm{supp}(\alpha) = S: (\alpha, \alpha_i) < 0 \;\forall i$

$\cup$

S hyperbolic $:\Longleftrightarrow$ S wild and every proper connected subgraph is of finite or tame type

$\Longleftrightarrow$ S is one of the following list

$n=2$: ≥ 3 edges; $n=3$; $n \geq 1$

$n=4$; $n=5$

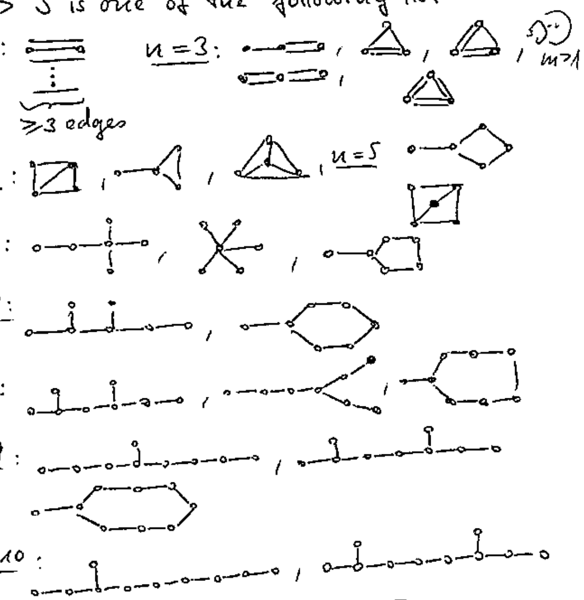
$n=6$

$n=7$

$n=8$

$n=9$

$n=10$



$$\boxed{\begin{aligned} \Delta_+^{re} &= \{\alpha \in \Gamma_+ \mid (\alpha,\alpha) = 1\} \\ \Delta_+^{im} &= \{\alpha \in \Gamma_+ \mid (\alpha,\alpha) \leq 0\} \end{aligned}}$$

$\exists$ complete list of $\Delta_+(S)$, S hyperbolic

see: ?

$\#W = \infty$

Examples by hand

$$\tilde{A}_2 : \begin{pmatrix} 1 & -\frac12 & -\frac12 \\ -\frac12 & 1 & -\frac12 \\ -\frac12 & -\frac12 & 1 \end{pmatrix} \underset{\substack{\text{"simultaneous"} \\ \text{Gauß-alg.} \\ \text{(for sym. matrices)}}}{=} \begin{pmatrix} 1 & 0 & 0 \\ -\frac12 & 1 & 0 \\ -\frac12 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \frac34 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac12 & -\frac12 \\ & 1 & -1 \\ & & 1 \end{pmatrix}$$

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \in \mathbb{N}_0^3 \setminus 0 : \quad (k_1 - \tfrac12 k_2 - \tfrac12 k_3)^2 + \tfrac34 (k_2 - k_3)^2 = \begin{cases} 1 & \text{for } \Delta_+^{re} \\ 0 & \text{for } \Delta_+^{im} \end{cases}$$

$$\Rightarrow \Delta_+^{im} = \mathbb{N} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \leftarrow \delta := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Delta_+^{re} : \quad (2k_1 - k_2 - k_3)^2 + 3\underbrace{(k_2 - k_3)^2}_{\in \{0,1\}} = 4 \Rightarrow \begin{cases} a)\ k_2 - k_3 = 0 \\ \text{or } b)\ k_2 - k_3 = 1 \\ \text{or } c)\ k_2 - k_3 = -1 \end{cases}$$

a) $k_2 = k_3 \Rightarrow (2k_1 - 2k_2) = \pm 2 \Rightarrow k_1 = \pm 1 + k_2 \rightsquigarrow \begin{pmatrix} k+1 \\ k \\ k \end{pmatrix}, \begin{pmatrix} k-1 \\ k \\ k \end{pmatrix}, k \in \mathbb{N}_0$

b) $k_2 - k_3 = 1 \Rightarrow (2k_1 + 1 - 2k_2) = \pm 1$

$\Rightarrow k_1 = \frac12(\pm 1 - 1 + 2k_2) \rightsquigarrow \begin{pmatrix} k \\ k \\ k-1 \end{pmatrix}, \begin{pmatrix} k-1 \\ k \\ k-1 \end{pmatrix}, k \in \mathbb{N}_0$

c) $k_2 - k_3 = -1 \Rightarrow (2k_1 - 1 - 2k_2) = \pm 1$

$\Rightarrow k_1 = \frac12(\pm 1 + 1 + 2k_2) \rightsquigarrow \begin{pmatrix} k \\ k \\ k+1 \end{pmatrix}, \begin{pmatrix} k+1 \\ k \\ k+1 \end{pmatrix}, k \in \mathbb{N}_0$

for $\Delta_+(A_2) := \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ we reobtain (cp. formular on list)

$$\Delta_+^{re}(\tilde{A}_2) = \{ \alpha + m\delta \mid \alpha \in \pm\Delta_+(A_2), m \geq 1 \} \cup \Delta_+(A_2)$$

$\circ\!\equiv\!\circ \quad \begin{pmatrix} 1 & -\frac32 \\ -\frac32 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac32 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -\frac54 \end{pmatrix} \begin{pmatrix} 1 & -\frac32 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \mathbb{N}_0^2 \setminus 0 : \quad (k_1 - \tfrac32 k_2)^2 - \tfrac54 k_2^2 \begin{cases} = 1 & \text{for } \Delta_+^{re} \\ \leq 0 & \text{for } \Delta_+^{im} \end{cases}$$

$W = \left\langle \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} \right\rangle \quad r_1 r_2 = \begin{pmatrix} 8 & -3 \\ 3 & -1 \end{pmatrix}, \ r_2 r_1 = \begin{pmatrix} -1 & 3 \\ -3 & 8 \end{pmatrix}, \ r_2 r_1 r_2 = \begin{pmatrix} 8 & -3 \\ 21 & -8 \end{pmatrix}, \dots$

$$\Delta_+^{re} = W \cdot \{e_1, e_2\} \cap \mathbb{N}_0^2 = \left\{ e_1, e_2, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 8 \\ 21 \end{pmatrix}, \dots \right\}$$

find recursive formular

$$= \left\{ \begin{pmatrix} c_j \\ c_{j+1} \end{pmatrix}, \begin{pmatrix} c_{j+1} \\ c_j \end{pmatrix} \,\middle|\, \begin{array}{l} c_0 = 0,\ c_1 = 1 \\ c_{j+2} = 3c_{j+1} - c_j \end{array} \right\}$$

$$\Delta_+^{im} : (2k_1 - 3k_2)^2 \leq 5k_2^2 \overset{k_2 \geq 0}{\iff} |2k_1 - 3k_2| \leq \sqrt5 k_2$$

$$\iff (2k_1 \leq 3k_2) \vee (3k_2 < 2k_1 \leq (\sqrt5 + 3)k_2)$$

$$\iff 2k_1 \leq (\sqrt5 + 3)k_2$$

$$\Delta_+^{im} = \left\{ \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \mathbb{N}_0^2 \setminus 0 \,\middle|\, 2k_1 \leq (\sqrt5 + 3)k_2 \right\}$$

How to calculate $m_\alpha = \dim \mathfrak{g}(A)_\alpha$?

<u>Kac '74</u> : $\underset{S(A)}{\overset{S}{\shortparallel}}$ connected ~~quiver~~ graph with edge loops. , $\# S_0 = n$

$\Gamma = \mathbb{Z}^n \circlearrowleft W$

extend W-action to $\mathbb{Z}^n \oplus \mathbb{Z}\rho$ via

$r_{\alpha_i}(\rho) := \rho - \alpha_i \qquad \Longrightarrow \rho - w(\rho) \in \Gamma_+ \setminus \{0\} \quad \forall w \in W$

for $\alpha = \sum k_i \alpha_i \in \Gamma_+ : X^\alpha := X^{k_1} \cdots X^{k_n}$

$$\Longrightarrow \sum_{w \in W} \det(w) X^{\rho - w(\rho)} = \prod_{\alpha \in \Gamma_+} (1 - X^\alpha)^{m_\alpha} \in \mathbb{Z}[\![X_1, \ldots, X_n]\!]$$

<u>example</u> : $S = A_2$, $W = S_3 = \{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \underset{r_1}{\begin{pmatrix} -1 & 1 \\ & 1 \end{pmatrix}}, \underset{r_2}{\begin{pmatrix} 1 & \\ 1 & -1 \end{pmatrix}}, \underset{r_1 r_2}{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}}, \underset{r_2 r_1}{\ldots} \quad \underset{r_1 r_2 r_1}{\}}$

$$1 - X_1 - X_2 + \underbrace{X_1 X_2^2}_{\substack{\shortparallel \\ X^{\rho - r_2 r_1(\rho)}}} + X_2 X_1^2 - X_1^2 X_2^2 = (1 - X_1)(1 - X_2)(1 - X_1 X_2)$$

<u>in general</u>: $m_\alpha = m_{w(\alpha)} \Longrightarrow m_\alpha = 1 \;\forall \alpha \in \Delta^{re}$

$*$ for $\tilde{X}_n \in \{\tilde{A}_n, \tilde{D}_n, \tilde{E}_n\}$, $\alpha \in \Delta^{im}(\tilde{X}_n)$: $m_\alpha = n$

$\alpha \in \Delta$: $\circ\ (\alpha, \alpha) = 0 \Longrightarrow \exists! \beta \in W\alpha$: $\mathrm{supp}(\beta)$ tame

$m_\alpha = m_\beta = \left(\#\,\text{ed vertices of } \mathrm{supp}(\beta)\right) - 1$

$\circ\ (\alpha, \alpha) < 0 \Longrightarrow m_{k\alpha}$ grows exponentially for $k \to \infty$

Kac proved his theorem even for graphs with edge loops.
Therefore we need to redefine: $\Pi, W, \Delta^{re}, \Delta^{im}, \Delta, \dots$

1. graphs and their root data

fix: $S = (S_0, S_1)$ connected graph, $S_0 = \{1, \dots, n\}$, $\# S_1 < \infty$

Def.: $\Gamma := \mathbb{Z}^n$ with standard basis $\alpha_1, \dots, \alpha_n$

$\cup$

$\Gamma_+ := \mathbb{N}_0^n$

$(\,,\,) : \Gamma \times \Gamma \longrightarrow \frac{1}{2}\mathbb{Z}$ bilinear

$(\alpha_i, \alpha_j) := \delta_{ij} - \frac{1}{2} b_{ij}$; $b_{ij} := \begin{cases} \text{no. of edges } i - j \text{ (number)}, & i \neq j \\ 2 \cdot (\text{no. of loops at } i), & i = j \end{cases}$

$\Pi := \{\alpha_i \mid \nexists \text{ loop at } i\} = \{\alpha_i \mid (\alpha_i, \alpha_i) = 1\}$ fundamental roots

$\alpha \in \Pi \Rightarrow r_\alpha \in \mathrm{Aut}(\Gamma)$ via

$$r_\alpha(\lambda) := \lambda - 2(\lambda, \alpha)\alpha$$

[check: $r_\alpha(\alpha) = -\alpha$
$(\lambda, \alpha) = 0 \Rightarrow r_\alpha(\lambda) = \lambda$
$(r_\alpha(\lambda), r_\alpha(\lambda)) = (\lambda, \lambda)$]

$W := \langle r_\alpha \mid \alpha \in \Pi \rangle \subset \mathrm{Aut}(\Gamma, (\,,\,))$ Weyl group
(rem: $\Pi = \emptyset \Rightarrow W = \{1\}$)

$\Delta^{re} := W\Pi$ real roots

$\cup$

$\Delta^{re}_+ := \Delta^{re} \cap \Gamma_+$

$\alpha \in \Gamma \Rightarrow \mathrm{supp}(\alpha) :=$ full subgraph of S with vertices $\{i \mid k_i \neq 0\}$, where $\alpha = \sum k_i \alpha_i$

$M := \{\alpha \in \Gamma_+ \setminus \{0\} \mid (\alpha, \beta) \leq 0 \ \forall \beta \in \Pi, \ \mathrm{supp}(\alpha) \text{ connected}\}$ fundamental set

$\Delta^{im} := \bigcup_{\alpha \in M \cup -M} W\alpha$ imaginary roots

$\cup$

$\Delta^{im}_+ := \Delta^{im} \cap \Gamma_+ = \bigcup_{\alpha \in M} W\alpha$

$\Delta := \Delta^{re} \cup \Delta^{im}$ roots

$\Delta_+ := \Delta^{re}_+ \cup \Delta^{im}_+$ positive roots

recall:

a) S finite $:\Longleftrightarrow$ $(\,,\,)$ positive definite

b) S tame $:\Longleftrightarrow$ $(\,,\,)$ positive semi-definite

c) S wild $:\Longleftrightarrow$ $(\,,\,)$ indefinite

next recall characterization, classification, examples for a), b), c) [if known]

Kac's theorem

$S = (S_0, S_1)$ connected graph, $S_0 = \{1, \dots, n\}$, $\# S_1 < \infty$
Q quiver with underlying graph S
$\mathbb{F} = \overline{\mathbb{F}}$ algebraically closed field
$\alpha = \sum k_i \alpha_i \in \Gamma = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$

recall: $\mathcal{M}^\alpha(Q) := \bigoplus_{i \to j \in Q_1} \mathrm{Hom}(\mathbb{F}^{k_i}, \mathbb{F}^{k_j}) \supset \mathcal{M}^\alpha_{ind}(Q) := \{M \in \mathcal{M}^\alpha(Q) \mid M \text{ indec.}\}$

$\circlearrowleft$
$G^\alpha := \prod_{i=1}^{n} Gl_{k_i}(\mathbb{F})$

$\mu_\alpha(Q) := \max\{\dim Z \mid Z \text{ irred., } Z \text{ is geometric subquotient of } \mathcal{M}^\alpha_{ind}(Q)\}$ $\circlearrowleft G^\alpha$

Consider $\mathbb{M}_{ind}(Q) := \bigcup_{\alpha \in \Gamma} \{G^\alpha\text{-orbits in } \mathcal{M}^\alpha_{ind}(Q)\} = \{\text{isomcl. of indec reps of } Q\}$

$\underline{\dim} : \mathbb{M}_{ind}(Q) \longrightarrow \Gamma$

$\Longrightarrow$ ① $\underline{\dim}(\mathbb{M}_{ind}(Q)) = \Delta_+(S)$

② $\underline{\dim}\big|_{\underline{\dim}^{-1}(\Delta_+^{re})}$ injective

③ $\alpha \in \Delta_+^{im} \Longrightarrow \mu_\alpha(Q) = 1 - (\alpha, \alpha) > 0$

$[\Longrightarrow \# \underline{\dim}^{-1}(\alpha) = \infty]$

For the proof assume the following three things

$\boxed{A}$ $\exists$ "reflection functors" inducing bijections on certain sets of orbits
$\Longrightarrow \forall i$ sink (or source), $\alpha \neq \alpha_i$: $\mu_\alpha(Q) = \mu_{r_{\alpha_i}(\alpha)}(Q)$,

$\boxed{B}$ a dense-subset lemma: $\alpha \in M$ s.t. $\exists\, i$ with $(\alpha, \alpha_i) < 0 \Longrightarrow$

a) $\mathcal{M}^\alpha_0(Q) := \{M \in \mathcal{M}^\alpha(Q) \mid \underbrace{\mathrm{End}(M) \cong \mathbb{F}}_{=:B}\}$ is open dense in $\mathcal{M}^\alpha(Q)$, G^α-invariant,

$\underbrace{\mu(G^\alpha, \mathcal{M}^\alpha_0(Q))}_{\text{max. dim. of geom. subquotient for } G^\alpha \circlearrowright \mathcal{M}^\alpha_0(Q)} = 1 - (\alpha, \alpha)$

b) $\mu(G^\alpha, \mathcal{M}^\alpha_{ind}(Q) \setminus \mathcal{M}^\alpha_0(Q)) < 1 - (\alpha, \alpha)$

$\boxed{C}$ a lemma proved by reduction mod p and counting points:
$\# \mathcal{M}^\alpha_{ind}(Q)$ and $\mu_\alpha(Q)$, if finite, are independent of the orientation of Q

sketch of proof: By $\boxed{A}$ and $\boxed{C}$ it holds

1. $\alpha \in \Delta_+^{re} \Longrightarrow$ wlog $\alpha \in \Pi$, but then $\#\underline{\dim}^{-1}(\alpha) = 1$ is clear. ②

2. $\alpha \in \Delta_+^{im} \Longrightarrow$ wlog $\alpha \in M$.
if $(\alpha, \alpha_i) = 0 \ \forall i \Longrightarrow \mathrm{supp}(\alpha)$ tame, $\alpha = k\delta$ $(k \in \mathbb{N})$
then show ③ in a case by case analysis [...]

else $\exists\, i : (\alpha, \alpha_i) < 0$. Apply $\boxed{B}$ to show ③.

1.+2. $\Longrightarrow \underline{\dim}(\mathbb{M}_{ind}(Q)) \supset \Delta_+$

For the other inclusion consider: $\alpha \in \Gamma_+ \setminus \Delta_+^{re}$, $\alpha \neq 0$ s.t. $\exists$ indec. rep of $\underline{\dim} = \alpha$
$\Longrightarrow \forall \beta \in W\alpha\ \exists$ indec. rep of $\underline{\dim} = \beta$ and $W\alpha \subset \Gamma_+ \setminus \Delta_+^{re}$

choose $\beta \in W\alpha$ of minimal height $\underset{[\dots]}{\Longrightarrow} \beta \in M$ and $\alpha \in \Delta_+^{im}$. $\Longrightarrow$ ①
↳ with $(\beta, \alpha_i) \leq 0 \ \forall i$

ad $\boxed{B}$

$M \in M^\alpha(Q) \Rightarrow \dim G^\alpha \cdot M = \dim M^\alpha(Q) - \dim \mathrm{End}(M)$

$$\Rightarrow \bigcup_{\substack{M \in M^\alpha(Q) \\ \dim G^\alpha \cdot M = \dim M^\alpha(Q) - 1}} G^\alpha \cdot M = B$$

clear: if $B \neq \emptyset \Rightarrow B$ open, G^α-inv., dense in $M^\alpha(Q)$ and $M^\alpha_{ind}(Q)$

$\Rightarrow \mu_\alpha(Q) = \mu_\alpha(G^\alpha, B) = \mu_\alpha(G^\alpha, M^\alpha(Q)) = 1 - (\alpha, \alpha)$

CB proved: • $\alpha \in M$ then

$\exists i : (\alpha, \alpha_i) < 0 \overset{(*)}{\Longleftrightarrow} \forall \alpha = \sum_{i=1}^{r} \beta_i$, $\beta_i \in \Gamma_+$: $(\alpha, \alpha) < \sum (\beta_i, \beta_i)$

• Say $\alpha = \beta + \gamma$, $\beta, \gamma \in \Gamma_+$, $\exists i: (\alpha, \alpha_i) < 0$. Consider

$\theta : Gl^\alpha \times M^\beta(Q) \times M^\gamma(Q) \longrightarrow M^\alpha(Q)$ $(g, x, y) \mapsto g \cdot (x \oplus y)$

$H = G^\beta \times G^\gamma$ operates freely on LHS

$(h, f) \circ (g, x, y) := (g [h^{-1} f^{-1}], hx, fy)$

$\Rightarrow \theta_{\beta,\gamma}$ constant on H-orbits

$\Rightarrow \dim \overline{\mathrm{Im}\, \theta_{\beta,\gamma}} \leq \dim (LHS) - \dim H$ $\qquad (\alpha, \alpha) = \dim G^\alpha - \dim M^\alpha(Q)$

$\Rightarrow \dim M^\alpha(Q) - \dim \overline{\mathrm{Im}\, \theta_{\beta\gamma}} \geq (\beta, \beta) + (\gamma, \gamma) - (\alpha, \alpha) \underset{(*)}{>} 0$

$\Rightarrow \overline{\mathrm{Im}\, \theta_{\beta,\gamma}}$ G^α-inv. proper closed subset.

$\Rightarrow \bigcup_{\alpha = \sum \beta_i} \overline{\mathrm{Im}\, \theta_{\beta_i}}$ G^α-inv. proper closed subset containing all decomposa

finite set

$\Rightarrow U := M^\alpha(Q) \setminus \bigcup \overline{\mathrm{Im}\, \theta_{\beta_i}}$ open G^α-invariant, dense contained in $M^\alpha_{ind}(Q)$, use this instead of B in the thm $\square$

(in particular $M^\alpha_{ind}(Q)$ irreducible)

Claudia

How to obtain a Lie algebra from a root category: a construction of Peng + Xiao.

Quiver Q $\rightsquigarrow$ associated Kac-Moody-Lie Algebra

Kac's Thm

$$\left\{\begin{matrix}\text{pos. roots of the}\\ \text{Kac-Moody Algebra}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{indecomposables}\\ \text{of } \operatorname{mod}(kQ)\end{matrix}\right\}$$

Question: Is it possible to recover the Lie Algebra from the data given by $\operatorname{mod} kQ$?

Yes. $\rightsquigarrow$ root category $\mathcal{D}^b(\operatorname{mod} kQ)/T^2$

$\rightsquigarrow$ Construction of the Lie Algebra $L(Q)$

Show: This is a Lie Algebra } this talk

From this we can obtain a Lie alg which is isom. to the Kac-Moody Alg. assoc. to Q. } next talk

Construction

k finite field, $q_k = |k|$

Let $\mathcal{T} = \mathcal{D}^b(\operatorname{mod} kQ)/T^2$ (or more generally, a skeletally small, finitary, 2 periodic triangulated category).

$\mathcal{G}(\mathcal{T})$ = Grothendieck group = free abelian group with gen. the iso. cl. of obj. / $[X]+[Y]-[L]$ whenever $\exists$ triangle $Y \to L \to X \to TY$

Denote by h_X the image of $[X]$ in $\mathcal{G}(\mathcal{T})$

W_{XY}^L := set of all triangles $Y \to L \to X \to TY$

The group $\operatorname{Aut}(X,Y) = \operatorname{Aut}(X) \times \operatorname{Aut}(Y)$ acts on W_{XY}^L via

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "y"'] \arrow[dr, phantom, "\circlearrowright"] & L \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[r, "h"] \arrow[d, "\xi"] \arrow[dr, phantom, "\circlearrowright"] & TY \arrow[d, "Ty"] \\
Y \arrow[r, "fy^{-1}"'] & L \arrow[r, "\xi g"'] & X \arrow[r, "(Ty)h\xi^{-1}"'] & TY
\end{tikzcd}$$

V_{XY}^{L} := the quotient

$F_{XY}^{L} := |V_{XY}^{L}| \in \mathbb{Z}^{+}$.

$\mathfrak{h}$:= subgroup of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}(\mathcal{J})$ generated by $\frac{h_X}{d(X)} =: \tilde{h}_X$, where X indec, $d(X) = \dim(\operatorname{End} X / \operatorname{rad} \operatorname{End} X)$

$\mathfrak{n}$:= free abelian group on generators u_X indexed by the iso classes of indec. objects X.

$$L(Q)\ (= L(\mathcal{J})) = \mathfrak{h} \oplus \mathfrak{n} = \underbrace{\mathfrak{n}^{+}}_{\substack{\text{gen. by} \\ u_X}} \oplus \mathfrak{h} \oplus \underbrace{\mathfrak{n}^{-}}_{\substack{\text{gen. by} \\ u_{TX}}}$$

Bilinear form on $\mathfrak{h} \times \mathfrak{h}$:

$$(h_X, h_Y) = \dim \operatorname{Hom}(X,Y) - \dim \operatorname{Hom}(X,TY) + \dim \operatorname{Hom}(Y,X) - \dim \operatorname{Hom}(Y,TX)$$

Lie bracket on $L(Q)$:

$$[u_X, u_Y] = \begin{cases} \sum_{L \text{ ind}} (F_{XY}^{L} - F_{YX}^{L})\, u_L & \text{if } X \not\cong TY \\ \tilde{h}_X & \text{if } X \cong TY \end{cases}$$

$$[\tilde{h}_X, u_Y] = -(\tilde{h}_X \mid h_Y)\, u_Y$$

$$[u_Y, \tilde{h}_X] = (\tilde{h}_X \mid h_Y)\, u_Y$$

$$[\tilde{h}_X, \tilde{h}_Y] = 0$$

<u>Thm</u>. Together with the bracket $[-,-]$, $L(Q)/(q_k - 1)$ is a Lie algebra over $\mathbb{Z}/(q_k - 1)\mathbb{Z}$.

The numbers F_{XY}^{L} and N_{XYZ}^{LM}

$\mathrm{Aut}(X,Y,Z,L) \hookrightarrow W_{XY}^{L} \times W_{LZ}^{M}$

$\mathrm{Aut}(X,Y,Z,L') \hookrightarrow W_{XL'}^{M} \times W_{YZ}^{L'}$

$\rightsquigarrow N_{XYZ}^{LM} :=$

$\hat{N}_{XYZ}^{ML'} :=$ number of orbits

Lemma X, Y, Z indec. Calculate the numbers

- $F_{LZ}^{M} = \dfrac{|W_{LZ}^{M}|}{|\mathrm{Aut}(Z,L)|}$ if $L \not\cong M \oplus TZ$

  $F_{M\oplus TZ}^{M} = 1$

- $F_{XY}^{X\oplus Y} = |\mathrm{Hom}(Y,X)|$ if $X \not\cong Y$

- $N_{XYZ}^{LM} = \dfrac{|W_{XY}^{L}|}{|\mathrm{Aut}(X,Y,Z)|}$ if $L \not\cong X\oplus Y$

  $\hat{N}_{XYZ}^{ML'} = \text{---}$ otherwise ...

⋮

Proof (of the first part). $L \not\cong M \oplus TZ$: Use the following fact for a fin. group G acting on a finite set X: $\dfrac{|X|}{|G|} = \sum_{x \in X/G} \dfrac{1}{|\mathrm{Stab}\, x|}$

$t = (l, m, n) \in W_{LZ}^{M}$. Show $\mathrm{Stab}(t) \cong$ v.sp

$S(t) := \{(\rho - 1, 0, \lambda - 1) \in \mathrm{End}\, t \mid (\rho, \lambda) \in \mathrm{Stab}\, t\}$

$\Rightarrow \dfrac{|W_{LZ}^{M}|}{|\mathrm{Aut}(Z,L)|} = \sum_{t \in V_{LZ}^{M}} 1 = F_{LZ}^{M}$

Let $(\bar{\rho}_i, 0, \bar{\lambda}_i) \in S(t)$, $i = 1, 2$

$L \not\cong M \oplus TZ$

$$\begin{tikzcd} Z \arrow[r, "l"] \arrow[d, "\rho-1"'] & M \arrow[r, "m"] \arrow[d, "0"] & L \arrow[r, "n"] \arrow[d, "\lambda-1"] & TZ \arrow[d] \\ Z \arrow[r, "l"'] & M \arrow[r, "m"'] & L \arrow[r, "n"'] & TZ \end{tikzcd}$$

$\Rightarrow l \neq 0 \Rightarrow \bar{\rho}_i$ nilpotent

$\Rightarrow \rho := 1 + \bar{\rho}_1 + \alpha \bar{\rho}_2$ iso

$\Rightarrow \lambda := 1 + \bar{\lambda}_1 + \alpha \bar{\lambda}_2$ iso $((\rho, 1, \lambda) \in \mathrm{End}\, t) \Rightarrow (\rho, \lambda) \in \mathrm{Stab}(t)$

$\Rightarrow (\rho - 1, 0, \lambda - 1) \in S(t)$

If $L \cong M \oplus TZ$

$$\Rightarrow \begin{tikzcd} M \arrow[r, "\binom{g_1}{g_2}"] \arrow[d, "\varphi_1", "\wr"'] & M \oplus TZ \arrow[r, "(h_1\, h_2)"] \arrow[d, "\varphi_2", "\wr"'] & TZ \arrow[r, "-Tf"] \arrow[d, "\varphi_3", "\wr"'] & TM \arrow[d, "\wr"] \\ M \arrow[r, "\binom{1}{0}"'] & M \oplus TZ \arrow[r, "(0\ 1)"'] & TZ \arrow[r, "0"'] & TM \end{tikzcd}$$

$$\rightsquigarrow \begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "T^{-1}\varphi_3"'] & M \arrow[r, "\binom{g_1}{g_2}"] \arrow[d, equal] & M \oplus TZ \arrow[r, "(h_1\, h_2)"] \arrow[d, "\varphi_2"] & TZ \arrow[d, "\varphi_3"] \\ Z \arrow[r, "0"'] \arrow[d, "\mathrm{id}"'] & M \arrow[r, "\binom{\varphi_1}{0}"] \arrow[d, equal] & M \oplus TZ \arrow[r, "(0\ 1)"] \arrow[d, "\begin{pmatrix}\varphi_1^{-1} & 0 \\ 0 & 1\end{pmatrix}"] & TZ \arrow[d, "T\mathrm{id}"] \\ Z \arrow[r, "0"'] & M \arrow[r, "\binom{1}{0}"'] & M \oplus TZ \arrow[r, "(0\ 1)"'] & TZ \end{tikzcd}$$

There is a unique orbit represented by $\longrightarrow$ the bottom row. $\boxtimes$

Proof of the theorem: Skew-symmetry ✓ by definition (Note that $h_X = -h_{TX}$).

We show the Jacobi identity

$$[[u_X, u_Y], u_Z] - [[u_X, u_Z], u_Y] - [[u_Z, u_Y], u_X] \equiv 0$$

in the case $X, Y \not\cong TZ$ and $X \not\cong TY$ (other cases similar).

Observe: L decomp or $0 \Rightarrow F_{XY}^L - F_{YX}^L \equiv 0$:

$$L \not\cong X \oplus Y \Rightarrow F_{XY}^L \equiv \frac{|W_{XY}^L|}{|\mathrm{Aut}(X,Y)|} = |\mathrm{Aut} Z|\, N_{XYZ}^{LM} \equiv 0$$

$$L \cong X \oplus Y,\ X \not\cong Y \Rightarrow F_{XY}^L = |\mathrm{Hom}(Y,X)| \equiv 1$$

$$[[u_X, u_Y], u_Z] = \underbrace{\sum_{\substack{L, M \text{ ind} \\ L \not\cong TZ}} (F_{XY}^L - F_{YX}^L)(F_{LZ}^M - F_{ZL}^M)\, u_M}_{(1)} \underbrace{- (F_{XY}^{TZ} - F_{YX}^{TZ})\, \tilde{h}_Z}_{(2)}$$

(1) M indec. $\sum_L (F_{XY}^L - F_{YX}^L)(F_{LZ}^M - F_{ZL}^M)$

using our knowledge about the numbers $F, N, \hat{N}$ $= \ldots = (F_{XY}^{M\oplus TZ} - F_{YX}^{M\oplus TZ}) F_{M\oplus TZ\, Z}^M - F_{X\, M\oplus TZ}^M (F_{YZ}^{M\oplus TX} - F_{ZY}^{M\oplus}$

$+ \ldots +$ terms involving N and $\hat{N}$ cancelling each other

$\equiv 0$

(2) Define an action ~~W_{XY}^{TZ}~~ of $\mathrm{Aut}(X,Y,Z)$ on W_{XY}^{TZ}

$$\Rightarrow \frac{|W_{XY}^{TZ}|}{|\mathrm{Aut}(X,Y,Z)|} = \sum_t \frac{1}{|\mathrm{Stab}(t)|} = \sum_t \frac{1}{|\mathrm{Aut}\, t|}$$

$$F_{XY}^{TZ} \equiv \frac{|W_{XY}^{TZ}|}{|\mathrm{Aut}(X,Y)|} = \frac{|W_{XY}^{TZ}|\,|\mathrm{Aut}(Z)|}{|\mathrm{Aut}(X,Y,Z)|} = \sum_t \frac{|\mathrm{Aut}\, Z|}{|\mathrm{Aut}\, t|} = \sum_t \frac{d(Z)}{d(t)}$$

$$t \in W_{XY}^{TZ} \Rightarrow h_X + h_Y + h_Z = 0$$

$$\Rightarrow \frac{d(X)}{d(t)}\tilde{h}_X + \frac{d(Y)}{d(t)}\tilde{h}_Y + \frac{d(Z)}{d(t)}\hat{h}_Z = 0$$

$$\Rightarrow -F_{XY}^{TZ}\tilde{h}_Z = -\sum_t \frac{d(Z)}{d(t)}\tilde{h}_Z = \sum_t\left(\frac{d(X)}{d(t)}\tilde{h}_X + \frac{d(Y)}{d(t)}\tilde{h}_Y\right)$$

$$= F_{YZ}^{TX}\tilde{h}_X + F_{ZX}^{TY}\tilde{h}_Y$$

$$\Rightarrow -(F_{XY}^{TZ} - F_{YX}^{TZ})\tilde{h}_Z + (F_{XZ}^{TY} - F_{ZX}^{TY})\tilde{h}_Y + (F_{ZY}^{TX} - F_{YZ}^{TX})\tilde{h}_X = 0$$

Construction of Peng and Xiao II — Karsten

Aim. Get a realization of all symmetrizable Kac-Moody Algebras from the root categories of all fin. dim. hereditary k-algebras.

Recall. The root cat $\mathcal{R}$ is divided by mod A into two parts mod A and $T(\operatorname{mod} A)$

$$\mathfrak{g}_{(q-1)} = K(\operatorname{mod} A)/(q-1)K(\operatorname{mod} A)$$

$$\oplus\; h/(q-1)h \;\oplus\; K(T(\operatorname{mod} A))/q-1$$

Now, suppose that E is a fin. field extension of k.

Put $V^E = E \otimes_k V$ for any k-vsp V

Clearly, A^E is a hereditary E-alg and for any A-module M, M^E caries a natural structure of an A^E-mod.

Def. 1) The field ext $k \subset E$ is called conservative for $X \in \operatorname{ind} A$ if $(\operatorname{End}_A X/\operatorname{rad}\operatorname{End}_A X)^E$ is a field

2) An indec. module X is called exceptional if $\operatorname{Ext}^1_A(X,X) = 0$

Prop (Ringel). Let $X \in \operatorname{ind} A$ be exceptional
$\Rightarrow \operatorname{End} X \cong \operatorname{End} S$ for some simple S.

Now we set

$$\Omega = \{E \mid k \subset E \subset \bar{k} \text{ fin. field extensions conservative for all exceptional } A\text{-modules}\}$$

Note. Ω is infinite!

Now, for any $E \in \Omega$, $K(\mathrm{mod}\,A^E) / (|E|-1) K(\mathrm{mod}\,A^E)$ is a Lie subalg of $\mathcal{H}(A^E) / (|E|-1) \mathcal{H}(A^E)$ over $\mathbb{Z}/(|E|-1)$.

Put $\Pi = \prod_{E \in \Omega} K(\mathrm{mod}\,A^E) / (|E|-1) K(\mathrm{mod}\,A^E)$

Notation: For $X \in \mathrm{mod}\,A$: $u_X := (u_{X^E})_{E \in \Omega} \in \Pi$

Let $S_1, \dots, S_n$ be the simple A-modules, and let $\mathcal{L}\mathcal{C}(A)$ be the Lie subalg of Π gen. by u_{S_i}.

Prop. Let $X \in \mathrm{ind}\,A$ exceptional $\Rightarrow \exists\, a \in \mathbb{Z} \setminus \{0\}$: $a \cdot u_X \in \mathcal{L}\mathcal{C}(A)$

Now, let $\mathcal{L}\mathcal{E}(A)$ be the Lie subalg of Π gen. by all u_X with X exceptional.

$\mathcal{L}\mathcal{C}(A) \subset \mathcal{L}\mathcal{E}(A)$

Cor. As Lie-subalg. over $\mathbb{Q}$

$\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{L}\mathcal{C}(A) \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{L}\mathcal{E}(A)$.

Come back to $\mathcal{R}$ and consider $\mathrm{mod}\,A$ as a fixed full subcat of $\mathcal{R}$.

An object X in $\mathcal{R}$ is called exceptional if $\mathrm{Hom}_{\mathcal{R}}(X, TX) = 0$. Clearly, X exceptional
$= \mathrm{Ext}^1(X,X)$ $\Leftrightarrow X \in \mathrm{ind}\,A$ or $TX \in \mathrm{ind}\,A$ exceptional

$\Omega = \{E \mid k \subset E \subset \bar{k}$ f.f. ext. conservative for all exc. objects X in $\mathcal{R}\}$
$= \{E \mid k \subset E \subset \bar{k} \dots$ cons. for exc. A-mod.$\}$ -74

Let R^E be the root cat. of $A^E \rightsquigarrow h^E, n^E$ and $\mathfrak{g}^E = h^E \oplus n^E$, $\mathfrak{g}^E_{(|E|-1)} = \mathfrak{g}^E / _{(|E|-1)}\mathfrak{g}^E$

Consider the direct product

$\prod_{E \in \Omega} \mathfrak{g}^E_{(|E|-1)}$ and let $\mathcal{L}\mathcal{E}(R)$ be the

Lie subalg. gen. by $\{u_X = (u_{XE})_{E \in \Omega} \mid X \text{ exc. in } R\}$

<u>Recall</u>. Given a symmetrizable GCM $C = (a_{ij})$, we have the complex Kac-Moody alg

${}_{\mathbb{C}}\mathfrak{g}(C)$ (rational...) defined as follows

$\mathfrak{g}(C)$ is a complex Lie alg. in $3n$ generators e_i, f_i, h_i subject to the relations, $i = 1, \dots, n$.

$$\begin{cases} [h_i, h_j] = 0, \quad [e_i, f_i] = h_i, \quad [e_i, f_j] = 0 \quad i \neq j \\ [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j \\ (\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \quad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0 \quad i \neq j \end{cases}$$

<u>Thm</u>. Let R be the root cat. of a fin. dim. hereditary k-alg and $C = (a_{ij})_{n \times n}$ the generalized symmetrizable Cartan matrix of A. Then, we have isomorphisms of Lie algebras

$$\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{L}\mathcal{E}(R) \simeq {}_{\mathbb{C}}\mathfrak{g}(C)$$

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{L}\mathcal{E}(R) \simeq {}_{\mathbb{Q}}\mathfrak{g}(C)$$

<u>Proof</u>. By the Corollary we know that $\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{L}\mathcal{E}(R)$ is the Lie subalg gen. by the u_{S_i} and u_{TS_i}

Put $h_{S_i} = (h_{S_i E})_{E \in \Omega} \in \prod \mathfrak{g}^E_{(|E|-1)}$ $\quad \forall i = 1, \dots, n$

One can check the following relations

(1) $[u_{S_i}, u_{TS_i}] = \frac{h_{S_i}}{d(S_i)} \in \mathcal{L}\mathcal{E}(R)$ $\forall i$ and

$$[u_{S_i}, u_{TS_j}] = 0 \quad \text{for } i \neq j$$

(2) $[h_{S_i}, h_{S_j}] = 0 \; \forall i, j$

(3) $\left[\frac{h_{S_i}}{d(S_i)}, u_{S_j}\right] = -a_{ij} u_{S_j}$, $\left[\frac{h_{S_i}}{d(S_i)}, u_{TS_j}\right] = a_{ij} u_{TS_j}$

Ringel: $(\operatorname{ad} u_{S_i})^{1-a_{ij}} u_{S_j} = 0 = (\operatorname{ad} u_{TS_i})^{1-a_{ij}} u_{TS_j}$

$\Rightarrow$ The map $\Phi : {}_{\mathbb{C}}\mathfrak{g}(C) \longrightarrow \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{L}\mathcal{E}(R)$

given by $\Phi(e_i) = u_{S_i}$, $\Phi(f_i) = -u_{TS_i}$ and

$\Phi(h_i) = -\frac{h_{S_i}}{d(S_i)}$ is an epi of Lie alg.

On the Cartan subalg Φ is easily seen to be injective.

Since any Lie-ideal which intersects trivially with the Cartan subalg is trivial, Φ is an isomorphism. $\square$

Overview: Hall algebras

Matthias

1. Intro to Hall alg's
2. Hall alg's of the Jordan quiver and sym fct's
3. Hall alg's of quivers and quantum groups
4. Hall alg's of w.p.l. and quantum loop alg's.

1. Hall alg's

Let $\mathcal{A}$ be small, finitary (i.e. $|\mathrm{Hom}(X,Y)|$, $|\mathrm{Ext}^1(X,Y)| < \infty$) hereditary, abelian cat. $K(\mathcal{A})$ Grothendieck group.

<u>Def</u>. The (square root of the) <u>mult. Euler form</u> is given by $\langle M,N\rangle_m = \sqrt{\frac{|\mathrm{Hom}(M,N)|}{|\mathrm{Ext}^1(M,N)|}}$ and gives a bilinear form $\langle -,-\rangle_m : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{C}$ and $(M,N)_m = \langle M,N\rangle_m \cdot \langle N,M\rangle_m$

<u>Def</u>. For any three objects $M, N, R \in \mathcal{A}$ let $\mathcal{P}^R_{M,N}$ be the (finite!) set of s.e.s.

$0 \longrightarrow N \longrightarrow R \longrightarrow M \longrightarrow 0$ and

$p^R_{M,N} := |\mathcal{P}^R_{M,N}|$, $\forall\ P \in \mathcal{A}$, $a_P = |\mathrm{Aut}\,P|$

$X := \mathrm{Ob}(\mathcal{A})/\sim$ set of iso classes

$\mathcal{H}_{\mathcal{A}} := \bigoplus_{M \in X} \mathbb{C}[M]$

<u>Prop</u> (Ringel).
$[M]\cdot[N] = \langle M,N\rangle_m \sum_R \frac{p^R_{M,N}}{a_M \cdot a_N} [R]$ turns $\mathcal{H}_{\mathcal{A}}$ in an ass. alg. with unit $[0]$.

Rem $\cdot$ $\frac{P^R_{M,N}}{a_M \cdot a_N} = \left| \{ L \subset R \mid L \cong N,\ R/L \cong M \} \right|$

- $\mathcal{H}_\mathcal{A}$ encodes extensions in $\mathcal{A}$.
- $\mathcal{H}_\mathcal{A}$ graded by $K(\mathcal{A})$.

Coproduct

Prop (Green). $\Delta([R]) = \sum_{M,N} \langle M,N \rangle_m \frac{1}{a_R} P^R_{M,N} [M] \otimes [N]$

$\in \mathcal{H}_\mathcal{A} \hat{\otimes} \mathcal{H}_\mathcal{A}$

defines a topological coass. coproduct on $\mathcal{H}_\mathcal{A}$, $\varepsilon([M]) = \delta_{M,0}$.

Bialgebra $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{A}$

- $\Delta(x \cdot y) = \Delta(x) \cdot \Delta(y)$
- Either twist multiplication in $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{A}$ or add "degree zero" part $\mathcal{K} = \mathbb{C}[K(\mathcal{A})]$

  $\alpha \in K(\mathcal{A}) \rightsquigarrow k_\alpha \in \mathcal{K}$

  $\tilde{\mathcal{H}}_\mathcal{A} = \mathcal{K} \otimes_\mathbb{C} \mathcal{H}_\mathcal{A}$ $\quad k_\alpha [M] k_\alpha^{-1} = (\alpha, M)_m [M]$,

  $\Delta \ldots$

Green's Thm $\Delta : \tilde{\mathcal{H}}_\mathcal{A} \longrightarrow \tilde{\mathcal{H}}_\mathcal{A} \otimes \tilde{\mathcal{H}}_\mathcal{A}$ is a morphism of algebras.

- skalar product
- Hopf alg. via antipode S.

2. Hall algebra of $\circlearrowleft$ and sym. functions.

- $\circlearrowleft$ nilp. rep. are encoded by the set Π of partitions $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Pi$

$\downarrow$

 $= M(\lambda) = \bigoplus_{i=1}^{m} M((\lambda_i))$

$\lambda_1 \; --- \; \lambda_m$

S
$\shortparallel$
$\square = (k, 0)$ unique simple

$\mathrm{Hom}(S,S) = \mathrm{Ext}^1(S,S) = k$

$\mathrm{rep}_k Q_0$ hereditary

$k = \mathbb{F}_q$, $\mathcal{H} := \mathcal{H}_{\mathrm{rep}_k Q_0}$

<u>Thm</u>. a) $\mathcal{H}$ is (co-) comm.

b) $\mathcal{H} \cong \mathbb{C}[\,[M((1))], [M((2))], \ldots]$

<u>Recall</u>. $\mathbb{C}[x_1, \ldots, x_n]^{S_n} =: \Lambda_n$

$\Lambda_{n+1} \longrightarrow \Lambda_n$

$x_{n+1} \longmapsto 0$

$\Lambda := \varprojlim \Lambda_n$

Λ also has a basis indexed by partitions,

$n \in \mathbb{N}$ $\quad e_n = \sum_{i_1 < \ldots < i_n} x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_n} \in \Lambda$

$\lambda \in \Pi$ $\quad e_\lambda := e_{\lambda_1} \cdot \ldots \cdot e_{\lambda_n}$

<u>Thm</u>. (Macdonald). $\{e_\lambda \mid \lambda \in \Pi\}$ is a basis of Λ, i.e. $\Lambda = \mathbb{C}[e_1, e_2, \ldots]$

<u>Cor / Thm</u>. $\Phi_{(q)} : \mathcal{H} \longrightarrow \Lambda$,

$\Phi_{(q)}([M_{((1^r))}]) = q^{-\binom{r}{2}} \cdot e_r$

is an iso. of bialg.

3. Q finite quiver without loops

$\operatorname{rep}_k Q$ = cat of nilp. reps $\qquad k = \mathbb{F}_q$

$\tilde{\mathcal{H}}_Q$ a Hopf algebra, set $\nu = q^{-\frac{1}{2}}$.

Recall $\mathfrak{g} \rightsquigarrow \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \rightsquigarrow U_v(\mathfrak{g}')$ gen. by $E_i, F_i, K_i^{\pm 1}$ subject to certain relations

"pos part" $U_v(b_+')$ gen by $E_i, K_i^{\pm 1}$, $i \in I$

specialize at $v = \nu$.

The rel. are e.g. $K_i E_j K_i^{-1} = v^{a_{ij}} E_j$

Thm (Ringel, Green)

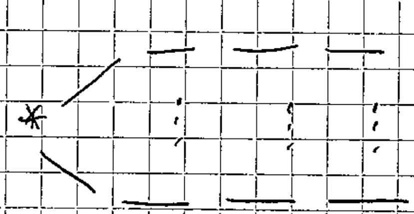
$E_i \longrightarrow [S_i]$, $K_i \longmapsto K_{S_i}$ $\quad i \in I$

extends to an embedding of Hopf alg's

$\Psi : U_\nu(b_+') \longrightarrow \tilde{\mathcal{H}}_Q$

Ψ isom. iff Q is of finite type $\iff \mathfrak{g}$ is a simple Lie alg.

4.



$U_v(\hat{\mathfrak{g}}_0)$, $\hat{\mathfrak{g}}_0 = \mathcal{L}\mathfrak{g}_0$, $\mathfrak{g}_0$ simple

$\rightsquigarrow$ def for $\mathfrak{g}$ general

$\rightsquigarrow \mathfrak{g}$

$U_v(\mathcal{L}\mathfrak{g}) \supset U_v(\mathcal{L}b_+)$

Thm $\exists$ alg. hom $\Psi : U_\nu(\mathcal{L}b_+) \longrightarrow \tilde{\mathcal{H}}_{X_{p,\lambda}}$.

This is an embedding in certain cases.

Proof of Kac's thm for w. p. l I   Dirk

Weighted proj. lines $\underline{p} = (p_1, \dots, p_t)$ $p_i \geq 2$

"weights"

$\underline{\lambda} = (\lambda_3 = 1, \lambda_4, \dots, \lambda_t)$ $\lambda_i \in k \setminus \{0\}$

pw distinct "parameters", $k = \bar{k}$

$S = S(p, \lambda) = k[x_1, \dots, x_t] / (x_i^{p_i} - x_1^{p_1} - \lambda_i x_2^{p_2} \mid i = 3, \dots, t)$

$\mathbb{X} = \mathbb{X}(p, \underline{\lambda}) = \mathrm{Proj}^{\mathbb{L}(p)}(S) \underset{k=\bar{k}}{=} \{(x_1^{p_1} - \lambda x_2^{p_2}) \mid \lambda \neq \lambda_i, \neq 0, \infty\}$

$\cup \{(x_i) \mid i = 1, \dots, t\}$

$\mathcal{H} = \mathrm{coh}(\mathbb{X}) = \cancel{\frac{\mathrm{mod}^{\mathbb{L}(p)}(S)}{\mathrm{mod}_0}}$

$$\frac{\mathrm{mod}^{\mathbb{L}(p)}(S)}{\mathrm{mod}_0^{\mathbb{L}(p)}(S)}$$

Insertion of weights.

Start with polyn. alg. $R = k[x_1, x_2]$, $\deg x_1 = 1 = \deg x_2$

$H = \mathbb{Z}$-graded.

$\pi \in R$ homog. prime element, of degree $h \in H$

$p \geq 2$ weight

$\bar{R} := R[\pi^{1/p}] = R[T]/(T^p - \pi)$, $\bar{\pi} := \pi^{1/p}$ = class of T

$\bar{H} = H[\frac{h}{p}] = H \oplus \mathbb{Z} / \langle (h, p) \rangle$, $\deg \bar{\pi} = \bar{h} :=$ class of $(0,1)$

Then,
- $\bar{R}|_H = R$
- $\bar{R} = R \oplus R\bar{\pi} \oplus \dots \oplus R\bar{\pi}^{p-1}$
- $\bar{R}$ is graded factorial
- $\mathrm{Proj}^{\bar{H}}(\bar{R}) = \mathrm{Proj}^H(R) \setminus \{\pi\} \cup \{\bar{\pi}\}$

can be iterated

Prop. $\pi_1 = x_1$, $\pi_2 = x_2$, $\pi_i = x_1 + \lambda_i x_2$

$\rightsquigarrow S(p, \underline{\lambda}) = k[x_1, x_2][\pi_1^{1/p_1}, \pi_2^{1/p_2}, \dots, \pi_t^{1/p_t}]$

p-cycles (Lenzing).

Start with $\mathcal{H} = \mathrm{coh}(\mathbb{P}^1)$. Fix $x \in \mathbb{P}^1$, weight $p \geq 2$.

There is a nat. Trafo $1_{\mathcal{H}} \xrightarrow{x} \sigma_x$ ← tubular mutation

$k = \bar{k} \Rightarrow \sigma_x \cong -(1)$ degree shift by 1

$E \longmapsto \sigma_x(E) =: E(x)$

e.g. $0 \to \mathcal{O}_{\mathbb{P}^1} \to \underset{\substack{\| \\ \mathcal{O}(x)}}{\mathcal{O}_{\mathbb{P}^1}(1)} \to S_x \to 0$

S_x ← unique sheaf conc. in x

p-cycle in x (Lenzing)

$$\cdots \to E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \to \cdots \to E_{p-1} \xrightarrow{f_{p-1}} E_p \to \cdots$$

$E_i \in \mathcal{H} = \mathrm{coh}(\mathbb{P}^1)$, $f_i : E_i \to E_{i+1}$ morph. in $\mathcal{H}$.

- $E_{i+p} = E_i(x)$ $\quad f_{i+p} = f_i(x)$
- $E_i \xrightarrow{f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i} E_i(x)$ $\quad = x_{E_i}$ $\quad \forall i \in \mathbb{Z}$

$\mathcal{H}(p_x)$ = cat of p-cycles in x (morph. def. naturally, kernel/image … exactness degreewise).

Thm. $\mathcal{H}(p_x) \cong \dfrac{\mathrm{mod}^{\mathbb{H}}(R[\pi_x^{1/p}])}{\mathrm{mod}_0^{\mathbb{H}}(R[\pi_x^{1/p}])}$

Cor $\mathrm{coh}(\mathbb{X}(p,\lambda)) = \mathrm{coh}(\mathbb{P}^1)\begin{pmatrix} p_1 & p_2 & \cdots & p_t \\ x_1 & x_2 & \cdots & x_t \end{pmatrix}$

$x_i \cong \lambda_i$

Vector bundles / parabolic bundles.

For simplicity, treat only p-cycle in one point over $\mathbb{P}^1$.

$$\begin{tikzcd} E_0 \arrow[r, "f_0"] \arrow[rrrr, bend right, "x_{E_0}"'] & E_1 \arrow[r, "f_1"] & \cdots \arrow[r] & E_{p-1} \arrow[r, "f_{p-1}"] & E_0(x) \end{tikzcd}$$

if all E_i are bundles, then all x_{E_i} are monos, hence, all f_i are monos

2) Since $\mathrm{rk}_{\mathbb{P}^1}(E_i) = \mathrm{rk}_{\mathbb{P}^1}(E_i(x))$, all E_i have the <u>same</u> rank.

<u>Def</u> $\mathrm{rk}_{\mathbb{X}}(E) = \mathrm{rk}_{\mathbb{X}}(E_i)$

$$\begin{tikzcd}
E_0 \arrow[r, equal] \arrow[d, hook] & E_0 \arrow[r, equal] \arrow[d, hook] & E_0 \arrow[r, equal] \arrow[d, hook] & \cdots \arrow[r, equal] & E_0 \arrow[r, equal] \arrow[d, hook] & E_0 \arrow[d, hook] \\
E_0 \arrow[r, hook, "f_0"] \arrow[d] & E_1 \arrow[r, hook] \arrow[d, two heads] & E_2 \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & E_{p-1} \arrow[r, hook, "f_{p-1}"] \arrow[d, two heads] & E_0(x) \arrow[d, two heads, "\simeq\ p\text{ cycl}"] \\
0 \arrow[r, hook] & E_1/E_0 \arrow[r, hook] & E_2/E_0 \arrow[r, hook] & \cdots \arrow[r, hook] & E_{p-1}/E_0 \arrow[r, hook] & E_0(x)/E_0
\end{tikzcd}$$

"$\dim E_i/E_0$" $= \text{length in } S_x$

$$n_{p-2} \leq n_{p-3} \leq \cdots \leq n_0 \leq \mathrm{rk}(E_0)$$

filtration of $E_0(x)/E_0$ in $\mathcal{S}_x = \mathrm{add}(S_x)$ semisimple cat with simple obj S_x

= parabolic bundle

$\pi: E_0 \to \mathbb{P}^1$ filtration of fibre $\pi^{-1}(x)$

<u>Fact.</u> $\mathrm{vect}(\mathbb{X}) \simeq$ cat. of parabolic bundles of $\mathbb{P}^1$.

$E_0 =:$ underlying bundle of E over $\mathbb{P}^1$.

Basis of ~~...~~ $K_0(\mathrm{coh}\,\mathbb{X}) = \hat{\Gamma}$

$\alpha_* = [\mathcal{O}]$, $\alpha_{ij} = [\tau^j S_i]$ $\quad 1 \leq i \leq t,\ 0 \leq j \leq p_i - 2$

$$\underbrace{0 \to \mathcal{O} \to \mathcal{O}(\vec{x}_i) \to S_i \to 0}$$

$\delta = [S_0]$

$\uparrow$ simple sheaf conc. in ordinary point

$$[\tau^{p_i-1} S_i] = \delta - \sum_{j=0}^{p_i-2} \alpha_{ij}$$

<u>Def</u>. $\underline{\dim}\, E := n_* \alpha_* + \sum_{i=1}^{t} \sum_{j=0}^{p_i-2} n_{ij} \alpha_{ij} \in \Gamma$

where $n_* = rk(E)$, $n_{ij} = \dim(E_j^{(i)}/E_0^{(i)})$

<u>Clearly</u>, $n_* \geq n_{i_0} \geq n_{i_1} \geq \dots \geq n_{i,p-2} \geq 0 \quad \forall i$

<u>Call</u> such dim. vector / root <u>strict</u>.

<u>Lemma</u>. $E \in \mathrm{vect}\, \mathbb{X}$ (a p-cycle $E_0 \hookrightarrow E_1 \hookrightarrow \dots \hookrightarrow E_0(x)$)

$\Rightarrow [E] = \underline{\dim}\, E + \deg_{\mathbb{P}^1}(E_0) \cdot \delta$

<u>Cor</u>. $E, F \in \mathrm{vect}\, \mathbb{X}$, $[E] = [F] \Rightarrow \deg_{\mathbb{P}^1}(E_0) = \deg_{\mathbb{P}^1}(F_0$

<u>Proof</u> (of Lemma). Check on basis

$[\mathcal{O}], \; [\mathcal{O}(l \cdot \vec{x}_i)], \; [\mathcal{O}(\vec{c})]$

$1 \leq i \leq t$

$0 \leq l \leq p_i - 1$ ... $\square$

<u>Thm</u> (CB, Kac's thm for w.p.l.)

$\phi \in \hat{\Gamma}$

1) ϕ is pos. root $\Leftrightarrow \exists$ ind $E \in \mathrm{coh}(\mathbb{X})$:

$[E] = \phi$

2) ϕ is (pos) real $\Leftrightarrow \exists!$ ——— " ———

ϕ is (pos) imaginary $\Rightarrow \exists \infty$ —— " ———

<u>Restatement</u>. $d \in \mathbb{Z}, \alpha \in \Gamma$

1) α strict $\Leftrightarrow \exists$ indec parabolic bundle E with

$\deg_{\mathbb{P}^1}(E_0) = d$ and $[E] = \alpha$

2) α strict real $\Rightarrow \exists!$ — " —

α strict imaginary $\Rightarrow \exists\infty$ — " —

$[E] = \underline{\dim}\, E + \deg(E_0)\cdot\delta = \alpha + d\cdot\delta$

Use: α strict $\iff \alpha + d\cdot\delta$ positive and $n_* > 0$

<u>Lemma 1.</u> $\phi \in \hat{\Gamma}$ fixed.

$E \in \mathrm{vect}(\mathbb{X})$ indec, with $[E] = \phi$

$E_0 = \bigoplus_{i=1}^{r} L_i$, $L_i \in \mathrm{coh}\,\mathbb{P}^1$ line bundles of degrees $n_1 \le n_2 \le \ldots \le n_r$

$\Rightarrow \mathrm{width}(E) := n_r - n_1$ bounded

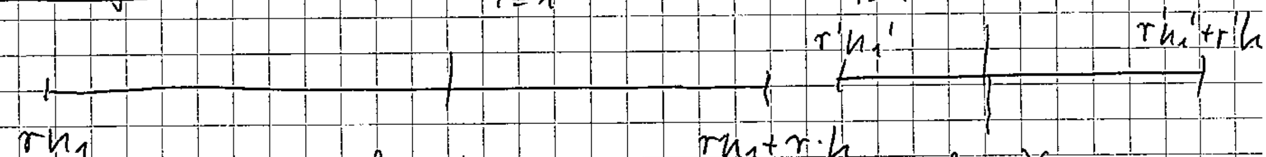
($\forall E$ ind.: $[E] = \phi$)

<u>Cor.</u> $\phi, \psi \in \hat{\Gamma}$

$\Rightarrow \exists\, h > 0$: $\dim \mathrm{Hom}(X,Y) \le h$ (hence also for Ext^1)

for all indec $X, Y \in \mathrm{vect}(\mathbb{X})$ with $[X] = \phi$, $[Y] = \psi$

<u>Proof of Cor.</u> $X_0 = \bigoplus_{i=1}^{r} L_i$ $\quad Y_0 = \bigoplus_{i=1}^{r'} L_i'$



$\deg X_0 = n_1 + n_2 + \ldots + n_r$ $\qquad \deg Y_0 = n_1' + n_2' + \ldots + n_r'$

$n_r \le n_1 + h$ (La 1) $\qquad n_r' \le n_1' + h$ (La 1)

$\Rightarrow |\deg L_i' - \deg L_j|$ bounded $\forall i, j$

$\Rightarrow \dim \mathrm{Hom}(X,Y) \le \dim \mathrm{Hom}(X_0, Y_0)$

$$= \sum_{ij} \dim \mathrm{Hom}(L_i, L_j')$$

$$\leq r \cdot r' \cdot c \cdot \max |\deg L_j' - \deg L_i|$$

bounded.

$\square$

Proof of Kac' thm for w.p.l II — Lingling

$\mathbb{X} = (\mathbb{P}^1, D, W)$ w.p.l.

$D = (a_1, \dots, a_k)$, $W = (w_1, \dots, w_k)$

k finite field

$\mathcal{R}$ triangulated cat, 2-periodic (i.e. $T^2 = 1$, T shift functor).

There is a bilinear form on $K_0(\mathcal{R})$:

$$\langle [X], [Y] \rangle = \dim \mathrm{Hom}(X, Y) - \dim \mathrm{Hom}(X, TY)$$

$$(-,-) = \langle [X], [Y] \rangle + \langle [Y], [X] \rangle$$

$\mathrm{ind}\,\mathcal{R}$ – representatives of iso classes of indec. obj. in $\mathcal{R}$.

Assume, $\mathcal{R}$ is finitary, i.e. it has finite Hom-space and $\{X \in \mathcal{R} \mid [X] = \Phi\}$ is finite for all $\Phi \in K_0(\mathcal{R})$

$d(X) = \dim(\mathrm{End}\,X / \mathrm{rad}\,\mathrm{End}\,X)$.

$K_0(\mathcal{R})$ is torsionfree, gen. by the indec. with $d(X) = 1$.

$[X]$ is divisible by $d(X)$ in $K_0(\mathcal{R})$ $\forall X \in \mathrm{ind}\,\mathcal{R}$.

Define $F^Z_{XY} = |\{\text{triangles } Y \to Z \to X \to\} / \mathrm{Aut}(X) \times \mathrm{Aut}(Y)|$

Let Λ be a comm. ring $|k| = 1$ in Λ ($\mathbb{Z}/q-1$).

[Peng, Xiao]: $L_\Lambda(\mathcal{R}) = (\Lambda \otimes_{\mathbb{Z}} K_0(\mathcal{R})) \oplus \bigoplus_{X \in \mathrm{ind}\,\mathcal{R}} \Lambda u_X$ becomes a Lie alg. over Λ via bracket

$$[u_X, u_Y] = \begin{cases} \sum_Z (F^Z_{XY} - F^Z_{YX}) u_Z, & X \not\cong TY \\ 1 \otimes \frac{[X]}{d(X)} & X \cong TY \end{cases}$$

$$[1 \otimes \Phi, u_X] = -(\Phi, [X]) u_X$$

$$[1 \otimes \Phi, 1 \otimes \Psi] = 0 \quad \forall \Phi, \Psi \in K_0(\mathcal{R})$$

Define the root cat. $\mathcal{R}_{\mathbb{X}} = D^b(\mathrm{coh}\,\mathbb{X}) / (T^2)$: 2 periodic objects ~

morphisms : $\mathrm{Hom}_{\mathcal{R}_{\mathbb{X}}}(X,Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^b(\mathrm{coh}\mathbb{X})}(X, T^{2n}Y)$

Since $\mathrm{coh}\mathbb{X}$ is hereditary,

$\mathrm{ind}\,\mathcal{R} = \mathrm{ind}\,\mathrm{coh}\mathbb{X} \cup \{TY \mid Y\in \mathrm{ind}\,\mathrm{coh}\mathbb{X}\}$

Since $X\to Y\to Z\to$ rotated $\to Y\to Z\to TX\to$:

Any triangle $X\to Y\to Z\to$ in $\mathcal{R}_{\mathbb{X}}$ with X,Y,Z indec. can be rotated s.t. $X,Z\in \mathrm{ind}\,\mathrm{coh}\mathbb{X}$, $Y\in\mathrm{ind}\,\mathrm{coh}\mathbb{X}$.

$\{\text{such triangles}\} \xleftrightarrow{1-1} \{0\to X\to Y\to Z\to 0 \mid X,Y,Z \text{ indec} \in \mathrm{ind}\,\mathrm{coh}\mathbb{X}\}$

$|k|=1$ in Λ : Lie alg. $\mathcal{L}_\Lambda(\mathcal{R}_{\mathbb{X}}) = \bigoplus_{X\in\mathrm{ind}\,\mathrm{coh}\mathbb{X}} \Lambda u_X \oplus (\Lambda\otimes\hat{\Gamma})$

$\oplus \bigoplus_{Y\in\mathrm{ind}\,\mathrm{coh}\mathbb{X}} \Lambda u_{TY}$

b_X, $X\in\mathrm{ind}\,\mathcal{R}_{\mathbb{X}}$

$\begin{cases} b_Y = u_Y \text{ for } Y\in\mathrm{ind}\,\mathrm{coh}\mathbb{X} \\ b_{TY} = -u_{TY} \end{cases}$

If S is a simple sheaf

$S[r]$ is the unique indec sheaf with length r $(r>0)$ and top S which is uniserial

$S[r] = TY$ where Y : length $-r$, $\mathrm{Ext}^1(Y,S)\neq 0$ $(r<0)$

$\Rightarrow \mathrm{Hom}(S[r],S)\neq 0$

Let $H_r = \{X\in\mathrm{ind}\,\mathcal{R}_{\mathbb{X}} \mid X \text{ is of type } r\delta,\ \mathrm{Hom}(X,S_{ij}) = \text{ for } 1\le i\le k,\ 1\le j\le w_i-1\}$

$= \{S_a[\tfrac{r}{d(S_a)}] \mid a\in\mathbb{P}^1\setminus D,\ \tfrac{r}{d(S_a)}\in\mathbb{Z}\} \cup \{S_{i0}[rw_i],\ 1\le i\le k\}$

<u>Thm2</u>: The following elements in $\mathcal{L}_\Lambda(\mathcal{R}_{\mathbb{X}})$ satisfy the relation for $\mathcal{L}\mathfrak{g}$:

$$e_{vr} = \begin{cases} b_{S_{ij}}[rw_i+1] & (v = ij) \\ b\,\mathcal{O}(r\vec{c}) & (v = *) \end{cases}$$

$$f_{vr} = \begin{cases} b_{S_{i,j-1}}[rw_i - 1] & (v = ij) \\ b_{\tau\mathcal{O}(-r\vec{c})} & (v = *) \end{cases}$$

$$c = -1 \otimes \delta$$

$$h_{vr} = \begin{cases} -1 \otimes \alpha_v & (r = 0) \\ b_{S_{ij}}[rw_i] - b_{S_{ij-1}}[rw_i] & (r \neq 0,\ v = ij) \\ h_r & (r \neq 0,\ v = *) \end{cases}$$

$$\mathrm{hom}: \ \mathcal{L}\mathfrak{g} \longrightarrow L_1(\mathcal{R}_*)$$

<u>Lemma 2</u>. If $0 \to X \to Y \to Z \to 0$ is a s.e.s. of indec. finite length sheaves, then up to auto of any two of X, Y, Z, any other s.e.s. with the same term is equivalent to this one.

<u>Proof</u>. Since Y is uniserial, there $\exists$ unique sub-sheaf Y' which is $\cong X$ ✓ for $\mathrm{Aut}(X) \times \mathrm{Aut}(Z)$

For $\mathrm{Aut}(X) \times \mathrm{Aut}(Y)$ we reduce to the case where X, Y, Z are fin. dim. modules for a fin. dim. serial alg. We can assume Y is projective.

Any two epi

$$\begin{array}{ccccccc} P_Y & \longrightarrow & Y & \xrightarrow{f} & Z & \longrightarrow & 0 \\ \downarrow ? & & \downarrow \mathrm{res} & & \| & & \\ P_Y & \longrightarrow & Y & \xrightarrow{f'} & Z & \longrightarrow & 0 \end{array}$$

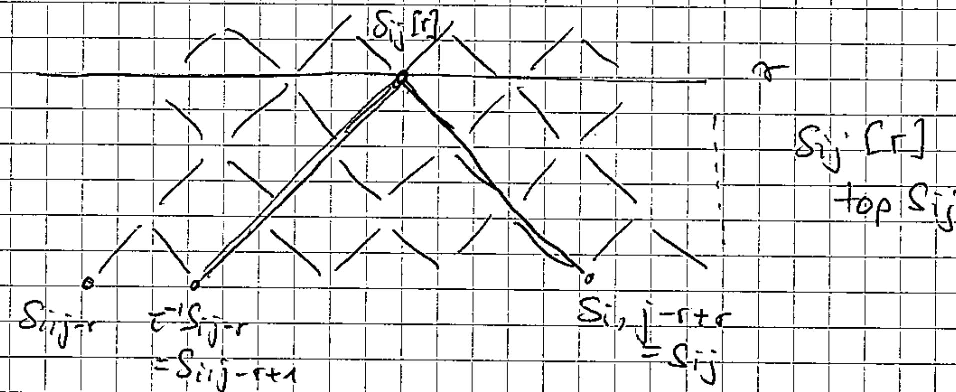
$\square$

<u>Lemma 3</u>. $\tau S_{ij}[r] = S_{i,j-r}[-r]$, $j - r \mod (w_i)$

<u>Proof</u>. Assume $r > 0$

$S_{i,j-r}[-r] = \tau Y$, Y: length r, $\mathrm{Ext}^1(Y, S_{i,j-r}) \neq 0$

$\Rightarrow \mathrm{Hom}(\tau^{-1} S_{i,j-r}, Y) \neq 0$



$Y = S_{ij}[r] \qquad S_{i,j-r}[r] = \tau S_{ij}[r]$ ∎

Lemma 4

$$[b_{S_{ij}[r]}, b_{S_{ik}[s]}] = \begin{cases} \delta_{j-r,k}\, b_{S_{ij}[r+s]} - \delta_{j,k-s}\, b_{S_{ik}[r+s]} & (r+s \neq 0) \\ -\delta_{j-r,k} \otimes [S_{ij}[r]] & (r+s=0) \end{cases}$$

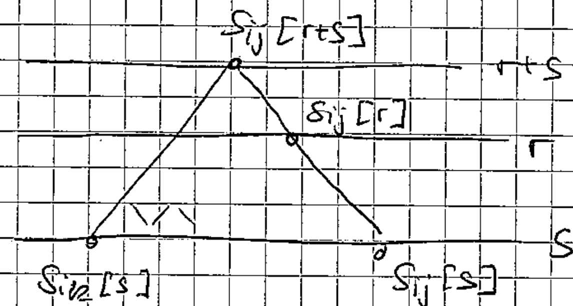
1) $r>0,\ s>0$

Proof. $b_{S_{ij}[r]} = u_{S_{ij}[r]} \qquad b_{S_{ik}[s]} = u_{S_{ik}[s]}$

$$[u_{\underbrace{S_{ij}[r]}_{X}}, u_{\underbrace{S_{ik}[s]}_{Y}}] = \sum_Z (F_{XY}^Z - F_{YX}^Z)\, u_Z$$

$F_{XY}^Z: \quad 0 \to S_{ik}[s] \to Z \to S_{ij}[r] \to 0$

$F_{YX}^Z: \quad 0 \to S_{ij}[r] \to Z \to S_{ik}[s] \to 0$



$Z = S_{ij}[r+s]$

$j-r=k$

$\delta_{j-r,k}\, S_{ij}[r+s]$

2) $r<0,\ s<0$ similar

3) ~~xxxxxx~~ $r\geq 0,\ s<0$

$b_{S_{ij}[r]} = u_{S_{ij}[r]}, \quad b_{S_{ik}[s]} = -u_{\tau Y} = -u_{S_{ik}[s]}$

$S_{ik}[s] = \tau Y, \quad Y$: length s, $\operatorname{Ext}^1(Y, S_{ik}) \neq 0$

$\Rightarrow Y = S_{i,k-s}[-s]$

$\Rightarrow \tau Y = S_{ik}[s]$ ∎

Lemma 5. There is a s.e.s.

$0 \to \mathcal{O}(r\vec{c}) \to X \to S_{ij}[s] \to 0$ with

X indec $\iff j \equiv s \mod w_i$. Then, $X = \mathcal{O}(r\vec{c} + s\vec{x_i})$

Proof. If X indec, $X = \mathcal{O}(\vec{x})$

$\vec{x} = r\vec{c} + s\vec{x_i}$ $\square$

Lemma 6. If $X, Y \in \operatorname{ind} \mathcal{R}_\chi$, $[X] = r\delta$, $[Y] = s\delta$

then $[b_X, b_Y] = 0$ if $X \neq \tau Y$

X, Y all of S_a or all of S_{ij} with fixed

$X = S_{ij}[r]$, $Y = S_{ik}[s]$

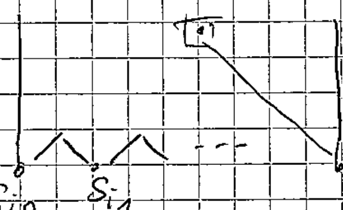
Lemma 7. $H_{-r} = \{\tau Y \mid Y \in H_r\}$ $\square$

Lemma 8. $\sum_{X \in H_r} d(X) = 2$ in Λ

Proof. We may assume $r > 0$.

If $r < 0$: $\sum_{X \in H_r} d(X) = \sum_{\substack{X = \tau Y \\ Y \in H_r}} d(\tau Y) = \sum_{Y \in H_{-r}} d(r)$.

$\operatorname{Hom}(X, S_{ij}) = 0$, $1 \le i \le b$, $1 \le j \le w_i - 1$



For those marked ~~marked~~ point each contribute at most one indec.

Thus, the point at ∞ contributes one indec sheaf and the rest to indec. r-dim modules for the polyn. ring $k[x]$.

Now absolutely indec. modules are given by the Jordan blocks, the number = the size of $k = 1$ in Λ.

$$m'(\Gamma, r\alpha, q) = \sum_{d \mid r} \frac{1}{d} \sum_{k \mid d} \mu(k)\, m(\Gamma, \tfrac{r}{d}\alpha, q)$$

Proof ... (ii) Gaodong

$\mathbb{X} = (\mathbb{P}^1, D, w)$

$D = \{a_1, \dots, a_k\} \subseteq \mathbb{P}^1$ distinct points, $a_i = [\lambda_i : \mu_i]$

$w = (w_1, \dots, w_k)$. Assume $w_i \geq 2$

$S(w, D) = k[u, v, x_1, \dots, x_k] / (x_i^{w_i} - \lambda_i u - \mu_i v)$

Simple sheaves over $\mathbb{X}$

- S_a $a \in \mathbb{P}^1 \setminus D$
- S_{ij} $1 \leq i \leq k$ $0 \leq j \leq w_i - 1$

$\dim \mathrm{Hom}(\mathcal{O}(r\vec{c}), S_{ij}) = \delta_{j0}$

$\dim \mathrm{Ext}^1(S_{ij}, \mathcal{O}(r\vec{c})) = \delta_{j1}$

The only ext between them are

$\dim \mathrm{Ext}^1(S_a, S_a) = d(S_a)$

$\dim \mathrm{Ext}^1(S_{ij}, S_{il}) = 1 \iff l = j - 1 \mod w_i$

$$S_a[r] = \left.\begin{bmatrix} S_a \\ \vdots \\ S_a \end{bmatrix}\right\} r \text{ times}$$

$\underline{r > 0}$ $$S_{ij}[r] = \begin{bmatrix} S_{ij} \\ S_{i,j-1} \\ \vdots \\ S_{i,j-r+1} \end{bmatrix} \qquad S_{ij}[r] = T\begin{bmatrix} S_{i,j-r} \\ \vdots \\ S_{i,j+2} \\ S_{i,j+1} \end{bmatrix}$$

Thm 2. Define in $L_1(\mathcal{R}_{\mathbb{X}})$

e_{vr}, f_{vr}, c

They satisfy relations in $\mathcal{L}\mathfrak{g}$

(i) c is central

(ii) $[e_{vr}, e_{vs}] = 0$

(iii) $[f_{vr}, f_{vs}] = 0$

(iv) $[h_{ur}, h_{us}] = r a_{uv} \delta_{r+s,0}\, c$

(v) $[e_{ur}, f_{vs}] = \delta_{uv}(h_{v,r+s} + r\delta_{r+s,0}\, c)$

(vi) $[h_{u,r}, e_{v,s}] = a_{uv}\, e_{v,r+s}$

(vii) $[h_{u,r}, f_{r,s}] = -a_{uv}\, f_{v,r+s}$

(viii) $(\operatorname{ad} e_{u0})^{1-a_{uv}}(e_{vs}) = 0$ for $u \neq v$

(ix) $(\operatorname{ad} f_{u0})^{1-a_{uv}}(f_{vs}) = 0$ for $u \neq v$

Proof. (ii) (a) $v = ij$ $(j \neq 0)$

$$[e_{ij,r}, e_{ij,s}] \overset{\text{def}}{=} [b_{S_{ij}[rw_i+1]}, b_{S_{ij}[sw_i+1]}]$$

$$\overset{\text{Lem 4}}{=} \begin{cases} \underset{\underset{\delta_{j-1,j}}{\shortparallel}}{\delta_{j-rw_i-1,j}}\, b_{S_{ij}[(r+s)w_i+2]} - \delta_{j,j-sw_i-1}\, b_{S_{ij}[(r+s)w_i+2]} & \text{if } (r+s)w_i+2 \neq 0 \\ -\underset{\shortparallel}{\delta_{j-rw_i-1,j}} \otimes [S_{ij}[rw_i+1]] & \text{if } (r+s)w_i+2 = 0 \end{cases}$$

$= 0$ since $w_i \geq 2$ $\forall i$

(b) $v = *$

$$[e_{*,r}, e_{*,s}] = [b_{G(r\vec{c})}, b_{G(s\vec{c})}]$$

$$= [u_{\overset{x}{G(r\vec{c})}}, u_{\overset{y}{G(s\vec{c})}}]$$

$$= \sum_{z \in \operatorname{ind} R_*} (F_{xy}^{z} - F_{yx}^{z})\, u_z$$

Fact $\{G(s\vec{c}) \mid s \in \mathbb{Z}\} \longleftrightarrow \mathcal{C} \leq \operatorname{coh}(\mathbb{X})$

$\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{coh}(\mathbb{P}^1)$ ($\mathcal{C} \cong \operatorname{coh}(\mathbb{P}^1)$)

$\Rightarrow \operatorname{rank} Z = 2$ contradiction!

(v) (a) $u = ij$ and $v = kl$

$$[e_{ij,r}, f_{kl,s}] = [b_{S_{ij}[rw_i+1]}, b_{S_{k,l-1}[sw_i-1]}]$$

$k \neq i$ Comp. factors of $S_{ij}[rw_i+1]$ are S_{ia}, τS_{ia}

$S_{k,l-1}[sw_i-1]$ ... S_{kb}, τS_{kb}

$$[e_{ij,r}, f_{kl,s}] = 0$$

Suppose now $k = i$

By Lem 4,

$\underline{r+s=0}$ $\quad (\Longleftrightarrow (rw_i+1)+(sw_i-1) = (r+s)w_i = 0)$

$[e_{ij,r}, f_{il,s}]$

$$\overset{\text{Lem 4}}{=} -\delta_{j-rw_i-1,l-1} \otimes [S_{ij}[rw_i+1]]$$

$$= -\delta_{j,l} \otimes (r\delta + \alpha_{ij})$$

$$= \delta_{jl}\left((-1\otimes\alpha_{ij} \ldots) + r(-1\otimes\delta)\right)$$

$$= \delta_{jl}\,(h_{ij,0} + rc)$$

$\underline{r+s\neq 0}$

$$[e_{ij,r}, f_{il,s}] \overset{\text{Lem 4}}{=} \delta_{j-rw_i-1,l-1}\, b_{S_{ij}[(r+s)w_i]}$$

$$- \delta_{j,l-1-sw_i+1}\, b_{S_{i,j-1}[(r+s)w_i]}$$

$$= \delta_{ji}\left(b_{S_{ij}[(r+s)m]} - b_{S_{i,j-1}[(r+s)w_i]}\right)$$

$$= \delta_{jl}\, h_{ij,r+s}$$

b) $v = *$

$$[e_{*,r}, f_{*s}] = [b_{\mathcal{O}(r\vec{c})}, b_{T\mathcal{O}(s\vec{c})}]$$

$$= -[u_{\mathcal{O}(r\vec{c})}, u_{T\mathcal{O}(-s\vec{c})}]$$

$$X = \mathcal{O}(r\vec{c}) = \underset{TY}{\underset{\shortparallel}{\mathcal{O}(-s\vec{c})}} \Longleftrightarrow r = -s$$

$\underline{r+s=0}$ $\quad [e_{x,r}, f_{x,s}] = -1 \otimes [\mathcal{O}(r\vec{c})]$

$\underline{r+s\neq 0}$ $\quad [e_{x,r}, f_{x,s}] = -\sum\limits_{z\in \operatorname{ind} R_x} (F_{XY}^{Z} - F_{YX}^{Z})\, u_Z$

$$= \sum (F_{YX}^{Z} - F_{XY}^{Z})\, u_Z$$

$$F_{YX}^{Z} \neq 0 \qquad \underset{\mathcal{O}(r\vec{c})}{\underset{\shortparallel}{X}} \longrightarrow Z \longrightarrow \underset{T\mathcal{O}(-s\vec{c})}{\underset{\shortparallel}{Y}} \longrightarrow$$

If $Z \in \operatorname{coh} \mathbb{X}$, get $\quad \xrightarrow{\sim} \mathcal{O} \to \mathcal{O}(-t\vec{c}) \to \mathcal{O} \to Z \to 0$

$$0 \to \mathcal{O}(-s\vec{c}) \to \mathcal{O}(r\vec{c}) \to Z \to 0 \qquad \boxed{t = r+s}$$

(1)

If $Z = \tau Z'$, $Z' \in \operatorname{coh}\mathbb{X}$, get

$$0 \to Z' \to \mathcal{O}(-s\vec{c}) \to \mathcal{O}(r\vec{c}) \to 0 \quad \text{impossible}$$

$F_{XY}^{Z} \neq 0 \qquad Y \to Z \to X \to$

$Y = T\mathcal{O}(-s\vec{c})$, $X = \mathcal{O}(r\vec{c})$

$$0 \to \mathcal{O}(t\vec{c}) \to \mathcal{O} \to \tau Z \to 0 \qquad (2)$$

In either case, get a positive contribution of b_Z.

<u>Suppose</u> $t > 0$

<u>Fact</u> $Z \in H_t \iff Z$ lies in a s.e.s of form (1)

(use [GL] (2.5.1), (2.5.2))

$$F_{YX}^{Z} = \left| \{\text{triangles } X \to Z \to Y \to\} \big/ \operatorname{Aut} X \times \operatorname{Aut} Y \right|$$

$$= \left| \{\text{ex. seq } 0 \to \mathcal{O}(-t\vec{c}) \to \mathcal{O} \to Z \to 0\} \big/ \operatorname{Aut}(\mathcal{O}(-t\vec{c})) \times \operatorname{Aut}\mathcal{O} \right|$$

$$= \frac{|\operatorname{Hom}^{epi}(\mathcal{O},Z)|\, \cancel{|\operatorname{Aut}(\mathcal{O}(-t\vec{c}))|}}{|\operatorname{Aut}\mathcal{O}|\, \cancel{|\operatorname{Aut}(\mathcal{O}(-t\vec{c}))|}} = \frac{q^t - q^{t-d}}{q-1}$$

$$= q^{t-d}(1 + q + \dots + q^{d-1})$$

$Z \in H_t \qquad = d(Z) = d \in \Lambda$

$\dim \operatorname{Hom}(\mathcal{O}, Z) = t \qquad d = \cancel{\ } \, d(Z)$

non epi $\mathcal{O} \to Z$ form a space of dim $t-d$

$\Rightarrow |\operatorname{Hom}^{epi}(\mathcal{O},Z)| = q^t - q^{t-d}$

$|\operatorname{Aut}\mathcal{O}| = q-1$

$[e_{x,r}, f_{x,s}] = \sum_{z \in H_t} d(z) u_z = h_t = h_{x,t}$

(c) $u = *$ $v = ij$ omitted

(viii)

(a) $u = ij \neq v = kl$

If $k \neq i$, $a_{uv} = 0$

$[e_{u0}, e_{vs}] = 0$

Suppose $i = k$

$$[e_{ij,0}, e_{il,s}] = [b_{s_{ij}[1]}, b_{s_{il}[sw_i+2]}]$$

$$\overset{\text{Lem4}}{=} \begin{cases} \delta_{j-1,l}\, b_{s_{ij}[sw_i+2]} - \delta_{j,l-1}\, b_{s_{il}[sw_i+2]} & \text{if } sw_i+2 \neq 0 \\ -\delta_{j-1,l} \otimes [\underbrace{s_{ij}[1]}_{s_{ij}}] & \text{if } sw_i+2 = 0 \end{cases}$$

$\Rightarrow$ if $l \neq j \pm 1$, $[e_{ij,0}, e_{il,s}] = 0$

(b) $u = *$, $v = ij$ with $j > 1$

$a_{uv} = 0$

(c) $[e_{ij,0}, [e_{ij,0}, e_{il,s}]] = 0$ for $l = j \pm 1$

use Lem4 $\Downarrow$

$a_{uv} = -1$

$1 - a_{uv} = 2$

(d) $[\underset{u}{e_{ij,0}}, [\underset{v}{e_{i1,0}}, e_{x,s}]] = 0$ $a_{uv} = -1$

(e) $[\underset{u}{e_{*,0}}, [e_{*,0}, \underset{v}{e_{i1,s}}]] = 0$

Computing $[e_{X,0}, e_{i,r,s}] = [b_{\mathcal{O}}, b_{S_{ij}[sw_i+1]}]$ one gets contributions from ex. seq.

$$0 \to \mathcal{O} \to \underset{\substack{\cap \\ \text{ind coh} \mathbb{X}}}{X} \to S_{ij}[sw_i+1] \to 0$$

Computing $[e_{X,0}, [e_{X,0}, e_{i,r,s}]]$, get contr. by from $0 \to X \to Y \to \mathcal{O} \to 0$

$$\rightsquigarrow 0 \to \mathcal{O} \to Y \to X \to 0$$

Fact. $\{\mathcal{O}, S_{ij}[sw_i+1], X, Y\} \subseteq \mathcal{C} \subseteq \operatorname{coh} \mathbb{X}$ — $\mathcal{C} = \operatorname{coh}(\mathbb{X}')$ with only one marked point a_i

indec. vectorbundle are line bundles in $\operatorname{coh}(\mathbb{X}')$

$\square$

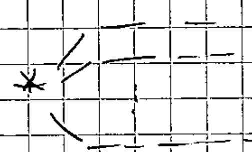
<u>Proof of - - IV</u> Fan

<u>Thm 1</u> There is an indec. coherent sheaf on $\mathbb{X}$ of type $\phi \in \hat{\Gamma} \iff \phi$ is a positive root in $\hat{\Delta}$. There is a unique indec. for a real root, infinitely many ~~for~~ an imaginary root.

$\mathbb{X}$ type (D, w) $\quad D = (a_1, \ldots, a_k) \quad a_i \in \mathbb{P}^1$

$w = (w_1, \ldots, w_k)$

Q_w : the star shaped quiver



$\mathfrak{g}$ Kac-Moody alg ass to Q_w

$L\mathfrak{g}$ 2-toroidal alg of $\mathfrak{g}$

$\hat{\Gamma} = \Gamma \oplus \mathbb{Z}\delta$

$\hat{\Delta} = \{\alpha + r\delta \mid \alpha \in \Delta, r \in \mathbb{Z}\} \cup \{r\delta \mid 0 \neq r \in \mathbb{Z}\}$

$K_0(\mathrm{coh}\,\mathbb{X}) = \hat{\Gamma}$

Idea: There is a Lie homom

$$L\mathfrak{g} \longrightarrow L_\Lambda(R_{\mathbb{X}}) = (\Lambda \otimes_{\mathbb{Z}} \hat{\Gamma}) \oplus \bigoplus_{[X] \in \mathrm{ind} R} \Lambda u_X$$

Need to show:

The number of iso classes of type $\phi = \alpha + r\delta$

= the number of iso classes of type $S_v(\phi)$

$= \phi - (\phi, \alpha_v)\alpha_v$

(* Define

fundamental domain: consists of $\alpha \neq 0$ with supp α connected and which are not made smaller by any reflection.

Assume $\phi \notin \hat{\Delta} \Rightarrow$ minimal height $\phi_0 \notin \hat{\Delta}$)

The first number $= \dim L_\wedge(R_\times)_\Phi$, for $k = \bar{k}$
the second number $= \dim L_\wedge(R_\times)_{s_\gamma(\Phi)}$

$L_\Phi \longleftrightarrow L_{s_\gamma(\Phi)}$ in complex Lie alg

$\rightsquigarrow$ reduce

for finite field

Recall Φ additive gp with symm. bilinear form $(-,-): \Phi \times \Phi \longrightarrow \mathbb{Z}$ s.t. $\alpha \in \Phi$,
$(\alpha, \alpha) = 2$
L Φ-graded complex Lie alg $e \in L_\alpha$,
$f \in L_{-\alpha}$, $h = [e, f]$
Assume that $\operatorname{ad} e : L \longrightarrow L$, $x \mapsto [e, x]$, $\operatorname{ad} f$ are loc. nilp.
For any $x \in L$ $\exists N(x)$ $(\operatorname{ad} e)^{N(x)}(x) = 0$

$\operatorname{ad} h$ acts on L_ψ, for $y \in L_\psi$,
$[h, y] = (\alpha, \psi) y$
$\exp(\operatorname{ad} e) = 1 + \operatorname{ad} e + \frac{(\operatorname{ad} e)^2}{2!} + \ldots + \frac{(\operatorname{ad} e)^k}{k!} + \ldots$

$\exp(\operatorname{ad} e) : L \xrightarrow{\sim} L$

Set $\theta = \exp(\operatorname{ad} e) \exp(-\operatorname{ad} f) \exp(\operatorname{ad} e) : L \xrightarrow{\sim} L$
$e \mapsto -f$
$f \mapsto -e$
$h \mapsto -h$

$\theta_\alpha(L_\Phi) = L_{s_\alpha(\Phi)}$ $\dim L_\Phi = \dim L_{s_\gamma \Phi}$
$\forall x \in L_\Phi$, $\theta_\alpha(x) = \sum_{r \in \mathbb{Z}} y_r$, $y_r \in L_{\Phi + r\alpha}$

$\theta([h, x]) = \theta((\alpha, \Phi) x) = \sum_{r \in \mathbb{Z}} (\alpha, \Phi) y_r$
$\quad \parallel$
$[\theta(h), \theta(x)] = [-h, \sum y_r] = \sum_{r \in \mathbb{Z}} (-\alpha, \Phi + r\alpha) y_r$ $\Rightarrow r = -(\alpha, \Phi)$

L over k char $k = l$

$\exp(\operatorname{ad} e)$ ~~...~~

$(\operatorname{ad} e)^{N(x)}(x) = 0$

$l > N(x)$

$(\exp \operatorname{ad} e)(x) = \left(1 + \dots + \frac{(\operatorname{ad} e)^{N(x)}}{N(x)!}\right)(x)$

Lemma. Given a function $\nu : \Phi \longrightarrow \mathbb{N}$ and $\phi \in \Phi$, there ex. l_0 s.t. the following property holds

For $l \geq l_0$, L is Φ-graded Lie alg over k, char $k = l$,

and $e \in L_\alpha$, $f \in L_{-\alpha}$ and $h = [e, f]$ have the property $(\operatorname{ad} e)^{\nu(\psi)}(x) = 0$, $(\operatorname{ad} f)^{\nu(\psi)}(x) = 0$, $(\operatorname{ad} h)(x) = (\alpha, \psi)(x)$ for all $x \in L_\psi$, then

$\dim L_\phi = \dim L_{s_\alpha(\phi)}$

$l_0' > \nu(\phi)$

$\theta_\alpha : L \longrightarrow L$

$L_\phi \longrightarrow L_{s_\alpha \phi}$ $\quad (\alpha, \phi) \neq 0 \quad , (\alpha, \phi + r\alpha)$

$L_\Lambda(R_X)$

Lemma 2 (Confirmation of the condition in Lemma 1).

Given a weight w and a vertex v, there is a function $\nu : \hat{\Pi} \longrightarrow \mathbb{N}$ s.t. for any w.p.l. X of type w over a finite field k the Lie alg $L = L_\Lambda(R_X)$ satisfies

$(\operatorname{ad} e_{v_0})^{\nu(\psi)}(x) = (\operatorname{ad} f_{v_0})^{\nu(\psi)}(x) = 0$ for all $\psi \in \hat{\Pi}$ and $x \in L_\psi$.

$$v = \begin{cases} ij & \rightsquigarrow S_{ij} \\ * & \rightsquigarrow 0 \end{cases}$$

$[X] \in L_{\alpha v_0}$ s.t. $Ext^1(X,X) = 0$, X indec

$[Y] \in L_\psi$ $\qquad (ad[X])([Y]) = [X,Y]$

$$= \sum_{Z \text{ ind}} (F_{XY}^Z - F_{YX}^Z) u_Z$$

$\eta : 0 \to X \to Z \to Y \to 0$ or $0 \to Y \to Z \to X \to 0$
non split

$Hom(X,-)$ to $\eta \Rightarrow Ext^1(X,Z) \cong Ext^1(X,Y)$

$Hom(-,X)$ to $\eta \Rightarrow Hom(X,X) \xrightarrow{f} Ext^1(Y,X)$

$$\to Ext^1(Z,X) \to 0$$

$f \neq 0 \Rightarrow \dim Ext^1(Y,X) > \dim Ext^1(Z,X)$

So $\dim Ext^1(X,Z) + \dim Ext^1(Z,X)$

$$< \dim Ext^1(X,Y) + \dim Ext^1(Y,X)$$

$$0 \to Z \to Z_1 \to Y \to 0$$

Choose $n > \dim Ext^1(X,Y) + \dim Ext^1(Y,X)$

$$(ad X)^{n(X,Y)}(Y) = 0$$

$n(X,Y) \rightsquigarrow$ depends on w and ψ (Lem 1)

<u>Lemma 3</u>. Suppose given a weight seq w and a vertex v. Let ~~...~~ $0 \neq \phi \in \hat{\Gamma}_+$. For any prime p, there is a power p^n s.t., if $\mathbb{X}$ is a w.p.l. of type w over a field k with $|k| = p^n$, then $\dim L_\phi = \dim L_{s_v(\phi)}$ (*)

<u>Pf</u>. Lem 2 $\Rightarrow \exists\ \nu : \hat{\Gamma} \to \mathbb{N}$ and L over smaller k_0

$\Downarrow$ Lem 1

$l \geq l_0$ + nilpotent $\Rightarrow (*)$

Choose n s.t. $l_0 \mid p^n - 1$ $\Rightarrow |k| = p^n$

$\Rightarrow |k| = 1$ in Λ

$L_\Lambda(R_*) \Rightarrow (*)$ $\blacksquare$

$k \rightsquigarrow \bar{k}$, char $k \neq 0$ $\rightsquigarrow K$ char $K = 0$

Case for quiver

$\operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n, x_{st}] \xrightarrow{f} \operatorname{Spec} \mathbb{Z}$

$f^{-1}(0)$ $\quad R(\alpha)(\bar{\mathbb{Q}})$

$f^{-1}(p)$ $\quad R(\alpha)(\bar{\mathbb{Z}}_p)$ $\quad R(\alpha)$ module variety of α

$\dim f^{-1}(0) = \dim f^{-1}(p)$

<u>Lemma</u>. Suppose given a weight seq ω and a vertex v. Let $\phi \neq 0 \in \hat{\Pi}_+$. If X is a w.p.l. of type ω over an alg. closed field, then,

$n(\phi) = \max_d \{\dim V_d^{ind} - d\}$

$d = \dim \mathcal{O}_x$ and $t(\phi)$ the number of irred. comp. of $\dim = n(\phi) + d$ are equal to $n(S_v^i(\phi))$ and $t(S_v(\phi))$

?? $\Rightarrow \dim L_\phi =$ ~~$n(\phi)$~~

the number of iso cl. of type ϕ over alg. cl. field

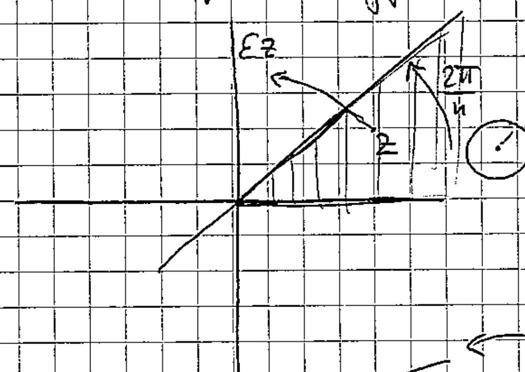
H Lenzing

Remarks on weighted proj. lines.

① Window to non-comm. alg. geometry.

Toy example. Weighted affine line

$k = \mathbb{C}$

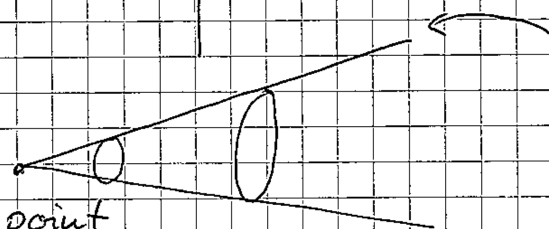


$\mathbb{C}$, $G = C_n = \langle \varepsilon \rangle$

$\varepsilon = \sqrt[n]{1}$

G acts on $\mathbb{C}$

$\mathbb{C}/G$ =



equipped with a metric

special point having weight n

3 math. objects for $\mathbb{C}/G$, i.e. $\mathrm{coh}(\mathbb{C}/G)$

- $\mathrm{coh}(\mathbb{C}) = \mathrm{mod}\text{-}\underbrace{\mathbb{C}[X]}_{A} \circlearrowleft G \quad x \mapsto \varepsilon x$

  $\mathrm{coh}(\mathbb{C}/G)$

  – $\mathrm{mod}\text{-}A[G]$ skew gp alg

  $A[G] = \bigoplus_{g \in G} A.g$

  $g \cdot a = {}^{g}a\, g \qquad {}^{g}a = g.a$

  – $A = \bigoplus_{n \in \mathbb{Z}_n} A_n$, $\deg(\mathbb{C}) = 0$
  $\deg X = [1] \in \mathbb{Z}_n$

  $\mathrm{mod}^{\mathbb{Z}_n} \mathbb{C}[X]$

  – $\mathrm{rep}_{\mathbb{C}}\, k$ $n \nearrow 1 \to 2 \searrow 3 \swarrow \dots \nwarrow$

Fact The 3 cat. are nat. eq. and serve as $\mathrm{coh}(\mathbb{C}/G)$.

The same question we ask to alg. geometry or invariant theory.

$$\left\{\begin{matrix}\text{cat. of}\\ \text{affine } \mathbb{C}\text{-spaces}\end{matrix}\right\}^{op} = \left\{\begin{matrix}\text{cat. of (f.g.)}\\ \text{comm. } \mathbb{C}\text{-algebras}\end{matrix}\right\}$$

$$G \circlearrowright X \longmapsto \mathbb{C}[X] \quad \text{coord. algebra}$$

$$\begin{tikzcd} X \arrow[rr, "g \in G"] \arrow[dr, "\pi"'] & \circlearrowleft & X \arrow[dl, "\pi"] \\ & X/G & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}[X] & \circlearrowleft & \mathbb{C}[X] \arrow[ll, "g"'] \\ A \arrow[u, "j"] \arrow[urr, "j'"'] & & \end{tikzcd} \qquad A = \mathbb{C}[X]^G$$

We are told by theory:

$$\begin{aligned} \mathrm{coh}(\mathbb{C}/G) &= \mathrm{mod}(\mathbb{C}[X]^G) \\ G \cong \mathbb{Z}_n \quad &= \mathrm{mod}(\mathbb{C}[X^n]) \\ &\cong \mathrm{mod}(\mathbb{C}[X]) \end{aligned}$$

<u>Explanation</u>. Have to pass to a larger category of non-comm. aff. $\mathbb{C}$-spaces and to form the quotient there

$$\left\{\begin{matrix}\text{non-comm.}\\ \text{affine}\\ \mathbb{C}\text{-spaces}\end{matrix}\right\} := \left\{\begin{matrix}\text{f.g. (non comm)}\\ \mathbb{C}\text{-alg.}\end{matrix}\right\}^{op}$$

<u>Conclusion</u>. Generally speaking: group quotients of alg. varieties / schemes / $k = \bar{k}$ lead to non-comm. spaces

<u>Back to example</u>. Not possible to interpret
$$\mathrm{mod}\left(k\left[\begin{tikzcd}[sep=small] 1 \arrow[r] & 2 \arrow[dl] \\ 3 \arrow[u] & \end{tikzcd}\right]\right)$$
as a comm. variety / scheme

<u>Reason</u>: $\exists$ simple objects $S_1 \not\cong S_2$ with $\mathrm{Ext}^1(S_1, S_2) \neq 0$

Lemma. A any comm. ring.
S, T simple A-modules, $S \not\cong T$
$\Rightarrow \operatorname{Ext}^i_A(S,T) = 0$

Proof. Write $S = A/\mathfrak{m}$, $T = A/\mathfrak{m}'$, $\mathfrak{m} \neq \mathfrak{m}'$.

$$\left.\begin{array}{l} \mathfrak{m} \text{ annihilates } S, \text{ hence } \operatorname{Ext}^i(S,T) \\ \mathfrak{m}' \quad '' \quad T, \quad '' \quad \operatorname{Ext}^i(S,T) \end{array}\right\} A = \mathfrak{m} + \mathfrak{m}'$$

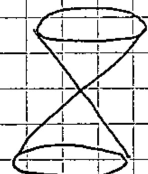
$\square$

② Trichotomy

$k = \bar{k}$, A f.d. k-alg.

$$\text{rep.type } A = \begin{cases} \text{finite} \\ \text{tame} \\ \text{wild} \end{cases} \qquad \text{(Drozd)}$$

- -200 Appolonius from Perge



$$\begin{array}{l} \text{inters} \\ \text{with} \\ \text{plane} \end{array} = \begin{cases} \text{ellipse} & \text{simple} \\ \text{parabola} & \text{medium} \\ \text{hyperbola} & \text{difficult} \end{cases}$$

- First half of 19^{th} century

$$\text{plane geometry}/\mathbb{R} = \begin{cases} \text{sperical} & \text{simple} \\ \text{euclidean} & \text{medium} \\ \text{hyperbolic} & \text{difficult} \end{cases}$$

- End of 19^{th} Century

$$\begin{array}{l} \text{Compact Riemann} \\ \text{Surfaces} = \text{Smooth} \\ \text{projective curves}/\mathbb{C} \end{array} = \begin{cases} S^2 = \mathbb{P}_1(\mathbb{C}) & \chi > 0 \\ \text{elliptic curves} = 2\text{-tori} & \chi = 0 \\ \text{the rest} & \chi < 0 \end{cases}$$

16/4

$$\left\{\begin{array}{l}\text{Compact}\\ \text{Riemann}\\ \text{surfaces}\end{array}\right\} \xrightarrow{\ \sim\ } \left\{\begin{array}{l}\text{smooth}\\ \text{proj. curves}\\ /\mathbb{C}\end{array}\right\}$$

X $\searrow \wr$ $\qquad$ $\wr \swarrow$ $\quad X \longmapsto \mathrm{coh}(X)$

$$\left\{\begin{array}{l}\text{function fields}\\ /\mathbb{C} \text{ in one}\\ \text{variable}\end{array}\right\} \qquad \longmapsto \frac{\mathrm{coh}(X)}{\mathrm{coh}_0(X)} \simeq \text{mod-}K \quad \text{field}$$

$\longleftarrow$ f.l.

$$\begin{array}{l} K \\ \mid \ \text{finite ext.} \\ K = \mathbb{C}(x) \\ \mid \ \ x \text{ transc.}/\mathbb{C} \\ \mathbb{C} \end{array}$$

$\mathbb{C}(X) = \{ f \mid f: X \to \mathbb{C} \cup \{\infty\} \text{ merom.}\}$ field

One non-trivial step: exhibit a non-constant merom. function.

$x: X \xrightarrow{\text{onto}} S^2$, finite # of sheets

$\mathbb{C}(x)$

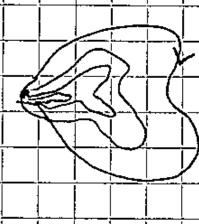
Relate {3 plane geometries} to X_x, X comp. Riem. surface!

Given compact Riemann surface X, we form the universal covering space $\tilde{X}$. (expl. of $\tilde{X}$ postponed)

$$\begin{array}{ccc} \tilde{x} & & \tilde{X} \\ \downarrow & & \downarrow \pi \\ x & & X \end{array}$$

- top. space Hausdorff
- π local homeom.
- $\tilde{X}$ becomes a Riemann surface

$\tilde{X}$ is simply connected



Riemann Mapping Thm:

Y Riemann surface simply connected

$$\Rightarrow Y = \begin{cases} \mathbb{S}^2 & \text{spherical plane} \\ \mathbb{C} & \text{euclidean plane} \\ \mathbb{H} = \{z \mid z \in \mathbb{C}, \operatorname{Im}(z) > 0\} & \text{hyperb. plane} \end{cases}$$

X comp. Riem. surface with conn. points, then univ. cover

$$\tilde{X} \text{ is } \begin{cases} \text{spherical} \iff \chi_X > 0 \\ \text{euclidean} \iff \chi_X = 0 \\ \text{hyperb.} \iff \chi_X < 0 \end{cases}$$

X comp. R. surface, $\tilde{X}$ univ. cover.

X & $\tilde{X}$ share the same local properties.

$$\tilde{X} = \begin{cases} \mathbb{S}^2 & \text{const. curv.} > 0 \\ \mathbb{C} & = 0 \\ \mathbb{H} & \text{const. curv.} < 0 \end{cases}$$

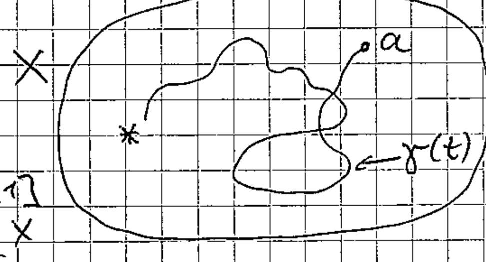
Relationship to Euler char. Gauß – Bonnet.

X compact RS

$$\chi_X = \frac{1}{2\pi} \int_X \kappa \, dX$$

Normalization of total surface volume assumed

Fundamental group & univ. cover



$\gamma : [0,1] \xrightarrow{\text{cont.}} X$

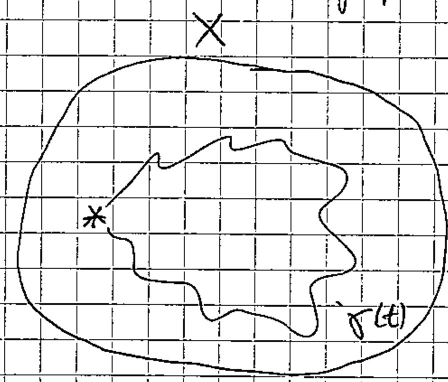
$*$ base point

Elements of $\tilde{X}$

homotopy classes (with fixed end points) of cont. curves starting at x

- $\pi : \tilde{X} \xrightarrow{\text{local hom.}} X$
  $[\gamma] \longmapsto \gamma(1)$
- $\tilde{X}$ is simply connected

Fundamental grp



$\gamma : [0,1] \longrightarrow X$

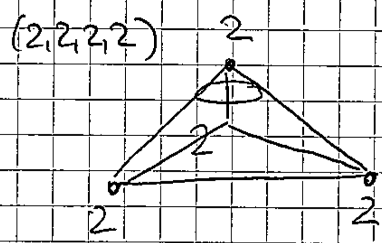
$\pi_1(X, *) \ni [\gamma]$

- γ closed curve $\gamma(0) = *$
- mult. by concatenation
- yields a usually non-comm. group $\pi_1(X)$
- $\pi_1(X)_{ab} = H_1(X)$ ← sing. hom

<u>Now weighted version</u>.

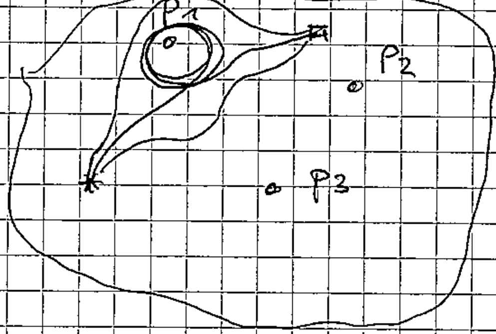
$$\left\{\begin{array}{l}\text{Compact}\\ \text{Riemann}\\ \text{Surfaces}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{smooth proj.}\\ \text{curves}/\mathbb{C}\\ \text{with a fin. \#}\\ \text{of weighted pts}\end{array}\right\}$$

with a fin. number of <u>cone points</u>



all 4 sides are congruent triangles

X with cone pts



$\pi_1(X, *)$

For homotopy the following rules are to be observed

- Any homotopy (with fixed end pts) in complement of marked pts o.k.
- Moving "around" marked pts affords cycling around p-times

(3) Representations of finite (and discrete) groups.

(p_1, p_2, p_3) weight type for a proj. line X

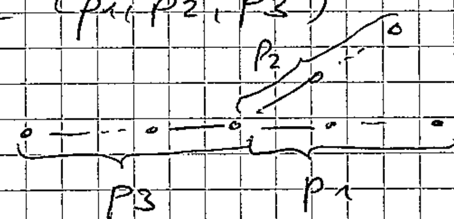
<u>Fact</u>. $\pi_1(X(p_1,p_2,p_3)) = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^{p_1} = \sigma_2^{p_2} = \sigma_3^{p_3} = \sigma_1\sigma_2\sigma_3 = 1 \rangle$

<u>Ex</u>. $(2,3,5)$ icosahedral gp $(\cong A_5)$
$(2,3,4)$ octahedral gp
$(2,3,3)$ tetrahedral gp

<u>Thm</u>. Assume $\{\pm 1\} < G < Sl_2(\mathbb{C})$, G finite

then, for some <u>Dynkin triple</u> (p_1, p_2, p_3)

$G \cong \pi_1(\mathbb{X}(p_1,p_2,p_3))$



<u>Note</u>. G naturally acts on $\mathbb{C}[X,Y] = \mathrm{Sym}(\mathbb{C}^2)$.
Action preserves the degree.

$\mathrm{coh}(\mathbb{P}_1\mathbb{C}) = \dfrac{\mathrm{mod}^{\mathbb{Z}}\,\mathbb{C}[X,Y]}{\mathrm{mod}_0^{\mathbb{Z}}\,\mathbb{C}[X,Y]}$

$\Rightarrow$ G naturally acts on $\mathrm{coh}(\mathbb{P}_1)$
(actually acts on $\mathbb{P}_1$ and then induces)

<u>Thm</u>. $\mathrm{coh}(\mathbb{X}(p_1,p_2,p_3)) = \mathrm{coh}(\mathbb{P}_1)[G]$

Dynkin (p_1,p_2,p_3)

- G-equiv. sheaves
- skew group category

$\mathcal{C}$ is an abelian category

G finite grp acting on $\mathcal{C}$: $(g \in G, X) \mapsto X^g$

$$(g \in G, X \xrightarrow{f} Y) \mapsto X^g \xrightarrow{f^g} Y^g$$

Now – objects of $\mathcal{C}[G]$

$(X, (u_g))$
- $X \in \operatorname{obj} \mathcal{C}$
- $u_g : X \longrightarrow X^g$
-
$$\begin{tikzcd} X \arrow[rr, "u_{gh}"] \arrow[dr, "u_h"'] & & X^{gh} \\ & X^g \arrow[ur, "(u_g)^h"'] & \end{tikzcd}$$

$$\begin{tikzcd} (X, (u_g)) \arrow[d] \\ (Y, (v_g)) \end{tikzcd}$$

- obvious notion of morph.

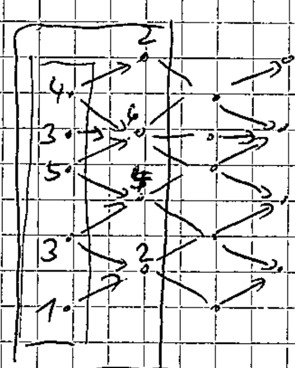
<u>Moral</u> $(\operatorname{coh} \mathbb{P}_1)[G]$ mixes "sheaf theory of $\mathbb{P}_1$" with "the $\mathbb{C}$-lin. rep. theory of G".

the important char. of the rep. theory of G = simple (= irred) reps + degree info.

Mc Kay correspondence

<u>Ex.</u> $(2,3,5)$ $G =$ binary icos. grp.

$\operatorname{ind} \operatorname{vect}(\mathbb{P}_1(2,3,5)) = \operatorname{mesh} \operatorname{cat}(\mathbb{Z}\tilde{E}_8)$



the slice $\boxed{}$ is in bij. corr. to the irred. reps of $G/\{\pm 1\}$.

The slice $\boxed{}$ is in bij. corr. to the irred. reps of G.

(4) Factoriality

$k=\bar{k}$

$R_1 = \dfrac{k[[x,y,z]]}{(x^2+y^3+z^5)}$ factorial

$R_2 = \dfrac{k[[x,y,z]]}{(x^2+x^3+z^7)}$ not factorial

? Why ?

"Answer"

$\mathrm{coh}(\mathbb{X}(2,3,5)) \longleftarrow$ tame domestic

$\mathrm{coh}(\mathbb{X}(2,3,7)) \longleftarrow$ wild

Thm (Kussin) $k=\bar{k}$. The coordinate algebras

$\left\| \begin{matrix} t \text{ weights} \\ p_1,\dots,p_t \end{matrix} \right| \begin{matrix} \lambda_3=1 \\ \lambda_4,\dots,\lambda_t \text{ p.w. dist.} \end{matrix} \quad S = \dfrac{k[x_1,\dots,x_t]}{(t-2)\text{ relations}} \quad \begin{matrix} x_3^{p_3} = x_2^{p_2} - \lambda_3 x_1^{p_1} \\ \vdots \\ x_t^{p_t} = x_2^{p_2} - \lambda_t x_1^{p_1} \end{matrix}$

of weighted proj. lines are exactly the following

- group graded, affine, comm. k-alg of Krull dimension two
- grading group is abelian of rk 1

Auslander-Reiten. Some general assumptions,
R $\mathbb{Z}$-graded comm., pos. $\mathbb{Z}$-graded
Then, the completion functor

$$CM^{\mathbb{Z}} R \xrightarrow{\ \wedge\ } CM(\hat{R})$$

preserves indec's, commutes with AR-transl.
Moreover, if R has finite CM-type, then "$\wedge$" is dense.

$CM^{\mathbb{Z}} R = \mathrm{vect}\, \mathbb{P}_1(2,3,5)$ $\qquad R = \dfrac{k[[x,y,z]]}{(x^2+y^3+z^5)}$

Up to τ-shift, we have only finitely many indec. $\qquad \cup$ f.g. free T-mod

$T = k[[u,v]]$

$\Rightarrow$ Setting of AR-thm $\Rightarrow$ $\wedge$: ind vect $\mathbb{P}_1(2,3,5)$
$\longrightarrow CM(\hat{R})$
is dense

$\Rightarrow$ $\hat{R}$ only inv. CM-module $\Rightarrow$ factorial.